Collect. Math. **44** (1993), 201–210

A commutator theorem with applications

MARIO MILMAN*

Department of Mathematics, Florida Atlantic University,
Boca Raton, Florida 33431, U.S.A.

ABSTRACT

We give an extension of the commutator theorems of Jawerth, Rochberg and Weiss [9] for the real method of interpolation. The results are motivated by recent work by Iwaniec and Sbordone [6] on generalized Hodge decompositions. The main estimates of these authors are based on a commutator theorem for a specific operator acting on L^p spaces and through the use of the complex method of interpolation. In this note we give an extension of the Iwaniec-Sbordone theorem to general real interpolation scales.

1. Introduction

In [13] and [9] Jawerth, Rochberg and Weiss initiated the study of second order and abstract commutator theorems for scales of interpolation spaces. Recall that given a compatible pair of Banach spaces the classical constructions of interpolation theory provide methods to obtain parameterized families of spaces with the interpolation property. That is if an operator T is bounded from a compatible pair $\overline{A}$ to another compatible pair $\overline{B}$ then T will also be bounded on the corresponding interpolation spaces. Jawerth, Rochberg, and Weiss (cf. [13] and [9]) have shown that associated with the classical methods of interpolation are certain operators, Ω, generally unbounded and non-linear, which can be obtained by differentiation with respect to certain parameters used in the specific method. These operators have the property that their commutator with a bounded operator T in the scale,

*Supported in part by NSF grant DMS–9100383

$[\Omega, T]$, is also bounded in the scale. This was shown in [13] for the complex method, and in [9] for the real methods. For example, in the case of L^p spaces one can use the operators $\Omega f = f \log(|f|/\|f\|_p)$. These results have interesting applications in analysis. We refer to these papers, and also to [2] and the survey [3] for a detailed account. We should also point out an interesting connection of the subject under consideration and the theory of logarithmic Sobolev inequalities (cf. [3]), in fact some of the basic ideas of the theory, for the complex method and in the L^p setting, are already implicit in Feissner's [5] study of higher order logarithmic Sobolev inequalities. In the setting of lattices these results have been considerably extended by Kalton (cf. [10] and the papers quoted therein) who has exhibited a large class of operators "Ω" which *commute* with bounded operators in an interpolation scale. The methods developed in Kalton's papers are very interesting and his results have many new applications. However, it is not yet clear how Kalton's methods can be incorporated in the general theory. In [12] a new approach to the abstract commutator theorems for the real method was given, showing, in particular, commutation relations with certain non-linear operators. We also mention [11] where a connection to the functional calculus for positive operators in Banach spaces is developed. The connections of this subject with "extrapolation theory" are also explored in [7] and [11]. A general unified approach to commutator theorems for the real and complex methods has been obtained in the forthcoming paper [4].

Recently in their study of minimizers for variational problems Iwaniec and Sbordone [6] have obtained and used the following commutator theorem using the complex method of interpolation.

Theorem 1

Let T be an operator $T: L^p \to L^p$, $p \in [r_1, r_2]$, where $1 \le r_1 < r_2 < \infty$, and let $\frac{p}{r_2} - 1 \le \varepsilon \le \frac{p}{r_1} - 1$. Define

$$\Omega_\varepsilon(f) = \left(\frac{|f|}{\|f\|_p}\right)^\varepsilon f.$$

Then,

$$\|[T, \Omega_\varepsilon]\|_{p/(1+\varepsilon)} \le c_p |\varepsilon| \, \|f\|_p \tag{1}$$

where

$$c_p = \frac{2p(r_2 - r_1)}{(p - r_1)(r_2 - p)} \sup_{r_1 \le s \le r_2} \|T\|_s.$$

This is a useful variant of the commutator theorem of [13] and can be obtained by the complex method. One of the main points in the applications of (1) is the fact that ε can be negative. Let us also point out that, as was observed in [6], letting $\varepsilon \to 0$ in (1) we obtain the Rochberg-Weiss [13] theorem in the context of L^p spaces

$$\|[T,\Omega]\|_p \leq c_p \|f\|_p \tag{2}$$

where $\Omega f = f \log\left(|f|/\|f\|_p\right)$.

