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Inequalities and interpolation

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Abstract

Some examples of the close interaction between inequalities and interpolation are presented and discussed. An interpolation technique to prove generalized Clarkson type inequalities is pointed out. We also discuss and apply to the theory of interpolation the recently found facts that the Gustavsson-Peetre class P^{+-} can be described by one Carlson type inequality and that the wider class P_0 can be characterized by another Carlson type inequality with "blocks".

0. Introduction

The first interpolation proof of an inequality (Hausdorff-Young's inequality) was given already in 1926 by M. Riesz [56]. He wanted to find a simple proof of the Hausdorff-Young inequality and this was the main reason to prove the convexity theorem of Riesz. Nowadays it is well-known that also most of the other classical inequalities (e.g. those by Paley, Young, Hölder, Minkowski, Beckenbach-Dresher, Clarkson, Carlson, Grothendieck etc.) can easily be proved by using such interpolation results. On the other hand, inequalities have been used to develop the theory of interpolation and its applications in various ways. In this paper we will present, discuss and complement some recently obtained examples of such interactions between inequalities and interpolation in both directions. Some new examples, proofs and results are also included.

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This paper contains the following contributions: For the reader's convenience and as an introduction of some ideas we use Section 1 to present and discuss four examples of well-known or "folklore" interpolation proofs of classical inequalities. In Section 2 we present an elementary interpolation technique to create Clarkson type inequalities and we also give some examples of results obtained in this way (see Theorems 1 and 2). In Section 3 we discuss some recently found results concerning generalized Carlson type inequalities (see [33]) namely that the Gustavsson-Peetre class P^{+-} can be exactly described by one Carlson type inequality (see Theorem 3) and that the wider class P_0 can be exactly characterized by another Carlson type inequality with "blocks" (see Th. 4). This exact information about inequalities gives us new information in the theory of interpolation e.g. concerning the $+-$ method by Gustavsson-Peetre (see [21]) and that the Peetre interpolation functor (see [45]) on a couple of Banach lattices can be characterized by the Calderón-Lozanovskii construction for every $\varphi \in P_0$. Ovchinnikov [43] was the first who used the famous Grothendieck inequality to prove that the Gagliardo completion $\varphi(\cdot)^c$ of the Calderón-Lozanovskii construction is an interpolation functor on a class of Banach function spaces. Finally, Section 4 is reserved for some concluding remarks and additional examples.

The fundamental interpolation theorems, e.g. the Riesz-Thorin and Marcinkiewicz interpolation theorems, and the basic results of the real interpolation method of Lions-Peetre and the complex interpolation method of Calderón can be found in the books of Bennett-Sharpley [6], Bergh-Löfström [8], Brudnyi-Krugljak [11], Krein-Petunin-Semenov [32] and Triebel [64].

CONVENTIONS. For $0 < p \leq \infty, p'$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$ ($p' = \infty$ for $p = 1$ and $p' = 1$ for $p = \infty$). Let f^* denote the nonincreasing rearrangement of a measurable function $|f|$ on a measure space (Ω, μ). The Lorentz $L_{p,q}$-spaces ($0 < p < \infty, 0 < q \leq \infty$) are defined by using the quasinorm (with the usual supremum interpretation when $q = \infty$)

$$\|f\|^*_{L_{p,q}} = \left(\int_0^\infty (f^*(t)t^{1/p})^q \frac{dt}{t}\right)^{1/q}.$$

1. Interpolation proofs of some classical inequalities

EXAMPLE 1 (Hausdorff-Young's and Paley's inequalities): Consider the Fourier transform

$$Ff(x) = \int_{\mathbb{R}^n} e^{-ixy} f(y)dy.$$

Then we have boundedness

$F: L_1 \to L_\infty$ with the norm $M_1 \le 1$, and

$F: L_2 \to L_2$ with the norm $M_2 = (2\pi)^{\frac{n}{2}}$ (the Parseval equality).

1^0. By using complex interpolation we obtain boundedness

$$F: L_p \equiv [L_1, L_2]_\theta \to [L_\infty, L_2]_\theta \equiv L_q,$$

where $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}$ and $\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$ (which gives $q = p'$) with the norm $M_\theta \le M_1^{1-\theta} M_2^\theta \le (2\pi)^{\frac{\theta n}{2}} = (2\pi)^{\frac{n}{p'}}$. Equivalently, we can formulate this as the *Hausdorff-Young inequality*:

$$\|Ff\|_{Lp'} \le (2\pi)^{n/p'} \|f\|_{Lp} \; \forall f \in L_p, 1 \le p \le 2. \tag{1.1}$$

2^0. By using real interpolation we obtain boundedness

$$F: L_{p,q} = (L_1, L_2)_{\theta,q} \to (L_\infty, L_2)_{\theta,q} = L_{p',q},$$

where $1 < p \le 2, 0 < q \le \infty$, with the norm $M_\theta \le C M_1^{1-\theta} M_2^\theta \le C(2\pi)^{\frac{\theta n}{2}}$.
Thus, in particular, we have proved the following version of the *Paley inequality* (sometimes also called the *Hardy-Littlewood inequality*):

$$\|Ff\|_{L_{p',p}} \le C\|f\|_{L_p} \; \forall f \in L_p, 1 < p \le 2. \tag{1.2}$$

Remark 1. The best constant in (1.1) is not $(2\pi)^{\frac{n}{p'}}$. In fact, Babenko and Beckner (cf. [2]) proved that the best constant is equal to $C_{p,n} = (A_p)^n (2\pi)^{\frac{n}{p'}}$, where $A_p = [p^{\frac{1}{p}}/p'^{\frac{1}{p'}}]^{\frac{1}{2}}$. The best constant $C = C_{p,n}$ in (1.2) is not known for $p \ne 2$.

