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Minimal projections onto subspaces of $l_\infty^{(n)}$ of codimension two

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Abstract

Let $Y \subset l_\infty^{(n)}$ be one of its subspaces of codimension two. Denote by $\mathcal{P}_Y$ the set of all linear projections going from $l_\infty^{(n)}$ onto Y. Put

$$\lambda_Y = \inf\left\{\|P\|: P \in \mathcal{P}_Y\right\}.$$

An operator $P_0 \in \mathcal{P}_Y$ is called a minimal projection if $\|P_0\| = \lambda_Y$. In this note we present a partial solution of the problem of calculation λ_Y as well as the problem of calculation of minimal projection. We also characterize the unicity of minimal projection.

1. Introduction

Let X be a Banach space and let $Y \subset X$ be one of its subspaces. A bounded, linear operator $P: X \to Y$ is called a projection if $Py = y$ for any $y \in Y$. Denote by $\mathcal{P}(X,Y)$ the set of all projections going from X onto Y. A projection P_0 is called minimal iff

$$\|P_0\| = \lambda(Y,X) = \inf\left\{\|P\|: P \in \mathcal{P}(X,Y)\right\}. \tag{1.1}$$

The significance of this notion can be illustrated by the following, well known inequality:

$$\|x - Px\| \le \|I - P\| \operatorname{dist}(x,Y) \le (1 + \|P\|) \operatorname{dist}(x,Y).$$

Unfortunately, in many important cases we do not know effective formulas for minimal projections. For more complete information concerning this subject the reader is referred to [1-3, 5, 6, 8].

In this note, if nothing special will be assumed, Y will be a subspace of $l_\infty^{(n)}, n \geq 3$ ($\mathbb{R}^n$ with the maximum norm) of codimension two. In Section 2 we find an effective formula for λ_Y (Theorem 2.5) and we calculate a minimal projection in this case. In Section 3 we deal with the problem of unicity of minimal projection (Theorems 3.1, 3.3, 3.4). Note that for Y being a hyperplane in $l_\infty^{(n)}$ the above problems were completely solved in [2].

Now we state a notation and some preliminary results which will be of use later. By $S(X)$ we denote the unit sphere in a normed space X and by $\text{ext}(X)$ the set of all its extreme points. We will write for brevity λ_Y instead of $\lambda(Y, l_\infty^{(n)}), \mathcal{P}_Y$ instead of $\mathcal{P}(l_\infty^{(n)}, Y)$. Let us denote

$$\mathcal{L}_Y = \{L \in \mathcal{L}(l_\infty^{(n)}, Y): L|_Y = 0\}. \tag{1.2}$$

It is easy to check that for any $P \in \mathcal{P}_Y$

$$\lambda_Y = \text{dist}(P, \mathcal{L}_Y). \tag{1.3}$$

In the sequel we also need

Lemma 1.1 (see [2])

Assume X is a normed space and let Y be one of its subspaces of codimension k, $Y = \bigcap_{i=1}^k ker f^i$, where $f^i \in X^*$ are linearly independent. Then there exist $y^1, \ldots, y^k \in X$ satisfying $f^i(y^i) = \delta_{ij}$ for $i, j = 1, \ldots, k$ such that

$$Px = x - \sum_{i=1}^k f^i(x)y^i \quad \text{for} \quad x \in X. \tag{1.4}$$

One the other hand, if $y^1, \ldots, y^k \in X$ satisfy $f^i(y_i) = \delta_{ij}$ then the operator $P = id - \sum_{i=1}^k f^i(\cdot)y^i$ belongs to $\mathcal{P}(X, Y)$.

At the end of this section we recall a notion of strong unicity. Let X be a normed space and let $Y \subset X$ be a nonempty subset. An element $y \in Y$ is called a strongly unique best approximation (briefly SUBA) to $x \in X$ iff there exists $r > 0$ such that for every $w \in Y$

$$\|x - w\| \geq \|x - y\| + r\|y - w\|. \tag{1.5}$$

In the case of projections, (1.5) suggests the following

DEFINITION 1.2. Let $P_0 \in \mathcal{P}_Y$. Then P_0 is called a strongly unique minimal projection (we will write a SUM projection for brevity) if and only if there is $r > 0$ such that for any $P \in \mathcal{P}_Y$

$$\|P\| \geq \|P_0\| + r\|P - P_0\|. \tag{1.6}$$

By [6,Th.2.3] it is easy to deduce

Theorem 1.3

Let $P_0 \in \mathcal{P}_Y$. Set

$$crit^*(P_0) = \{i \in \{1, \dots, n\}: \|e_i \circ P_0\| = \|P\|\} \tag{1.7}$$

where

$$e_i(x) = x_i \quad \text{for} \quad x \in l_\infty^{(n)} \tag{1.8}$$

and

$$\mathcal{A}_i = \{x \in ext(l_\infty^{(n)}): e_i(P_0 x) = \|P_0\|\}. \tag{1.9}$$