The purpose of this note is to point out that the Iwaniec-Sbordone result can be incorporated to the general theory of commutator inequalities for real method of interpolation. Thus, we exhibit a general class of operators Ω_ε which commute, with bounded operators T acting on the initial pairs, in the sense that an estimate of the type (1) holds for $[\Omega_\varepsilon, T]$ inside the real interpolation scale. When specialized to the L^p setting our results give the Iwaniec-Sbordone theorem with a less precise constant.

We assume that the reader is familiar with the basic results of interpolation theory as developed in [1], where we refer for background information. In order to make the paper self contained we have included a brief summary of the necessary definitions concerning the theory of real interpolation commutators.

2. Quasi-logarithmic operators associated to real interpolation

In this section we briefly review the relevant definitions from interpolation theory and introduce the relevant operators that we shall study in this paper. We refer to [1], [9] and [3] for more details.

Let $\overline{A} = (A_0, A_1)$ be a Banach pair, $a \in \Sigma(\overline{A}) = A_0 + A_1$, and recall that the K functional of a is defined, for $t > 0$, by

$$K(t,a;\overline{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}.$$

The interpolation spaces $\overline{A}_{\theta,q;K}, 0 < \theta < 1,\ 0 < q \leq \infty$, are defined by

$$\overline{A}_{\theta,q;K} = \left\{a : \|a\|_{\overline{A}_{\theta,q;K}} = \left\{\int_0^\infty [t^{-\theta}K(t,a,\overline{A})]^q \frac{dt}{t}\right\}^{1/q} < \infty\right\}. \tag{3}$$

We shall be concerned with the process of computing these interpolation norms. We say that the decomposition $a = a_0(t) + a_1(t)$ is almost optimal for the K method if

$$\|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1} \leq cK(t,a;\overline{A})$$

where c is a constant fixed before hand, say $c = 2$. We then write $D_K(t;\overline{A})a = D_K(t)a = a_0(t)$. The operator Ω associated with this decomposition is defined by

$$\Omega_{\overline{A};K}a = \int_0^1 D_K(t)a\frac{dt}{t} - \int_1^\infty (I - D_K(t))a\frac{dt}{t}. \tag{4}$$

Similarly, we can define the corresponding operators Ω associated with the J and E methods. Recall that given a Banach pair $\overline{A}$ the spaces $\overline{A}_{\theta,q;J}$, $0 < \theta < 1$, $0 < q \leq \infty$, are defined using the quasi-norms

$$\|a\|_{\overline{A}_{\theta,q;J}} = \inf\left\{\left[\int_0^\infty \left(J(s,u(s);\overline{A})s^{-\theta}\right)^q\frac{ds}{s}\right]^{1/q} : a = \int_0^\infty u(s)\frac{ds}{s}\right\}$$

where $u: (0,\infty) \to \Delta(\overline{A})$, and the J functional is defined for $h \in \Delta(\overline{A})$, $t > 0$, by

$$J(t,h;\overline{A}) = \max\{\|h\|_{A_0}, t\|h\|_{A_1}\}.$$

We shall say that $u(t)$ is an almost optimal decomposition of a for the J method, and write $D_J(t,\overline{A})a = D_J(t)a = u(t)$, if

$$a = \int_0^\infty u(s)\frac{ds}{s}, \ \|a\|_{\overline{A}_{\alpha,q;J}} \approx \left\{\int_0^\infty [J(s,u(s);\overline{A})s^{-\theta}]^q\frac{ds}{s}\right\}^{1/q}. \tag{5}$$

The corresponding Ω_J operator is defined by

$$\Omega_J a = \int_0^\infty D_J(t)a\log t\frac{dt}{t}. \tag{6}$$

For the E method we have a similar definition. Recall that

$$E(t,a;\overline{A}) = \inf_{\|a_1\|_{A_1}\leq t}\{\|a_0\|_{A_0} : a = a_0 + a_1\}.$$