Remark 2. Some generalizations of the Hausdorff-Young inequality for the Fourier transform on Orlicz spaces are done by Luxemburg [35] and Jodeit-Torchinsky [27], and on the rearrangement-invariant spaces by Bennett [4], [5]. Moreover, Russo [58] has obtained some generalizations of the Hausdorff-Young theorem on integral operators.

Remark 3. According to 1^0 and 2^0 we see that, for $1 < p \le 2$,

$$f \in L_p \Rightarrow Ff \in L_{p'} \tag{$*$}$$

and

$$f \in L_p \Rightarrow Ff \in L_{p',p}, \tag{$**$}$$

respectively. Moreover, since $L_{p',p}$ is continuously and properly embedded in $L_{p'}$, we see that (**) is a sharper criterion than (*). Moreover, it can be confusing to compare these criterions with the following criterion (see [47]):

$$f \in L_p \Rightarrow \int_0^\infty |Ff|^{p'} h\left(\max\left(|Ff|, \frac{1}{|Ff|}\right)\right)^{(2-)p/(p-1)} dx < \infty, \qquad (***)$$

for some function $h \geq 1, 1/th(t) \in L_1(1,\infty)$ such that $h(x)x^a$ is a decreasing or increasing function of x for some real number a $(1 < p \leq 2)$.
It is possible to prove directly that (**) and (* * *) are, in a way, equivalent (see [48]). Another way to understand this fact is to use interpolation in the following way: it is well-known that the Lorentz space $L_{p',p}$ coincides with the interpolation space $(L_{p_0}, L_{p_1})_{\eta,p}, 0 < \eta < 1, 1/p' = (1-\eta)/p_0 + \eta/p_1$. Moreover, by restricting the general descriptions of real interpolation spaces in off-diagonal cases obtained in [48] to this case we find that $(L_{p_0}, L_{p_1})_{\eta,p}$ also coincides with the spaces described by the right hand side of (* * *) (see [49], Corollary 3.3 and cf. also [38]).

Remark 4. This interpolation proof shows that both (1.1) and (1.2) are true not only for the Fourier operator but also for any operators bounded from L_1 into L_∞ and from L_2 into L_2.

EXAMPLE 2 (Young's inequality): Here we consider the convolution operator

$$Tf(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy = k * f(x), \quad \text{with} \quad k \in L_q(\mathbb{R}^n), 1 \leq q \leq \infty.$$

Then we have boundedness $T: L_{q'} \to L_\infty$ with the norm $\leq \|k\|_{L_q}$ (by the Hölder inequality), and $T: L_1 \to L_q$ with the norm $\leq \|k\|_{L_q}$ (by a generalized Minkowski inequality).
By using complex interpolation we obtain boundedness

$$T: L_p \equiv [L_{q'}, L_1]_\theta \to [L_\infty, L_q]_\theta \equiv L_r,$$

where $\frac{1}{p} = \frac{1-\theta}{q'} + \frac{\theta}{1}$ and $\frac{1}{r} = \frac{1-\theta}{\infty} + \frac{\theta}{q}$ (which gives $1 \leq p \leq q'$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$) with the norm $\leq \|k\|_{L_q}$.
Thus we have proved the *Young inequality*: if $1 \leq q \leq \infty, 1 \leq p \leq q'$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, then

$$\|k * f\|_{L_r} \leq \|k\|_{L_q} \|f\|_{L_p}. \qquad (1.3)$$

By instead using real interpolation we obtain the inequality

$$\|k * f\|_{L_{r,s}} \leq C\|k\|_{L_q} \|f\|_{L_{p,s}}, \qquad (1.4)$$

$1 < q < \infty, 1 < p < q', \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and $s > 0$.

Remark 5. Beckner [2] found the following sharp form of the Young inequality (1.3):

$$\|k * f\|_{L_r} \leq (A_p A_q A_{r'})^n \|k\|_{L_q} \|f\|_{L_p} \quad \forall f \in L_p(\mathbb{R}^n) \cdot \forall k \in L_q(\mathbb{R}^n),$$

where $A_s = [s^{\frac{1}{s}}/s'^{\frac{1}{s'}}]^{\frac{1}{2}}$ and where $(A_p A_q A_{r'})^n$ is the best constant. The best constant $C = C_{p,q,r,n}$ in (1.4) is not known when either $r \neq p$ or $r \neq q$.

Remark 6. The inequality (1.4) is still true even if we only assume that $k \in L_{q,\infty}(\mathbb{R}^n)$. This sharper result is due to O'Neil [42]. All such inequalities are very useful for numerous applications, e.g. in fractional differentiation, imbeddings between Sobolev spaces etc.

Remark 7. A bilinear interpolation theorem for the general K-method of interpolation (generated by the Banach sequence lattices Φ, not only by $l_p(2^{-n\theta})$) is equivalent to the boundedness of the convolution operator from $\Phi_0 \times \Phi_1$ into Φ (see [1] and [36]). In this connection we remark that Cwikel and Kerman [18] recently have used interpolation to prove some new interesting inequalities of Young type to hold for the more general case with positive multilinear operators acting on weighted L_p-spaces.