Then P_0 is a minimal projection (P_0 is a SUM projection with a constant $r > 0$ resp.) if and only if for every $L \in \mathcal{L}_Y$ there exists $i \in crit^*(P_0)$ such that

$$\inf_{x \in \mathcal{A}_i} e_i(Lx) \le 0 \ (\le -r\|L\| \ resp.) \tag{1.10}$$

Theorem 1.4 (see [7])

Let X be a finite dimensional real normed space and let $Y \subset X$ be one of its subspaces. Assume that $card(ext(X)) < \infty$. Then $x \in X$ has a unique best approximation in Y iff X possesses a SUBA *element in Y.*

Section 2

First we prove some preliminary results. We start with

Lemma 2.1.

Let $Y \in l_\infty^{(n)}$ be one of its subspaces of codimension two. Then for every $i \in \{1, \dots, n\}$ with $e_i \notin Y$ there exists one exact to a constant, $f^i \in l_1^{(n)} \setminus \{0\}$ such that $f^i|_Y = 0$ and $f_i^i = 0$.

Proof. Put $Y_i = Y \oplus [e_i]$. Since $e_i \notin Y, dimY_i = n - 1$. Consequently, there exists exactly one to a constant $f^i \in l_1^{(n)}$ satisfying $f^i|_{Y_i} = 0$, as desired. □

Lemma 2.2

Let $Y \subset ker(f)$ for some $f \in S(l_1^{(n)})$. If there exists $i \in \{1, \dots, n\}$ satisfying

$$|f_i| \ge \sum_{k \ne i} |f_k| \tag{2.1}$$

then for every $L \in \mathcal{L}_\infty^{(n)}(l_\infty^{(n)}, ker(f))$

$$\|e_i \circ L\| \le \max_{j \ne i} \|e_j \circ L\|. \tag{2.2}$$

Moreover, if in (2.1) we have the strict inequality then the same holds in (2.2).

Proof. Let us take any $f \in S(l_1^{(n)})$ such that in (2.1) we have the strict inequality. Then it is easy to deduce that $\|e_i|_{ker(f)}\| < 1$. Hence

$$\|e_i \circ L\| \leq \|e_i|_{ker(f)}\| \; \|L\| < \|L\|$$

and consequently

$$\|e_i \circ L\| < \max_{l \neq i} \|e_l \circ L\|.$$

If in (2.1) we have equality, then we can approximate f by a sequence $\{f^n\} \in l_1^{(n)}$ such that f^n satisfies the strict inequality in (2.1). From this, it is easy to derive that (2.2) holds true for any $L \in \mathcal{L}(l_\infty^{(n)}, ker(f))$. □

Lemma 2.3

Let $Y = ker(f^1) \cap ker(f^2)$, where $f^1, f^2 \in l_1^{(n)}$ are linearly independent. Let $P \in \mathcal{P}_Y, P = id - (f^1(\cdot)y^1 + f^2(\cdot)y^2)$ where $y^1, y^2 \in l_\infty^{(n)}$. Then

$$\|P\| = \max_{i=1,\dots,n} |1 - f_i^1 y_i^1 - f_i^2 y_i^2| + \sum_{j \neq i} |f_j^1 y_i^1 + f_j^2 y_i^2| \,. \tag{2.3}$$

Proof. Note that $\|P\| = \max_{i=1,\dots,n} \|e_i \circ P\|$. So to finish the proof, it is sufficient to demonstrate that for each $i \in \{1, \dots, n\}$

$$\|e_i \circ P\| = |1 - f_i^1 y_i^1 - f_i^2 y_i^2| + \sum_{j \neq i} |f_j^1 y_i^1 + f_j^2 y_i^2| \,. \tag{2.4}$$

To do this, take any $x \in l_\infty^{(n)}$. Then

$$\begin{aligned}
(e_i \circ P)x &= x_i - f^1(x) y_i^1 - f^2(x) y_i^2 \\
&= x_i - \Big(\sum_{j=1}^n f_j^1 x_j\Big) y_i^1 - \Big(\sum_{j=1}^n f_j^2 x_j\Big) y_i^2 \\
&= x_i(1 - f_i^1 y_i^1 - f_i^2 y_i^2) - \sum_{j \neq i} x_j (f_j^1 y_i^1 + f_j^2 y_i^2) \\
&\leq |1 - f_i^1 y_i^1 - f_i^2 y_i^2| + \sum_{j \neq i} |f_j^1 y_i^1 + f_j^2 y_i^2| \,.
\end{aligned}$$