The corresponding interpolation spaces $\overline{A}_{\theta,q;E}$, $0 < \theta < \infty$, $0 < q \leq \infty$, are defined using the quasi-norms

$$\|f\|_{\overline{A}_{\theta,q;E}} = \left\{\int_0^\infty [t^\theta E(t,f,\overline{A})]^q\frac{dt}{t}\right\}^{1/q}. \tag{7}$$

Let $D_E(t;\overline{A}) = D_E(t)a = a_0(t)$, for an almost optimal decomposition, that is such that

$$E(t,a;\overline{A}) \approx \|D_E(t)a\|_{A_0}. \tag{8}$$

Then, the corresponding $\Omega's$ are defined by

$$\Omega_E a = \int_0^1 D_E(t) a \frac{dt}{t} - \int_1^\infty (I - D_E(t)) a \frac{dt}{t}. \tag{9}$$

The main result of [9] is that if T is a bounded operator $T: \overline{A} \to \overline{B}$, and F denotes any of these methods of interpolation, then there exists a constant $c(F)$ such that if we let $[\Omega_F, T] = \Omega_{F(\overline{B})} T - T\Omega_{F(\overline{A})}$, then

$$\|[\Omega_F, T] f\|_{F(\overline{B})} \leq c\|f\|_{F(\overline{A})}.$$

This result also holds for the complex method (cf. [13]).

In the next sections we consider variants of these operators and commutator theorems for them.

3. A commutator theorem for the E method

We consider first variants of the Ω operators associated with the E method since it is the method that will provide us with an appropriate generalization of Theorem 1. Let $\alpha \in (-1, 1)$, $\alpha \neq 0$, and define

$$\Omega_{E,\alpha} a = \Omega_\alpha a = \alpha \Big(\int_1^\infty D_E(t) a t^\alpha \frac{dt}{t} - \int_0^1 (I - D_E(t)) a t^\alpha \frac{dt}{t} \Big).$$

Theorem 2

Let $\overline{A}$ and $\overline{B}$ be a Banach pairs, $T: \overline{A} \to \overline{B}$ be a bounded operator, then there exists a constant $c > 0$ such that if $\theta + \alpha > 0$,

$$\|[\Omega_\alpha, T] f\|_{(B_0, B_1)_{\theta/(\alpha+1), q; E}} \leq \frac{c}{\theta} |\alpha| (2c_\alpha)^{\theta/(\alpha+1)} (\alpha + 1)^{1/q} \|f\|_{(A_0, A_1)_{\theta+\alpha, q; E}}.$$

Proof. It is easy to see that according to our definitions for any Banach pair $\overline{H}$, and for $t > 0$, we have

$$\Omega_{\alpha,\overline{H}} a + a\varphi_\alpha(t) = \alpha \Big(\int_t^\infty D_{E,\overline{H}}(s) a s^\alpha \frac{ds}{s} - \int_0^t (I - D_{E,\overline{H}}(s)) a s^\alpha \frac{ds}{s} \Big) \tag{10}$$

where $\varphi_\alpha(t) = 1 - t^\alpha$.

Let $\widetilde{a}_1(t) = \frac{1}{\alpha}(\int_0^t (I - D_E(s))as^\alpha \frac{ds}{s})$, then

$$\|\widetilde{a}_1(t)\|_{H_1} \leq |\alpha|\Big(\int_0^t \|(I - D_E(s))a\|_{H_1} s^\alpha \frac{ds}{s}\Big) \leq \frac{|\alpha|}{(\alpha+1)} t^{\alpha+1}. \tag{11}$$

Thus, letting $c_\alpha = |\alpha|(\alpha+1)^{-1}$ and combining (10), (11), and (8), we get

$$E\big(c_\alpha t^{\alpha+1}, \Omega_\alpha a + \varphi_\alpha(t)a; \overline{H}\big) \leq |\alpha|\Big(\int_t^\infty E(s, a, \overline{H}) s^\alpha \frac{ds}{s}\Big). \tag{12}$$