EXAMPLE 3 (Hölder's inequality): We note that the multiplication operator $T(f,g) = fg$ is a bilinear bounded operator from $L_\infty \times L_1$ into L_1 and from $L_1 \times L_\infty$ into L_1, and that

$$\|T(f,g)\|_{L_1} \leq \|f\|_{L_\infty} \|g\|_{L_1} \quad \forall f \in L_\infty \quad \forall g \in L_1,$$
$$\|T(f,g\|_{L_1} \leq \|f\|_{L_1} \|g\|_{L_\infty} \quad \forall f \in L_1 \quad \forall g \in L_\infty.$$

Using the interpolation theorem for bilinear operators in complex spaces (Calderón theorem [12]) we find that

$$T: [L_1, L_\infty]_\theta \times [L_\infty, L_1]_\theta \to [L_1, L_1]_\theta \text{ with the norm } \leq 1.$$

Since $[L_1, L_\infty]_\theta \equiv L_p \ (p = \frac{1}{\theta}), [L_\infty, L_1]_\theta \equiv [L_1, L_\infty]_{1-\theta} \equiv L_{p'} \quad (p' = \frac{1}{1-\theta})$ and $[L_1, L_1]_\theta \equiv L_1$ we obtain the *Hölder inequality*

$$\|fg\|_{L_1} = \|T(f,g)\|_{L_1} \leq \|f\|_{L_p} \|g\|_{L_{p'}} \quad \forall f \in L_p \quad \forall g \in L_{p'}. \tag{1.5}$$

Remark 8. It is well-known that the *Minkowski* and *Beckenbach-Dresher inequalities* follow from the Hölder inequality (1.5). Here we remark that also the *Carlson inequality* (see [14])

$$\sum_{n=1}^{\infty} a_n \leq \sqrt{\pi} \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/4} \left(\sum_{n=1}^{\infty} n^2 a_n^2 \right)^{1/4}, \tag{1.6}$$

where a_n are positive numbers, follows from the Hölder inequality in the following way (cf. Hardy [22]):
Let $\alpha = \sum_{n=1}^{\infty} n^2 a_n^2$ and $\beta = \sum_{n=1}^{\infty} a_n^2$. Then

$$\begin{aligned}\left(\sum_{n=1}^{\infty} a_n\right)^2 &= \left(\sum_{n=1}^{\infty} a_n \sqrt{\alpha + \beta n^2} \frac{1}{\sqrt{\alpha + \beta n^2}}\right)^2 \\ &\leq \sum_{n=1}^{\infty} a_n^2 (\alpha + \beta n^2) \sum_{n=1}^{\infty} \frac{1}{\alpha + \beta n^2} \\ &\leq 2\alpha\beta \int_0^{\infty} \frac{1}{\alpha + \beta x^2} dx = \pi \sqrt{\alpha\beta} = \pi \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{1/2}.\end{aligned}$$

(Carlson in his original paper [14] suggested that (1.6) does not follow from the Hölder inequality). It is obviously easy to generalize this proof in various directions. In our Section 3 we discuss some new precise forms of Carlson type inequalities, which recently have been proved partly guided by the Hardy idea presented above (see [33]).

EXAMPLE 4 (Clarkson's inequalities): Let $l_p^{(2)}$ be the 2-dimensional complex l_p-space. Clearly, $l_p^{(2)}$ can be identified with $\mathbb{C}^2$ endowed with the p-norm

$$\|(a, b)\|_p = (|a|^p + |b|^p)^{1/p}.$$

We consider an elementary operator $T: l_p^{(2)} \to l_q^{(2)}$ given by

$$T(a, b) = (a + b, a - b).$$

By the triangle inequality, $T: l_1^{(2)} \to l_\infty^{(2)}$ has norm 1 and, by the parallelogram law, $T: l_1^{(2)} \to l_2^{(2)}$ has norm $2^{\frac{1}{2}}$. Therefore, by using complex interpolation, we find that, for $1 \leq p \leq 2, T: l_p^{(2)} \to l_{p'}^{(2)}$ has norm $\leq 2^{\frac{1}{p'}}$, i.e., for $1 \leq p \leq 2$ and all $a, b \in \mathbb{C}$,

$$(|a + b|^{p'} + |a - b|^{p'})^{1/p'} \leq 2^{1/p'} (|a|^p + |b|^p)^{1/p}. \tag{1.7}$$

By integrating (1.7) and using standard arguments we obtain the *Clarkson inequalities*:

$$((\|x + y\|_{L_p})^{p'} + (\|x - y\|_{L_p})^{p'})^{1/p'} \leq 2^{1/p'} ((\|x\|_{L_p})^p + (\|y\|_{L_p})^p)^{1/p}, \ 1 \leq p \leq 2,$$

$$((\|x+y\|_{L_p})^p + (\|x-y\|_{L_p})^p)^{1/p} \le 2^{1/p}(\|x\|_{L_p})^{p'} + (\|y\|_{L_p})^{p'})^{1/p'}, \ p \ge 2.$$

Remark 9. Consider $A_p = (|a|^p + |b|^p)^{\frac{1}{p}}$ and $B_p = (|a+b|^p + |a-b|^p)^{\frac{1}{p}}$. It is well-known that A_p, B_p are nonincreasing in p and that $2^{-1/p}A_p, 2^{-1/p}B_p$ are nondecreasing in p. By using these facts together with the (Clarkson-Hausdorff-Young) estimate (1.7) we obtain the following inequality:
Let $r \in \mathbb{R}, r \neq 0, s > 0, q = \min(2, s)$ if $r \le 2$ and $q = \min(r', s)$ if $r > 2$. Then, for $a, b \in \mathbb{C}$ (for the case $r \le 0$ we put by definition $0^r = 0$),

$$(|a+b|^r + |a-b|^r)^{1/r} \le 2^{\gamma}(|a|^s + |b|^s)^{1/s}, \quad \gamma = \frac{1}{r} - \frac{1}{s} - \frac{1}{q}.$$

This estimate was proved in [39]. The case $r > 0$ with a different proof is due to Koskela [31].