Hence

$$\|e_i \circ P\| \leq |1 - f_i^1 y_i^1 - f_i^2 y_i^2| + \sum_{i \neq j} |f_j^1 y_i^1 + f_j^2 y_i^2|.$$

If we take $x = (x_1, \dots, x_n)$ where $x_i = sgn(1 - f_i^1 y_i^1 - f_i^2 y_i^2), x_j = -sgn(f_j^1 y_i^1 + f_j^2 y_i^2)$ for $j \neq i$ we get (2.4) which completes the proof of Lemma 2.3. □

Lemma 2.4

Let f^1, f^2 be as in Lemma 2.3. Put

$$g^1 = \big(sgn(f_1^2)f_1^1, \dots, sgn(f_n^2)f_n^1\big)$$

and

$$g^2 = (|f_1^2|, \dots, |f_n^2|)\,. \tag{2.5}$$

Then the set $\mathcal{P}_Y$ is linearly isometric to $\mathcal{P}_{Y_1}$ where $Y_1 = ker(g^1) \cap ker(g^2)$.

Proof. It is easily seen that the mapping $L: l_\infty^{(n)} \to l_\infty^{(n)}$ defined by

$$Lx = \big(x_1 sgn(f_1^2), \dots, x_n sgn(f_n^2)\big)$$

is a linear isometry such that $L(Y_1) = Y$. Hence the mapping $\Psi(P) = L^{-1} \circ P \circ L$ for $P \in \mathcal{P}_Y$ is a required linear isometry between $\mathcal{P}_Y$ and $\mathcal{P}_{Y_1}$. □

Now we can state the main result of this section

Theorem 2.5

Let $Y \subset l_\infty^{(n)}$ be one of its subspaces of codimension two. Assume furthermore that there is $i_0 \in \{1, \dots, n\}$ and $f^{i_0} \in l_1^{(n)} \setminus \{0\}, f_{i_0}^{i_0} = 0, f^{i_0}|_Y = 0$ such that

$$|f_j^{i_0}| \geq \sum_{k \neq j} |f_k^{i_0}| \quad \text{for some} \quad j \in \{1, \dots, n\}\,. \tag{2.6}$$

Let $f^j \in S(l_1^{(n)})$ satisfy the assumptions of Lemma 2.1 for $i = j$. Then if there exists $j_0 \in \{1, \dots, n\}$ such that

$$|f_{j_0}^j| \geq \sum_{k \neq j_0} |f_k^j| \tag{2.7}$$

then $\lambda_Y = 1$. In the opposite case

$$\lambda_Y = 1 + \Big(\sum_{i=1}^{n} |f_i^j|/(1 - 2|f_i^j|)\Big)^{-1}.$$

Proof. First we consider the case described in (2.6) and (2.7). Let us define $y_j = (y_1^j, \dots, y_n^j)$ by

$$y_k^j = \begin{cases} 0 & \text{if } k \neq j_0, j \\ 1/f_{j_0}^j & \text{if } k = j_0 \\ -f_{j_0}^{i_0}/f_{j_0}^j f_j^{i_0} & \text{if } k = j \end{cases} \tag{2.8}$$

and $y^{i_0} = (y_1^{i_0}, \dots, y_n^{i_0})$ by

$$y_k^{i_0} = \begin{cases} 0 & \text{if } k \neq j \\ 1/f_j^{i_0} & \text{if } k = j. \end{cases} \tag{2.9}$$

Consider the operator

$$P = id - f^{i_0}(\cdot)y^{i_0} - f^j(\cdot)y^j. \tag{2.10}$$

Since $f^j(y^{i_0}) = f^{i_0}(y^j) = 0$ and $f^j(y^j) = f^{i_0}(y^{i_0}) = 1$ the operator P belongs to $\mathcal{P}_Y$. In view of Lemma 2.3,

$$\|P\| = \max\left\{1, \|e_{j_0} \circ P\|, \|e_j \circ P\|\right\}.$$

According to (2.6), (2.7) and Lemma 2.2, $\|P\| = 1$ and consequently, $\lambda_Y = 1$.
Now let f^{i_0} satisfy (2.6) and let f^j does not satisfy (2.7). In view of Lemma 2.4 we may assume that $f^j \geq 0$. Put $w^j = (w_1^j, \dots, w_n^j)$ where

$$w_k^j = \begin{cases} (\Sigma)^{-1}/(1 - 2f_k^j) & \text{if } k \neq j \\ -\Sigma_{l \neq j}(f_l^{i_0} \cdot (\Sigma)^{-1}/((1 - 2f_l^j)f_k^{i_0}) & \text{if } k = j \end{cases} \tag{2.11}$$