Therefore if $T: \overline{A} \to \overline{B}$, then we can estimate $E(2c_\alpha t^{\alpha+1}, \Omega_{\alpha,\overline{B}} Ta - T\Omega_{\alpha,\overline{A}} a; \overline{B})$ as less than or equal to

$$E\big(c_\alpha t^{\alpha+1}, \Omega_{\alpha,\overline{B}} Ta + \varphi_\alpha(t) Ta; \overline{B}\big) + E\Big(c_\alpha t^{\alpha+1}, T\big(\Omega_{\alpha,\overline{A}} a + \varphi_\alpha(t)a\big); \overline{B}\Big).$$

Using the fact that T is bounded, and applying (12) to each of these terms we get

$$E(2c_\alpha t^{\alpha+1}, \Omega_{\alpha,\overline{B}} Ta - T\Omega_{\alpha,\overline{A}} a; \overline{B}) \leq c|\alpha|\Big(\int_t^\infty E(s, a, \overline{A}) s^\alpha \frac{ds}{s}\Big)$$

where c depends only on the norm of T on the initial pair. An application of Hardy's inequality (cf. [14]) now yields

$$\Big\{\int_0^\infty [t^\theta E(2c_\alpha t^{\alpha+1}, \Omega_{\alpha,\overline{B}} Ta - T\Omega_{\alpha,\overline{A}} a; \overline{B})]^q \frac{dt}{t}\Big\}^{1/q} \\ \leq \frac{c|\alpha|}{\theta}\Big\{\int_0^\infty [E(s, a, \overline{A}) s^{\alpha+\theta}]^q \frac{ds}{s}\Big\}^{1/q}$$

and therefore we finally get

$$\|[\Omega_\alpha, T]a\|_{(B_0,B_1)_{\theta/(\alpha+1),q;E}} \leq \frac{c|\alpha|}{\theta}(2c_\alpha)^{\theta/(\alpha+1)}(\alpha+1)^{1/q}\|a\|_{(A_0,A_1)_{\theta+\alpha,q;E}}. \square$$

We consider now in detail the special case of L^p spaces. Although the calculation of the interpolation spaces in this case is well known we include the details for the sake of completeness and the reader's convenience. The E functional for the pair (L^1, L^∞) is well known and easy to compute (cf. [1], [8])

$$E(t, f, L^1, L^\infty) = \int_t^\infty \lambda_f(s) ds \tag{13}$$

and an approximate optimal decomposition is given by $f = f\chi_{\{|f|>t\}} + f\chi_{\{|f|\leq t\}}$. (In fact an optimal decomposition is given by $f = (f-t)^+ + t\chi_{\{|f|\leq t\}})$. The interpolation spaces for the E method can be determined using this formula. In fact if we recall the formula

$$\|f\|_p = \left\{ p \int_0^\infty \lambda_f(s) s^{p-1} ds \right\}^{1/p}$$

we see, using (13) and integration by parts that

$$\|f\|_{(L^1, L^\infty)_{p-1,1;E}} = [(p-1)p]^{-1} \|f\|_p^p.$$

A calculation using (9) gives

$$\Omega_\alpha f = f|f|^\alpha - f.$$

Let us set $S_\alpha f = f|f|^\alpha$, then we clearly have $[T, \Omega_\alpha] = [T, S_\alpha]$. Now to apply Theorem 2 we let

$$\frac{\theta}{\alpha+1} = \frac{r}{1+\alpha} - 1, \quad \text{then} \quad \theta + \alpha = r - 1$$

and the previous discussion gives

$$\|TS_\alpha f - S_\alpha Tf\|_{r/1+\alpha}^{r/1+\alpha} \leq c2^{(r-1-\alpha)/(\alpha+1)} \left(\frac{|\alpha|}{(\alpha+1)} \right)^{r/\alpha+1} \frac{1}{(r-1)} \|f\|_r^r.$$

Raising both members of the previous inequality to the power $\frac{1+\alpha}{r}$ gives an estimate of Iwaniec-Sbordone type,

$$\left\| T\left(\frac{|f|^\alpha f}{\|f\|_r^\alpha} \right) - \frac{|Tf|^\alpha Tf}{\|f\|_r^\alpha} \right\|_{r/1+\alpha} \leq c \left(\frac{|\alpha|}{(\alpha+1)} \right) \left(\frac{1}{(r-1)} \right)^{(\alpha+1)/r} \|f\|_r. \qquad (14)$$