2. An interpolation technique to obtain Clarkson type inequalities

The idea to prove Clarkson type inequalities presented in Example 4 is easy to generalize in various directions, e.g. to more dimensions. Here we present some results which can be obtained in this way.
Let $m, n \in \mathbb{N}$ and consider any linear bounded operator $T: l_1^{(n)} \to l_\infty^{(m)}$ with the norm M_1 which is also bounded $T: l_2^{(n)} \to l_2^{(m)}$ with the norm M_2. By using complex interpolation we obtain the (Hausdorff-Young type) estimate

$$\|T(\overline{a})\|_{l_{p'}^{(m)}} \le M_1^{1-1/p'} M_2^{2/p'} \|\overline{a}\|_{l_p^{(n)}}, \ 1 \le p \le 2, \tag{2.1}$$

which may be seen as a genuine generalization of (1.7). By using this interpolated estimate with different operators and using the monotonicity arguments discussed in Remark 9 we can prove the following Clarkson type inequality, which, in particular, unifies and generalizes some earlier results of this kind (see e.g. [15], [29], [31], [39], [51] and [66]).

Theorem 1

Let T be any linear bounded operator $T: l_1^{(n)} \to l_\infty^{(m)}$ with the norm M_1 and $T: l_2^{(n)} \to l_2^{(m)}$ with the norm M_2. Put $M_u = M_1^{1-\frac{2}{u}} M_2^{\frac{2}{u}}$ and assume that $r \in \mathbb{R}, r \neq 0, s > 0, u = \max(r, 2), q = \min(s, 2)$ if $r \le 2$ and $q = \min(r', s)$ if $r > 2$. Then, for any complex numbers $\overline{a} = (a_1, a_2, \ldots, a_n)$,

$$\Big(\sum_1^m |(T(\overline{a}))_i|^r\Big)^{1/r} \le M_u m^{1/r-1/u} n^{1/q-1/s} \Big(\sum_1^n |a_i|^s\Big)^{1/s}. \tag{2.2}$$

Proof. We put

$$A_r = \Big(\sum_1^m |(T(\overline{a}))_i|^r\Big)^{1/r} \quad \text{and} \quad B_r = \Big(\sum_1^n |a_i|^r\Big)^{1/r}$$

and note that, by (2.1), $A_{p'} \leq M_{p'}B_p$. Moreover, by using this estimate together with the well-known facts that A_r, B_r are nonincreasing in r and $A_r m^{-\frac{1}{r}}, B_r n^{-\frac{1}{r}}$ are nondecreasing in r (see e.g. [39, Lemma 1]), we obtain the following estimates:

$$\begin{aligned}
&r \leq 2, s \leq 2\colon A_r \leq m^{1/r-1/2}A_2 \leq M_2 m^{1/r-1/2}B_2 \leq M_2 m^{1/r-1/2}B_s,\\
&r \leq 2, s \leq 2\colon A_r \leq m^{1/r-1/2}A_2 \leq M_2 m^{1/r-1/2}B_2 \leq M_2 m^{1/r-1/2}n^{1/2-1/s}B_s,\\
&r \leq 2, s \leq 2\colon A_r \leq M_r B_{r'} \leq M_r B_s,\\
&r \leq 2, s \leq 2\colon A_r \leq M_r B_{r'} \leq M_r n^{1/r'-1/s}B_s.
\end{aligned}$$

The proof of (2.2) follows by combining these inequalities. □

EXAMPLE 5 (see [39]): Consider the operator $T\colon \overline{a} \to \big(\sum_1^n a_i, \ldots, \sum_1^n \varepsilon_i a_i, \ldots, \sum_1^n -a_i\big)$, where $\varepsilon_i = \pm 1, 1 \leq i \leq n$ (each coordinate of the vector $T(\overline{a}) \in \mathbb{R}^m, m = 2^n$, is equal to a sum of the type $\sum_1^n \varepsilon_i a_i$). It is easy to see that here $M_1 = 1$ and $M_2 = 2^{\frac{n}{2}}$ and, thus, according to Theorem 1, we obtain that

$$\left(\sum_{\varepsilon_i=\pm 1} 2^{-n}\Big|\sum_1^n \varepsilon_i a_i\Big|^r\right)^{1/r} \leq n^{1/q-1/s}\left(\sum_1^n |a_i|^s\right)^{1/s},$$

for every $r \in \mathbb{R}, r \neq 0, s > 0, q = \min(2, s)$ for $r \leq 2$ and $q = \min(r', s)$ for $r > 2$. For the case $r > 0$ this inequality can be rewritten as

$$\left(\int_0^1 \Big|\sum_1^n \varphi_i(t)a_i\Big|^r dt\right)^{1/r} \leq n^{1/q-1/s}\left(\sum_1^n |a_i|^s\right)^{1/s},$$

where $\varphi_i(t)$ =sign $(\sin(2^i\pi t))$ are the usual Rademacher functions. For the case $n = 2$ the last estimate coincides with a result of Koskela [31, Th. 1] and for the case $s = r', r > 2$ and $s = r \leq 2$ another proof has been done by Williams-Wells [66, formulas (26)-(27)].