(we will write

$$\left(\sum\right)^{-1} = \left(\sum_{l=1}^{n} |f_l^j|/(1 - 2|f_l^j|)\right)^{-1} \tag{2.12}$$

for brevity) (compare with [2]). It is easy to check that $f^j(w^j) = 1$ and $f^{i_0}(w^j) = 0$. Let us define

$$P^1 = id - f^{i_0}(\cdot)y^{i_0} - f^j(\cdot)w^j. \tag{2.13}$$

In view of Lemma 2.3 one can check that

$$\|P^1\| = \max\left\{\|e_j \circ P^1\|, 1 + \left(\sum_{l=1}^{n} |f_l^j|/(1 - 2|f_l^j|)\right)^{-1}\right\}.$$

According to Lemma 2.2, $\|e_j \circ P^1\| \le \max_{l\neq j} \|e_l \circ P^1\|$. Hence

$$\|P^1\| = 1 + \Big(\sum_{k=1}^{n} |f_k^j|/(1-2|f_k^j|)\Big)^{-1}$$

and $\lambda_Y \le 1 + (\sum_{k=1}^{n} |f_k^j|/(1-2|f_k^j|))^{-1}$.

To prove the opposite inequality, we apply Theorem 1.3. First observe that $crit^*(P^1) \supset D = \{i: f_i^j > 0\}$. According to the proof of Lemma 2.3 and (2.13) for each $i \in crit^*(P^1) \setminus \{j\}$

$$\{x^{i,1} = (x_1^{i,1}, \ldots, x_n^{i,1}),\ x^{i,2} = (x_1^{i,2}, \ldots, x_n^{i,2})\} \subset \mathcal{A}_i$$

(see Theorem 1.3), where

$$x_l^{i,1} = \begin{cases} 1 & \text{if } l = j, i \text{ or } f_l^j = 0 \\ -1 & \text{if } f_l^j > 0 \text{ and } l \neq j, i \end{cases} \tag{2.14}$$

and

$$x_l^{i,2} = \begin{cases} 1 & \text{if } l = i \text{ or } (f_l^j = 0, l \neq j) \\ -1 & \text{if } l = j \text{ or } (f_l^j > 0, l \neq i). \end{cases} \tag{2.15}$$

Now we will show that for every $L \in \mathcal{L}_Y$ there exists an index $i \in D$ such that $e_i(Lx^{i,k}) \le 0$ for $k = 1$ or $k = 2$. Fix $L \in \mathcal{L}_Y$. According to Lemma 1.1 and (1.2) $L = f^{i_0}(\cdot)z^{i_0} + f^j(\cdot)z^j$ for some $z^{i_0}, z^j \in Y$. Observe that by (2.6) for any $i \in D$

$$f^{i_0}(x^{i,1})z_i^{i_0} \le 0 \quad \text{or} \quad f^{i_0}(x^{i,2})z_i^{i_0} \le 0. \tag{2.16}$$

Hence, if $z_i^j = 0$ for some $i \in D$ then

$$e_i(Lx^{i,1}) \le 0 \quad \text{or} \quad e_i(Lx^{i,2}) \le 0.$$

Now let $z_i^j \neq 0$ for any $i \in D$. Since $z^j \in Y, f_j^j = 0$ and $\|f^j\| = 1, z_i^j > 0$ for some $i \in D$. Moreover, according to (2.14) and (2.15) $f^j(x^{i,1}) = f^j(x^{i,2})$. Hence for $k = 1, 2$

$$f^j(x^{i,k})z_i^j = \Big(\sum_{l\neq j} f_l^j x_l^{i,k}\Big) z_i^j = z_i^j\Big(|f_i^j| - \sum_{l\neq i,j} |f_l^j|\Big) < 0, \tag{2.17}$$

since f^j does not satisfy (2.7). Consequently, by (2.16) and (2.17) there exists $i \in D \subset crit^*(P^1)$ and $k \in \{1, 2\}$ such that $e_i(Lx^{i,k}) \le 0$. Since for $k = 1, 2$ $x^{i,k} \in \mathcal{A}_i$, according to Theorem 1.3, the proof of Theorem 2.5 is complete. □

Corollary 2.6

Let f^{i_0} satisfy (2.6) and f^j (2.7). Then the projection P defined by (2.10) is a minimal projection in $\mathcal{P}_Y$. If f^j does not satisfy (2.7) the same holds for P^1 defined by (2.13).