In order to obtain a version of Theorem 1 we argue that

$$\left\| T\left(\frac{|f|^\alpha f}{\|f\|_r^\alpha} \right) - \frac{|Tf|^\alpha Tf}{\|Tf\|_r^\alpha} \right\|_{r/1+\alpha} \leq \left\| T\left(\frac{|f|^\alpha f}{\|f\|_r^\alpha} \right) - \frac{|Tf|^\alpha Tf}{\|f\|_r^\alpha} \right\|_{r/1+\alpha} + \left\| \frac{|Tf|^\alpha Tf}{\|f\|_r^\alpha} - \frac{|Tf|^\alpha Tf}{\|Tf\|_r^\alpha} \right\|_{r/1+\alpha} = I + II, \text{ say.}$$

I is controlled by (14) while II can be readily computed

$$II = \|Tf\|_r \left| \left(\frac{\|Tf\|_r}{\|f\|_r} \right)^\alpha - 1 \right|.$$

Let $x = \|Tf\|_r$, $y = \|f\|_r$, $u = y/x$, $\varphi(u) = u^{\alpha+1} - u$, and assume, as we may, that $\|T\|_{r\to r} \leq 1$, then $u \in [0,1]$, and we have reduced everything to prove that there exists $c > 0$, such that $\forall u \in [0,1]$

$$|\varphi(u)| \leq c|\alpha|. \tag{15}$$

We study φ using calculus and we see that (15) holds with $c = (\frac{1}{1+\alpha})^{\frac{1+\alpha}{\alpha}}$. We conclude the analysis by observing that the factor $1/(1+\alpha)$ is under control by r_2/r. By collecting estimates we see that we have thus obtained an end point version of Theorem 1 by real methods with a somewhat worst constant, but with the right control when $\alpha \to 0$. By reiteration we may obtain the full result.

In a similar fashion we can deal with the family of error functionals E_β introduced in [9], this is particularly useful when dealing with pairs of weighted L^p spaces. As an example when dealing with the pair $(L^{p_0}(w_0(x)dx),\ L^{p_1}(w_1(s)dx))$ the corresponding $\Omega's$ can be chosen to be of the form

$$\Omega f = f\left(\frac{w_0}{w_1}\right)^{\varepsilon} - f.$$

For brevity sake we refer to [9] for other possible applications of Theorem 2, and where similar calculations are performed. Using these methods one can also deal with operators T that are not necessarily linear (cf. [12] for a detailed treatment of non-linear operators in the context of the K method).

4. Remarks on the K and J methods

There are many variants of the results of the previous section. We can consider the K and J methods, or consider variants of the E method (as in [9]), etc. However, since the analysis of these methods is similar to the one we developed in detail in the previous section we shall be rather brief here. In fact in the case of the K method the analysis follows closely the one given in [12]. We consider operators defined by

$$\Omega_\alpha a = \alpha\Big(\int_0^1 D_K(s)as^\alpha\frac{ds}{s} - \int_1^\infty \big(I - D_K(s)\big)as^\alpha\frac{ds}{s}\Big).$$

Then, as before we see that

$$\Omega_\alpha a - \varphi_\alpha(t)a = \alpha\Big(\int_0^t D_K(s)as^\alpha\frac{ds}{s} - \int_t^\infty \big(I - D_K(s)\big)as^\alpha\frac{ds}{s}\Big).$$

This leads to the estimate

$$K(t,\Omega_\alpha a+\varphi_\alpha(t)a,\overline{A})\leq c|\alpha|\int_0^\infty \min\left\{1,\frac{t}{s}\right\}K(s,a;\overline{A})s^\alpha\frac{ds}{s}. \tag{16}$$

Thus, if $T:\overline{A}\to\overline{B}$, we see, using the cancellation property for commutators in the usual fashion, the estimate (16), and Hardy's inequality, that

$$\|[T,\Omega_\alpha]f\|_{\overline{B}_{\theta,q;K}}\leq c(\theta,q)|\alpha|\ \|f\|_{\overline{A}_{\theta-\alpha,q;K}}.$$