EXAMPLE 6: Consider the Littlewood-Walsh matrices $A_{2^n} = (\varepsilon_{ij}), 1 \leq i,j \leq 2^n$, defined recursively in the following way:

$$A_{2^1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ldots, A_{2^n} = \begin{pmatrix} A_{2^{n-1}} & A_{2^{n-1}} \\ A_{2^{n-1}} & -A_{2^{n-1}} \end{pmatrix}, \quad n = 2, 3, \ldots.$$

By applying Theorem 1 to the operator
$T: \overline{a} \to \left(\sum_1^{2^n} \varepsilon_{1j} a_j, \sum_1^{2^n} \varepsilon_{2j} a_j, \ldots, \sum_1^{2^n} \varepsilon_{2^n j} a_j \right)$ from $\mathbb{R}^{2^n}$ to $\mathbb{R}^{2^n}$, we find that

$$\left(\sum_1^{2^n} \left| \sum_{j=1}^{2^n} \varepsilon_{ij} a_j \right|^r \right)^{1/r} \leq 2^{n(1/r - 1/s + 1/q)} \left(\sum_1^{2^n} |a_j|^s \right)^{1/s},$$

for every $r \in \mathbb{R}, r \neq 0, s > 0, q = \min(2, s)$ if $r \leq 2$ and $q = \min(r', s)$ if $r > 2$. For some special cases this estimate was also proved by Pietsch [51, p. 15] (see also Kato [29, p. 164]).

Remark 10. Gurarii-Kadec-Macaev [20] used also an interpolation proof in the estimate of the Littlewood-Walsh matrix operator between l_p-spaces. Their interest was to find the order of the Banach-Mazur distance between the spaces l_p^n and l_q^n.

EXAMPLE 7: We note that for the operator

$$T: \overline{a} \to \left(\sum_1^n a_i, a_1 - a_2, a_1 - a_3, a_1 - a_4, \ldots, a_1 - a_n, \ldots, a_{n-1} - a_n \right)$$

from $\mathbb{R}^n$ to $\mathbb{R}^m, m = \frac{n(n-1)}{2} + 1$, we have $M_1 = 1$ and $M_2 = n^{\frac{1}{2}}$ and, thus, by Theorem 1, it holds that if $r \in \mathbb{R}, r \neq 0$ and $s > 0$, then

$$\left(\left| \sum_1^n a_i \right|^r + \sum_{1 \leq i < j \leq n} | a_i - a_j |^r \right)^{1/r} \leq C \left(\sum_1^n | a_i |^s \right)^{1/s},$$

where $C = m^{\frac{1}{r} - \frac{1}{2}} n^{\frac{1}{2} + \frac{1}{q} - \frac{1}{s}}$, $q = \min(2, s)$ for $r \leq 2$ and $C = n^{\frac{1}{r} + \frac{1}{q} - \frac{1}{s}}, q = \min(s, r')$ for $r > 2$. In particular, for the case $r = s = p \geq 2$, this estimate reads

$$\left| \sum_1^n a_i \right|^p + \sum_{1 \leq i < j \leq n} |a_i - a_j|^p \leq n^{p-1} \sum_1^n |a_i|^p.$$

For the case $n = 3$ Shapiro proved (by interpolation) this and some similar estimates already 1973 in a talk at the meeting of the Swedish Mathematical Society (see [61]).

It is well-known that local versions of the Clarkson type inequalities always imply some corresponding global versions yielding e.g. for L_p-spaces or more general function spaces. Here we consider the following situation: Let T denote a linear class of functions $f = f(t)$ defined on a non-empty set E. Moreover, we let $B: \Gamma \to \mathbb{R}$, denote an isotone sublinear functional ("isotone" means that for every $f, g \in L$ such that $|f(t)| \geq |g(t)|$ on E it holds that $L(f) \geq L(g)$). We also say that $f \in B_p$ if $B_p(f) = (B(|f|^p))^{\frac{1}{p}} < \infty, 0 < p < \infty$. Our global version of Theorem 1 reads

Theorem 2

Let T be any linear operator such that $T: l_1^{(n)} \to l_\infty^{(m)}$, with the norm M_1 and $T: l_2^{(n)} \to l_2^{(m)}$, with the norm M_2. Put $M_u = M_1^{1-\frac{2}{u}} M_2^{\frac{2}{u}}$ and assume that $r \in \mathbb{R}, r \neq 0, p, s > 0, u = \max(r, 2)$ and $q = \min(p, p', r', s)$ with the conventions that p' is omitted if $0 < p \leq 1$ and r' is omitted if $r \leq 1$. Then, for any isotone linear functional B and any $f_1, f_2, \ldots, f_n \in B_p$, it holds that

$$\left(\sum_1^m B_p^r((T(\overline{f}))_i)\right)^{1/r} \leq M_u m^{1/r-1/u} n^{1/q-1/s} \left(\sum_1^m B_p^s(f_i)\right)^{1/s}.$$

By applying Theorem 1 to various operators, e.g. those considered in Examples 5-7, we obtain Clarkson type inequalities, for example those obtained in [29], [31], [39], [50]. Here we only give the following example:

EXAMPLE 8 (see [39, Theorem 4.1]): We apply Theorem 2 with the operator considered in Example 6 and obtain the following result: Let $0 < p, s < \infty$ and $r \in \mathbb{R}, r \neq 0$. Then, for any $n \in \mathbb{Z}_+$ and any $f_1, f_2, \ldots, f_{2^n} \in B_p$,

$$\left(\sum_1^{2^n} B_p^r\left(\sum_1^{2^n} \varepsilon_{ij} f_j\right)\right)^{1/r} \leq 2^{n(1/r-1/s+1/q)} \left(\sum_1^{2^n} B_p^s(f_j)\right)^{1/s},$$

where $q = \min(p, p', s, r')$ with the convention that p' is omitted if $0 < p \leq 1$ and r' is omitted if $r \leq 1$.