EXAMPLE 2.7: Let $f = (0, 1/3, 0, 1/3, 0, 1/3)$ and $g = (1/3, 0, 1/3, 0, 1/3, 0)$. Then $Y = ker(f) \cap ker(g)$ does not satisfy the assumption of Theorem 2.5.

Section 3

In this section we present necessary and sufficient conditions for Y under which P and P^1 given by (2.10) and (2.13) are unique minimal projections. We start with

Theorem 3.1

Let f^{i_0} satisfy (2.6) and f^j (2.7). Then the projection P defined by (2.10) is a unique minimal projection if and only if

$$|f_{j_0}^{i_0}| = |f_j^{i_0}| \quad \text{and} \quad |f_{j_0}^j| > |f_l^j| \quad \text{for} \quad l \neq j_0. \tag{3.1}$$

or

$$|f_j^{i_0}| > |f_l^{i_0}| \quad \text{for} \quad l \neq j \quad \text{and} \quad |f_{j_0}^j| > |f_l^j| \quad \text{for} \quad l \neq j_0 \tag{3.2}$$

(the indexes i_0, j, j_0 are the same as in Theorem 2.5).

Proof. Assume that (3.2) holds. We will show that P is a SUM projection. According to (2.10) and Lemma (2.3) the set $\{i : i \neq j, j_0\} \subset crit^*(P)$. Moreover, for each $i \neq j, j_0$ the set $\mathcal{A}_i$ (see Th. 1.3) contains all the vectors from the set $ext(l_\infty^{(n)})$ having the i-th coordinate equal to 1. Since f^{i_0} and f^j satisfy (3.2), there exists two different indexes $i_1, i_2 \neq j, i_0$ such that $f_k^{i_0} \neq 0$ for $k = i_1, i_2$. Analogously, there exist two different indexes $j_1, j_2 \neq j, j_0$ such that $f_k^j \neq 0$ for $k = j_1, j_2$. Now take any $L \in \mathcal{L}_Y \setminus \{0\}$ (see (1.2)). We will show that there exist $i \neq j, j_0$ and $x \in \mathcal{A}_i$ such that $e_i(Lx) < 0$. By Lemma 1.1 and (1,2)

$$L = f_{i_0}(\cdot)z^{i_0} + f^j(\cdot)z^j$$

where $z_{i_0}, z^j \in Y$. Since $L \neq 0, z_i^{i_0} \neq 0$ or $z_i^j \neq 0$ for some $i \in \{1, \dots, n\}$. Moreover, since $f_j^j = 0, f_j^{i_0} \neq 0$ and $f_{j_0}^j \neq 0$, we may assume that $i \neq j, j_0$. Now we divide the proof onto two cases.

Case 1: $z_i^{i_0} \neq 0$ and $z_i^j = 0$. Since $f_{i_1}^{i_0} \neq 0$ and $f_{i_2}^{i_0} \neq 0$ for two different indexes $\notin \{j, i_0\}$, there exist $x^1, x^2 \in \mathcal{A}_i$ such that $f^{i_0}(x^1)f^{i_0}(x^2) < 0$. Consequently, $e_i(Lx^k) < 0$ for $k = 1$ or $k = 2$.

Case 2: $z_i^j \neq 0$. Reasoning as above, we can find $x^1, x^2 \in \mathcal{A}_i$ such that $f^j(x^1)f^j(x^2) < 0$. Since $j_1, j_2, i \notin \{j_0, j\}$, modifying the j-th coordinate of x^1 and x^2

if necessary, we may assume that $f^{i_0}(x^1)z_i^{i_0} \le 0$ and $f^{i_0}(x^2)z_i^{i_0} \le 0$. Consequently, $e_i(Lx^k) < 0$ for $k = 1$ or $k = 2$.