We formally state this result as,

Theorem 3

Let $\overline{A},\overline{B}$, be Banach pairs, let $T:\overline{A}\to\overline{B}$, be a bounded operator, and let $\alpha\in(-1,1)\setminus\{0\}$, $0<\theta<1, 0<q\leq\infty$, and suppose that $0<\theta-\alpha<1$. Then there exists a constant $c=c(\theta,q)$ such that

$$\|[T,\Omega_\alpha]f\|_{\overline{B}_{\theta,q;K}}\leq c|\alpha|\ \|f\|_{\overline{A}_{\theta-\alpha,q;K}}.$$

Let us remark that the operators Ω_α for this method are different than those for the E method (cf. [9]). In the familiar examples of the theory they can be easily calculated by trivial modifications to the calculations of Ω in [9] and [3]. For example, for the pair (L^1,L^∞) a possible choice of Ω_α is given by $\Omega_\alpha f=f(r_f)^\alpha-f$, where r_f is the "rank function" of f defined by $r_f(x)=|\{y:|f(y)|>|f(x)|$ or $|f(y)|=|f(x)|$ and $y\leq x\}|$ (cf. [3]).

The J method admits a similar treatment and analogous results. For example a class of operators that can be treated by these methods is given by (cf. [3])

$$\Omega_\alpha a=\int_0^\infty t^\alpha D_J(t)a\frac{dt}{t}.$$

The relationship to the corresponding $\Omega_{\alpha;K}$ operators is, as usual, given by the fact that the fundamental lemma of interpolation theory implies that we can take $D_J(t)a=t\frac{d}{dt}D_K(t)a$. We also point out that the resulting theory is closely related to the functional calculus associated with positive operators in Banach spaces (cf. [11] and the references therein) and Zafran's work [15].

We shall deal elsewhere with the complex method and with applications to weighted norm inequalities for classical operators (cf. also [9], [12], [13]).

References

1. J. Bergh and J. Löfström, *Interpolation Spaces: An introduction*, Springer-Verlag, Berlin, Heidelberg and New York, 1976.
2. M. Cwikel, B. Jawerth, and M. Milman, The domain spaces of quasilogarithmic operators, *Trans. Amer. Math. Soc.* **317** (1990), 599–609.
3. M. Cwikel, B. Jawerth, M. Milman, and R. Rochberg, *Differential estimates and commutators in interpolation theory*, "Analysis at Urbana II", London Math. Soc., Cambridge Univ. Press 1989, pp. 170–220.
4. M. Cwikel, N. Kalton, M. Milman, and R. Rochberg, *in preparation*.
5. F. Feissner, Hypercontractive semigroups and Sobolev's inequality, *Trans. Amer. Math. Soc.* **210** (1975), 51–62.
6. T. Iwaniec, and C. Sbordone, *Weak minima of variational integrals*, University of Naples, preprint 1992.
7. B. Jawerth and M. Milman, New results and applications of extrapolation theory, Interpolation Spaces and Related Topics, M. Cwikel et al editors, *Israel Math. Conf. Proc.* **5** (1992), 81–105.
8. B. Jawerth and M. Milman, Interpolation of weak type spaces, *Math. Z.* **201** (1989), 509–519.
9. B. Jawerth, R. Rochberg, and G. Weiss, Commutator and other second order estimates in real interpolation theory, *Ark. Mat.* **24** (1986), 191–219.
10. N. Kalton, *Differentials of complex interpolation process for Köthe function spaces*, to appear.
11. M. Milman, *Extrapolation and Optimal Decompositions with Applications to Analysis*, Lecture Notes in Math., Springer, 1994.
12. M. Milman and T. Schonbek, Second order estimates in interpolation theory and applications, *Proc. Amer. Math. Soc.* **110** (1990), 961–969.
13. R. Rochberg and G. Weiss, Derivatives of analytic families of Banach spaces, *Ann. Math.* **118** (1983), 315–347.
14. E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, 1970.
15. M. Zafran, Spectral theory and interpolation of operators, *J. Funct. Anal.* **36** (1980), 185–204.