Remark 11. We can e.g. use Example 8 with $B_p(f) = \|f\|_{X^p}$ (X is any Banach function space) and, thus, in particular, with $B(f) = \int_\Omega |f(t)| d\mu(t)$ (f is a measurable function on a measure space (Ω, Σ, μ)) so that $B_p(f) = \|f\|_{L^p}$. Therefore

Example 8 generalizes some previous results by Kato [29, Th. 1], Koskela [31, Th. 2] and Persson [50, Th. 5.1].

Proof of Theorem 2. 1^0. Let $0 < s \leq p \leq r < \infty$. According to the well-known facts that $B_{\frac{p}{r}}$ is superadditive and $B_{\frac{p}{s}}$ is subadditive (see e.g. [46, Lemma 1.2]) we find that

$$\sum_1^m B_p^r((T(\overline{f}))_i) = \sum_i^m B_{p/r}(|T(\overline{f}))_i|^r) \leq B_{p/r}\left(\sum_1^m |T(\overline{f}))_i|^r\right) \tag{2.3}$$

and

$$\begin{aligned} B_{p/r}\left(\left(\sum_1^n |f_i|^s\right)^{r/s}\right) &= \left(B_{p/s}\left(\sum_1^n |f_i|^s\right)\right)^{r/s} \\ &\leq \left(\sum_1^n B_{p/s}(|f_i|^s)\right)^{r/s} = \left(\sum_1^n (B_p(f_i))^s\right)^{r/s} . \end{aligned} \tag{2.4}$$

In view of (2.3) and (2.4) we can use Theorem 1 and the isotonity of the functional B to obtain that

$$\left(\sum_1^m B_p^r(T(\overline{f}))_i\right)^{1/r} \leq M_u m^{1/r-1/u} n^{1/q-1/s} \left(\sum_1^n B_p^s(f_i)\right)^{1/s}, \tag{2.5}$$

where $q = s$ for $0 < r \leq 2$ and $q = \min(s, r')$ for $r \geq 2$.
2^0. Let $s, r \leq p$. We use (2.5) for the case 1^0 with $r = p$ and also the well-known fact that $m^{-\frac{1}{r}}\left(\sum_1^m B_p^r(T(\overline{f}))_i\right)^{\frac{1}{r}}$ is nondecreasing in r to see that in this case (2.5) holds with $q = s$ for $0 < r \leq 2$ and $q = \min(p, r')$ for $r \geq 2$.
3^0. Let $p \leq s, r$. Here we use (2.5) for the case 1^0 with $s = p$ and find as above that in this case (2.5) holds with $q = p$ for $0 < r \leq 2$ and $q = \min(p, r')$ for $r \geq 2$. The proof is complete. □

3. Some exact Carlson type inequalities and interpolation

Let P denote the set of all positive concave functions $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ $(\mathbb{R}_+ = (0, \infty))$. It is easy to see that $\varphi(t)$ is nondecreasing and that $\varphi(t)/t$ is nonincreasing. For $\varphi \in P$ we consider

$$s_\varphi(t) = \sup_{s>0}\left(\frac{\varphi(st)}{\varphi(s)}\right), \; 0 < t < \infty .$$

We also define the following subsets of P:

$$P_0 = \left\{\varphi \in P | \lim_{t\to 0+} \varphi(t) = \lim_{t\to\infty} \frac{\varphi(t)}{t} = 0\right\},$$

$$P^{+-} = \left\{\varphi \in P | \lim_{t\to 0+} s_\varphi(t) = \lim_{t\to\infty} \frac{s_\varphi(t)}{t} = 0\right\}.$$

For our discussions later on it is important to note that P^{+-} is a genuine subset of P_0. For example, the functions $\varphi_1(t) = \min(1,t)$ and $\varphi_2(t) = \min(1,t^{\frac{1}{2}})$ belong to P_0 but not to P^{+-}. Moreover, in the sequel we let $\varphi(s,t)$ denote a 1-homogeneous function of two variables defined as $\varphi(s,t) = s\varphi(\frac{t}{s})$ if $s,t > 0$ and $\varphi(s,t) = 0$ if $s = 0$ or $t = 0$.

Next, we remark that some straightforward calculations show that the Carlson type inequality

$$\sum_{n=1}^{\infty} a_n \leq C \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/4} \left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{1/4}$$

can equivalently be rewritten as

$$\|\{a_n\}\|_{l_1} \leq C\varphi\Big(\Big\|\Big\{\frac{a_n}{\varphi(2^n)}\Big\}\Big\|_{l_p}, \Big\|\Big\{\frac{2^n a_n}{\varphi(2^n)}\Big\}\Big\|_{l_q}\Big), \tag{3.1}$$

where $\varphi(t) = t^{\frac{1}{2}}$ and $p = q = 2$.

In 1977 Gustavsson and Peetre [21] proved (in connection to some problems in the theory of interpolation between Orlicz spaces) that (3.1) holds for any $\varphi \in P^{+-}$ and for all $p,q > 1$ (cf. also [37, pp. 143-145]). The following recent result by Krugljak-Maligranda-Persson [33] shows that, in fact, the (Carslon-Gustavsson-Peetre) inequality (3.1) is exactly equipped with the class P^{+-} in the following way:

Theorem 3

Let $\varphi \in P_0$ and let $1 < \min(p,q) \leq \infty$. The following statements are equivalent:

(i) $\varphi \in P^{+-}$.