Now consider the function

$$F(L) = \min_{i \neq j, j_0} \left(\min_{x \in \mathcal{A}_i} (e_i(Lx)) \right) \tag{3.3}$$

for $L \in \mathcal{L}_Y$. It is easily seen that F is a continuous function as a minimum of a finite number of continuous functions. Moreover, by the reasoning presented above for every $L \neq 0$ $F(L) < 0$. Since $\mathcal{L}_Y$ is finitely dimensional, the constant $\gamma = \sup_{L \in S(\mathcal{L}_Y)} F(L)$ is strictly negative. We will show that P given by (2.10) is a SUM projection with a constant $r = -\gamma$. To do this, take any $L \in \mathcal{L}_Y \setminus \{0\}$. Then there exist $i \neq j, j_0$ and $x \in \mathcal{A}_i$ with

$$e_i(L/\|L\|)x = F(L/\|L\|).$$

Hence

$$e_i(Lx) \le -r\|L\|$$

which, according to Theorem 1.3, completes the proof of this part.

Now assume that (3.1) holds. Define

$$g^{i_0} = f^{i_0} + (-f_{j_0}^{i_0}/f_{j_0}^{j}) \cdot f^j.$$

Then $Y = ker(g^{i_0}) \cap ker(f^j)$ and according to (3.1) g^{i_0} and f^j satisfy (3.2) which completes the proof of this part.

Now let (3.1) and (3.2) do not hold. We first consider the case: $|f_k^{i_0}| = |f_j^{i_0}|$ for some $k \neq i_0, j_0, j$. Let us define $z = (z_1, \dots, z_n) \in l_\infty^{(n)}$ by

$$z_l = \begin{cases} 1/f_k^{i_0} & \text{if } l = k \\ -f_k^j/(f_k^{i_0} f_{j_0}^j) & \text{if } l = j_0 \\ 0 & \text{if } l \neq k, j_0. \end{cases}$$

Set

$$Q = id - f^{i_0}(\cdot)z - f^j(\cdot)y^j$$

where y^j is given by (2.8). Since $|f_{j_0}^j|, |f_j^{i_0}|, |f_k^{i_0}| \ge 1/2$, according to Lemma 2.3, $\|e_l \circ Q\| \le \max_{n \neq l} \|e_n \circ Q\|$ for $l = j, j_0, k$. Observe that for $l \neq j, j_0, k, e_l \circ Q = e_l$. Consequently, $\|Q\| = 1$. Since $z \neq y^{i_0}$ (see (2.9)), Q is a minimal projection different from P, which completes the proof of this case.

Now let $|f_k^j| = |f_{j_0}^j|$ for some $k \neq j, j_0$. Changing the role of f^{i_0} and f^j and by the above reasoning we can construct a projection Q of a norm one different from P. The proof of Theorem 3.1 is fully complete. □

Lemma 3.2

Assume $Y = ker(f^{i_0}) \cap ker(f^j)$ where f^j and f^{i_0} are as in Theorem 2.5. Assume furthermore that

$$f_k^j \neq 0 \quad \text{for} \quad k \neq j$$

and

$$1/2 = |f_j^{i_0}| > |f_k^{i_0}| \quad \text{for} \quad k \neq j.$$

Then $\|e_j \circ P^1\| < \|P^1\|$ where P^1 is defined by (2.13).

Proof. In view of Lemma 2.4, we may assume $f_l^j > 0$ for $l \neq j$. Observe that by Lemma (2.3), (2.9) and (2.11)

$$\|e_j \circ P^1\| = \sum_{k \neq j} \Big| f_k^{i_0} - f_k^j \cdot \Big(\sum_{l \neq i_0, j} f_l^{i_0}/(1 - 2f_l^j) \Big) \cdot \Big(\sum\Big)^{-1} \Big| / |f_j^{i_0}|.$$

If we fix f^j and $f_j^{i_0}$ the above formula may be considered as a convex function (we will denote it by φ) of variables $f_k^{i_0}, k \neq j$, satisfying $\sum_{k \neq j} |f_k^{i_0}| = 1/2$. Hence to finish the proof, it is sufficient to show that φ is a strictly convex function. To do this, we demonstrate that for $\alpha, \beta \neq 0, |\alpha| + |\beta| = 1/2$, and two different indexes $k, l \neq i_0, j$

$$\varphi(\alpha \cdot e_k + \beta \cdot e_l) < \|P^1\| = 1 + \Big(\sum\Big)^{-1}.$$

Observe that

$$\begin{aligned}
\varphi(\alpha \cdot e_k + \beta \cdot e_l) = 1/f_j^{i_0} \cdot \Bigg(& \Big| \alpha - f_k^j (\alpha/(1 - 2f_k^j) \\
& + \beta/(1 - 2f_l^j)) \cdot \Big(\sum\Big) - 1 \Big| \\
& + \Big| \beta - f_l^j (\alpha/(1 - 2f_k^j) + \beta/(1 - 2f_l^j)) \cdot \Big(\sum\Big)^{-1} \Big| \\
& + \sum_{m \notin \{j,l,k\}} |f_m^j| \Big| (\alpha/(1 - 2f_k^j) + \beta/(1 - 2f_l^j)) \cdot \Big(\sum\Big)^{-1} \Big| \Bigg).
\end{aligned}$$