(ii) The following inequality holds:

$$\|\{a_n\}\|_{l_1} \leq C\varphi\Big(\Big\|\Big\{\frac{a_n}{\varphi(2^n)}\Big\}\Big\|_{l^p}, \Big\|\Big\{\frac{2^n a_n}{\varphi(2^n)}\Big\}\Big\|_{l_q}\Big).$$

(iii) The following inequality holds:

$$\|\{\varphi(a_n,b_n)\}\|_{l_1} \leq C\varphi\left(\Big\|\Big\{\sum_{k:(a_k,b_k)\in S_n} a_k\Big\}\Big\|_{l_p}, \Big\|\Big\{\sum_{k:(a_k,b_k)\in S_n} b_k\Big\}\Big\|_{l_q}\right),$$

where S_n, $n = 1,2,\ldots$, are the areas in $\mathbb{R}_+^2$ between the lines $y = 2^n x$ and $y = 2^{n+1}x$.

For any couple of Banach lattices $\vec{X} = (X_0, X_1)$ we consider the Calderón-Lozanovskii interpolation space $\varphi(\vec{X})$ equipped with the norm

$$\|f\|_{\varphi(\vec{X})} = \inf \max(\|f_0\|_{X_0}, \|f_1\|_{X_1}),$$

where the infimum is taken over all representations of $|f|$ in the form $|f| = \varphi(|f_0|, |f_1|), f_i \in X_i, i = 0, 1$.

Remark 12. By using Theorem 3 it is possible to prove that, for any $\varphi \in P^{+-}$, the Peetre interpolation functor $G^0_\varphi(\vec{X})$ (see [44]) and its Gagliardo closure $[G^0_\varphi(\vec{X})]^c$ can be identified with a Calderón-Lozanovskii interpolation space (cf. also Remark 14).

In order to be able to generalize the statements in Theorem 3 and Remark 12 to the "final" case $\varphi \in P_0$, Krugljak-Maligranda-Persson [33] considered a tricky increasing sequence $\{t_n\}$ constructed by Brudnyi-Krugljak already in 1981 (in connection to their final solution of the K-divisibility problem in interpolation theory, cf. [11]) and having the useful properties that, for $\varphi \in P_0$,

$$\varphi(t) \approx \sum \varphi(t_{2n+1}) \min\left(1, \frac{t}{t_{2n+1}}\right) \quad \text{and} \quad \varphi(t) \approx \max_n \left(\varphi(t_{2n+1}) \min\left(1, \frac{t}{t_{2n+1}}\right)\right),$$

where the constants of equivalences are independent of φ and t. More exactly, the following Carlson type inequality with "blocks" was proved in [33]:

Theorem 4

Assume that $1 < p, q \le \infty, \varphi \in P_0$ and let $\chi_n = [t_{2n}, t_{2n+2})$. Then, for any positive sequence $\{a_n\}$, it holds that

$$\|\{a_n\}\|_{l_1} \le C\varphi\Big(\Big\|\Big\{\sum_{k:2^k \in X_n} \frac{a_k}{\varphi(2^k)}\Big\}\Big\|_{l^p}, \Big\|\Big\{\sum_{k:2^k \in X_n} \frac{2^k a_k}{\varphi(2^k)}\Big\}\Big\|_{l_q}\Big), \tag{3.2}$$

with the constant C not depending on $\{a_n\}$. Moreover, the inequality (3.2) is equivalent to the following inequality:

$$\|\{\varphi(a_n, b_n)\}\|_{l_1} \le C\varphi\Big(\Big\|\Big\{\sum_{k:(a_n, b_k) \in T_n} a_k\Big\}\Big\|_{l_p}, \Big\|\Big\{\sum_{k:(a_k, b_k) \in T_n} b_k\Big\}\Big\|_{l_q}\Big), \tag{3.3}$$

where $T_n, n = 1, 2, \ldots$, are the areas in $\mathbb{R}^2_+$ between the lines $y = t_{2n}x$ and $y = t_{2n+2}x$.

Our proof of Theorem 4 shows that the constants in the inequalities (3.2) and (3.3) can be estimated by $(1+\sqrt{2})^2$ and $2(1+\sqrt{2})^2$, respectively.

Remark 13. By using Theorem 4 it is possible to prove the following generalization of an interpolation result by Ovchinnikov [43] and Nilsson [41] (see [33]): *If $\varphi \in P_0$, then, for any couple of Banach lattices $\vec{X}$, $G^0_\varphi(\vec{X}) = \varphi(\vec{X})^0$, and $[G^0_\varphi(\vec{X})]^c = [\varphi(\vec{X})]^c$ and the constants in the equivalence of the norms in these equalities do neither depend on the couple $\vec{X}$ nor on the function φ* (here, as usual, for any intermediate space A, A^0 denotes the closure of $A_0 \cap A_1$ in A and A^c denotes the Gagliardo closure with respect to $A_0 + A_1$).

Remark 14. The main interest to consider the Peetre functor $G^0_\varphi(\vec{X})$ is inspired by the fact that if $\varphi(t) = t^\theta$ $(0 < \theta < 1)$, then, on couples of Banach lattices $\vec{X}$, and their retracts, this functor coincides with the complex method (see [62], [41], [44], [11]) and, thus, it may be regarded as a "real version" of the complex method of interpolation.