Hence if $\varphi(\alpha e_k + \beta e_l) = \|P^1\|$ then

$$\begin{aligned}
& \Big| \alpha \Big(1 - (f_k^j/(1 - 2f_k^j)) \cdot \Big(\sum\Big)^{-1}\Big) - (\beta f_k^j/(1 - 2f_l^j)) \cdot \Big(\sum\Big)^{-1} \Big| \\
& \quad = \Big| \alpha \Big(1 - (f_k^j/(1 - 2f_k^j)) \cdot \Big(\sum\Big)^{-1}\Big) \Big| + \Big| (\beta f_k^j/(1 - 2f_l^j)) \cdot \Big(\sum\Big)^{-1} \Big|
\end{aligned} \tag{3.4}$$

and

$$|\alpha/(1 - 2f_k^j) + \beta/(1 - 2f_l^j)| = |\alpha/(1 - 2f_k^j)| + |\beta(1 - 2f_l^j)|. \tag{3.5}$$

Consequently, according to (3.5), $sgn(\alpha) = sgn(\beta)$. But, by (3.4), $sgn(\alpha) = -sgn(\beta)$; a contradiction $(\alpha, \beta \neq 0)$. The proof of Lemma 3.2 is complete. □

Theorem 3.3

Let f^{i_0}, f^j be as in Theorem 2.5. Assume furthermore that f^j does not satisfy (2.7) and

$$|f_k^{i_0}| = |f_j^{i_0}| = 1/2, \quad \text{for some} \quad k \neq j, i_0 \tag{3.6}$$

or

$$f_l^j = 0 \quad \text{for some} \quad l \neq j. \tag{3.7}$$

Then the projection P^1 defined by (2.13) is not a unique minimal projection.

Proof. According to Lemma 2.4 we may assume that $f^j \geq 0$. Let f^{i_0} satisfy (3.6). For $\alpha \in \mathbb{R}$ put

$$u_\alpha^j = w^j - (\alpha e_{i_0} + \beta e_j + \gamma e_k),$$

where w^j is given by (2.11) and β, γ are chosen so that $f^j(u_\alpha^j) = 1, f^{i_0}(u_\alpha^j) = 0$. Define for $x \in l_\infty^{(n)}$

$$Q^\alpha x = x - f^{i_0}(x)y^{i_0} - f^j(x)u_\alpha^j,$$

where y^{i_0} satisfies (2.9). By Lemma 1.1, Q^α belongs to $\mathcal{P}_Y$. We will show that for α sufficiently small $\|Q^\alpha\| = \lambda_Y$. According to (2.4) and (2.11) $\|e_{i_0} \circ Q^\alpha\| < \lambda_Y$ for $\alpha \geq 0$ sufficiently small. By Lemma 2.2,

$$\|e_l \circ Q^\alpha\| \leq \max_{u \neq l} \|e_u \circ Q^\alpha\|$$

for $l = j, k$. For $l \neq i_0, j, k$, in view of (2.3) and (2.11), $\|e_l \circ Q^\alpha\| = \lambda_Y$. (Since f^j does not satisfy (2.7) such an index exists.) Consequently, $\|Q^\alpha\| = \lambda_Y$ which proves that P^1 is not a unique minimal projection.
Now let f^j satisfy (3.7). For $\alpha \in \mathbb{R}$ define

$$z_\alpha^j = w^j - (\alpha e_i + \beta e_j),$$

where w^j is given by (2.11) and β is chosen so that $f^j(z_\alpha^j) = 1$ and $f^{i_0}(z_\alpha^j) = 0$. Define for $x \in l_\infty^{(n)}$

$$Z^\alpha x = x - f^{i_0}(x)y^{i_0} - f^j(x)z_\alpha^j.$$

By Lemma 1.2, Z^α belongs to $\mathcal{P}_Y$. Reasoning in a similar way as in the case of Q^α we can show that $\|Z^\alpha\| = \lambda_Y$ for α sufficiently small. The proof of Theorem 3.3 is complete. □

Before presenting the next result of this section let us set for $i \neq j$

$$E_1 = \{i : f^{i_0}(x^{i,1}) = 0\} \tag{3.8}$$

and

$$E_2 = \{i : f^{i_0}(x^{i,2}) = 0\} \tag{3.9}$$

where $x^{i,1}, x^{i,2}$ are defined by (2.14) and (2.15).

Theorem 3.4

Let f^{i_o}, f^j be as in Theorem 2.5. Assume furthermore that f^{i_o} does not satisfy (3.6) and f^j does not satisfy (3.7). Then P^1 is not a unique minimal projection if and only if E_1 and E_2 are nonempty sets.

Proof. In view of Lemma 2.4, we can assume that $f^j \geq 0$ and $f_j^{i_o} > 0$. Suppose E_1, E_2 are nonempty sets. By [4,Th. 2.2a], $card(ext(\mathcal{L}^*(l_\infty^{(n)})))$ is finite. Hence, according to Theorem 1.4, it is sufficient to show that P^1 is not a SUM projection. To do this, take $y \in S(Y)$ such that $y_k < 0$ for $k \in E_1, y_k > 0$ for $k \in E_2$ and $y_k = 0$ for $k \notin E_1 \cup E_2 \cup \{j\}$. Put $L = f^{i_o}(\cdot)y$. We will show that for any $i \in crit^*(P^1)$ (see (1.7))

$$\inf_{x \in \mathcal{A}_i} e_i(Lx) \geq 0. \tag{3.10}$$

Following Lemma 3.2 and (2.13) $crit^*(P^1) = \{1, \ldots, n\} \setminus \{j\}$. Since $f_j^{i_o} > 0$, for any $k \in E_1$

$$f^{i_o}(x^{k,l}) \leq 0 \quad (l = 1, 2)$$

or for any $k \in E_2$

$$f^{i_o}(x^{k,l}) \geq 0 \quad (l = 1, 2).$$

Consequently, for any $i \neq j$, $l = 1, 2$ $f^{i_o}(x^{i,l})y_i \geq 0$. Since f^j does not satisfy (3.7) $\mathcal{A}_i = \{x^{i,1}, x^{i,2}\}$ for $i \neq j$. Hence, by the above reasoning, (3.10) holds true. In view of Theorem 1.3, P^1 is not a SUM projection as desired.

To prove the converse, suppose that $E_1 \neq \emptyset$ and $E_2 = \emptyset$. Take any $L \in \mathcal{L}_Y \setminus \{0\}$ (see 1.2). We will show that there is $i \in crit^*(P^1)$ such that

$$\inf_{x \in \mathcal{A}_i} e_i(Lx) < 0.$$

According to Lemma 1.2 and (1.2) $L = f^{i_o}(\cdot)z^{i_o} + f^j(\cdot)z^j$. Since $L \neq 0, z_i^{i_o} \neq 0$ or $z_i^j \neq 0$ for some $i \in \{1, \ldots, n\}$. Moreover, since $f_j^{i_o} \neq 0$, we may assume that $i \neq j$. If $z^j \neq 0$, reasoning as in Theorem 2.5, we can show that $e_i(Lx^{i,1}) < 0$ or $e_i(Lx^{i,2}) < 0$. In the opposite case, since $f_i^j > 0$ for $i \neq j, z_i^{i_o} > 0$ for some $i \in E_1$ or $z_i^{i_o} \neq 0$ for some $i \notin E_1 \cup \{j\}$. Since E_2 is an empty set, it is easy to check that $\inf_{x \in \mathcal{A}_i} e_i(Lx) < 0$, where the index i is defined as above. To finish the proof of this part, let us consider the function F defined by (3.3). Applying Theorem 1.3 and reasoning as in Theorem 3.1, we get that P^1 is a SUM projection.

If $E_2 \neq \emptyset$ and $E_1 = \emptyset$ or $E_1, E_2 = \emptyset$, reasoning in the same manner as in the previous part of the proof, we get that P^1 is a SUM projection. The proof of Theorem 3.4 is complete. □

EXAMPLE 3.5 (nonuniqueness): Let

$$f^{i_o} = (1/2, 0, 1/4, -1/4),\ f^j = (0, 1/3, 1/3, 1/3).$$

Then it is easy to check that $E_1 = \{4\}, E_2 = \{3\}$. Hence, in view of Theorem 3.4, P^1 is not a unique minimal projection.

EXAMPLE 3.6 (uniqueness): Let

$$f^{i_o} = (1/2, 0, 1/4, 1/4),\ f^j = (0, 1/3, 1/3, 1/3).$$

Then $E_1 = \{2\}, E_2 = \emptyset$. By Theorem 3.4, P^1 is a SUM projection.

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