4. Concluding remarks

1^0. Calderón [13] generalized the classical Marcinkiewicz interpolation theorem. He even pointed out the maximal operator $(1 \le p_0 < p_1 \le \infty, 1 \le q_0 \ne q_1 \le \infty)$

$$Sf(t) = \int_0^\infty \min(t^{-1/q_0} s^{1/p_0}, t^{-1/q_1} s^{1/p_1}) f(s) \frac{ds}{s},$$

which is bounded from $L_{p_i,1}$ to $L_{q_i,\infty}$ and for any operator acting from $L_{p_i,1}$ to $L_{q_i,\infty}$ we have the estimate $(Tf)^*(x) \le CSf^*(x)$, where C depends only on p_i and q_i $(i = 0, 1)$. The Calderón operator can be written as

$$Sf(t) = t^{-1/q_0} \int_0^{t^m} s^{1/p_0} f(s) \frac{ds}{s} + t^{-1/q_1} \int_{t^m}^\infty s^{1/p_1} f(s) \frac{ds}{s},$$

where $m = \frac{\frac{1}{q_0} - \frac{1}{q_1}}{\frac{1}{p_0} - \frac{1}{p_1}}$. Therefore, in order to obtain generalized forms of the Marcinkiewicz interpolation theorem, we only need to investigate the boundedness of the Hardy operator H and its dual H^*, defined by

$$Hg(x) = x^{-1} \int_0^x g(s)ds \quad \text{and} \quad H^*g(x) = x^{-1} \int_x^\infty g(s)ds,$$

in the function spaces we are interested in. These types of the Hardy inequalities can be found in many places (cf. [32], [10], [7] and the literature given there).

2^0. Interpolation theorems can also be used to obtain estimates of operators between vector-valued spaces (see e.g. [19]). For example, if $T: L_p \to L_p$, is bounded then the natural vector-valued extension $T_E: L_p(L_p) \to L_p(L_p)$ given by

$$(T_E f)(x, y) = Tf(\cdot, y)(x)$$

has the same norm. On the other hand, according to the Marcinkiewicz-Zygmund theorem, T_E has also the same norm on $L_p(L_2)$. Thus, by using interpolation, we find that T_E has the same norm on $L_p(L_q)$ for all q satisfying $\min(p, 2) \leq q \leq \max(p, 2)$.

3^0. Jameson [23] proved the famous Grothendieck inequality by using an interpolation technique. Moreover, Pisier [54] used the Riesz-Thorin interpolation theorem to obtain the following estimate of the complex Grothendieck constant: $K_G^C \leq e^{1-\gamma} = 1.527\ldots$, where γ is the Euler constant.

4^0. Interpolation techniques can also be used to find the "regularity" of the solutions of boundary-initial value problems. For example, if we consider the Klein-Gordon equation (KG) in $\mathbb{R}^3$ given by

$$u_{tt} - \Delta_x u + u = 0, u(x, 0) = 0, u_t(x, 0) = f(x),$$

and the operator $T_t f = u(\cdot, t)$, then $T_t: L_p \to L_{p'}$ is bounded if and only if $4/3 \leq p \leq 2$ (see [40]). This statement in one direction can easily be proved by interpolation (for $p = 2$ we use the energetic identity and for $p = 4/3$ we use the fact that the KG equation is connected with the wave equation $u_{tt} = \Delta_x u$ and in this case it is well-known that we have boundedness of T_t). Some similar results connected to the Korteweg-De Vries equation and other equations can be found in [9], [34] and [63].

5^0. Karadzov (1973), Birman-Solomjak (1977), König (1978) and Pietsch (1980) have used interpolation techniques to improve s-number estimates or estimates of entropy numbers or eigenvalues of operators in Banach spaces. For all these results and related references we refer to the books of König [30] and Pietsch [52], [53].

6^0. Rochberg-Weiss [57], Jawerth-Rochberg-Weiss [26] and Kalton [28] used the complex and real methods of interpolation to obtain nonlinear inequalities of the form ($1 \leq p_0 < p < p_1 < \infty$)

$$\|T(f \log |f|) - Tf \log |Tf|\|_{L_p} \leq C_p \|f\|_{L_p},$$

for operators T which maps L_{p_i} boundedly into L_{p_i} for $i = 0, 1$.

7^0. Wells-Williams [65] used a generalized Riesz-Thorin interpolation theorem to obtain the exact value of so-called packing constant for the space $L_p(\mu)$.

8^0. Interpolation results for vector-valued spaces give us information about the type and cotype of concrete spaces because a Banach space X has Rademacher type $p, 1 \leq p \leq 2$, if the operator $T: l_p(X) \to L_p(X)$ given by $T((x_k)) = \sum x_k r_k(t)$ is bounded. This possibility to use interpolation was investigated by Cobos [16], [17].

9^0. Pisier [55] used the complex method of interpolation to prove the inverse Brunn-Minkowski inequality due to V.D. Milman.

10^0. In connection to the real method of interpolation there is also an extrapolation theory giving some old and new inequalities like those by Yano (1951), Stein (1969), Moser (1971) etc. (cf. Jawerth-Milman [25]).

11^0. Another example of a new interesting inequality where interpolation is used in a crucial way can be found in Semenov's paper [60] in this volume.

12^0. For concrete operators, e.g. the average operator, the Hardy maximal operator, the conjugate operator, the Hilbert transform, the singular integral operators, etc., we can use Riesz-Thorin or Marcinkiewicz interpolation theorems to prove their boundedness between L_p-spaces and more general function spaces. This boundedness can be interpreted as an inequality. Such simple interpolation techniques to prove inequalities can be avoided in many concrete cases but to the cost of some more complicated computations.

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