*Collect. Math.* **44** (1993), 135–146


# On extreme points of Orlicz spaces with Orlicz norm

HENRYK HUDZIK AND MAREK WISŁA

*Institute of Mathematics, A. Mickiewicz University,*

*Matejki 48/49 Poznań, 60-769 Poland*

ABSTRACT

In the paper we consider a class of Orlicz spaces equipped with the Orlicz norm over a non-negative, complete and $\sigma$-finite measure space $(T, \Sigma, \mu)$, which covers, among others, Orlicz spaces isomorphic to $L^\infty$ and the interpolation space $L^1 + L^\infty$. We give some necessary conditions for a point $x$ from the unit sphere to be extreme. Applying this characterization, in the case of an atomless measure $\mu$, we find a description of the set of extreme points of $L^1 + L^\infty$ which corresponds with the result obtained by R.Grząślewicz and H.Schaefer [3] and H.Schaefer [13].

The aim of this paper is to extend some known descriptions of the set of extreme points of Orlicz spaces yielded with the Orlicz norm (cf., e.g., [7], [15], [6]) to the case that covers classical Banach spaces like $L^\infty$ and the interpolation space $L^1 + L^\infty$ with the norm

$$\|x\|_{L^1+L^\infty} = \inf\left\{\|y\|_1 + \|z\|_\infty : y + z = x, y \in L^1, z \in L^\infty\right\}.$$

The point is that in the previous papers on this subject the authors have assumed that the function $\Phi$ generating the Orlicz space $L^\Phi$ is an N-function, i.e., $\Phi : \mathbb{R} \to [0, \infty)$ is even, convex, continuous, vanishing at 0 function satisfying $\Phi(u)/u \to 0$ as $u \to 0$ and $\Phi(u)/u \to \infty$ as $u \to \infty$.

Keywords: extreme point, Orlicz space, space $L^1 + L^\infty$.

AMS Subject Classification: 46B20.

In this paper we will take into consideration a more general class of the functions $\Phi$. We shall assume that $\Phi : \mathbb{R} \to [0, \infty]$ (so $\Phi$ can take infinity value), $\Phi$ vanishes at 0, it is even, convex, left-continuous on $[0, \infty)$, nonidentically equal to 0 and such that $0 \leq \Phi(u) < \infty$ for some $u > 0$. To motivate this sort of conditions, let us consider the following function $\Phi : \mathbb{R} \to [0, \infty)$:

$$\Phi(u) = \begin{cases} 0 & \text{if } |u| \leq 1 \\ |u| - 1 & \text{otherwise.} \end{cases} \tag{1}$$

Then, an easy calculation shows that the space $L^1 + L^\infty$ is equal (as a set) to the space $L^\Phi$ of all those measurable functions $x : T \to \mathbb{R}$ for which $I_\Phi(\lambda x) := \int_T \Phi(\lambda x(t))\, d\mu < \infty$ for some $\lambda > 0$ (depending on $x$). The space $L^\Phi$, and thus $L^1 + L^\infty$, is, in fact, an Orlicz space generated by the function $\Phi$ (cf. [8], [12], [4], [11]). It occurs that the norm $\|\cdot\|_{L^1 - L^\infty}$ can be described by means of the function $\Phi$ as well, namely $\|\cdot\|_{L^1+L^\infty}$ is equal to the Orlicz norm $\|\cdot\|_\Phi^0$ given by

$$\|x\|_\Phi^0 = \sup\left\{\int_T |x(t)y(t)|\, d\mu : y \in L^{\Phi^*}, I_{\Phi^*}(y) \leq 1\right\}, \tag{2}$$

where $\Phi^*$ denotes the complementary function to $\Phi$ in the Young sense, i.e.,

$$\Phi^*(u) = \sup\{uv - \Phi(v) : v \geq 0\}$$

(cf. [12]) and $\Phi$ is defined by (1). It is easy to show that , if $\Phi$ is given by (1) then

$$\Phi^*(u) = \begin{cases} |u| & \text{if } |u| \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, $L^{\Phi^*} = L^1 \cap L^\infty$ (as sets) and the classical norm $\|\cdot\|_{L^1 \cap L^\infty} = \max(\|\cdot\|_1, \|\cdot\|_\infty)$ coincides with the Luxemburg norm $\|\cdot\|_{\Phi^*}$ on $L^{\Phi^*}$ defined by

$$\|\cdot\|_{\Phi^*} = \inf\{\lambda > 0 : I_\Phi(x(t)/\lambda) \leq 1\}.$$

Thus (2) follows from the well-known formula

$$\|x\|_{L^1+L^\infty} = \sup\left\{\int_T |x(t)y(t)|\, d\mu : y \in L^1 \cap L^\infty, \|y\|_{L^1 \cap L^\infty} \leq 1\right\}$$

(cf. [1], [5]).

If we consider the function $\Phi$ given by (1) then, obviously, $\Phi(u)/u \to 1$ as $u \to \infty$, thus the above mentioned results cannot be applied. In fact, a description which covers all the cases of Orlicz functions is not known yet. (Let us mention that the similar problem concerning the description of the set of extreme points of Orlicz spaces yielded with the Luxemburg norm and Lorentz spaces have been already solved – cf. [2], [14]).

The Orlicz norm given by (2) is not easy to deal with. It is far more convenient to make use of the Amemiya formula:

$$\|x\|_\Phi^0 = \inf_{0<k<\infty} \frac{1}{k}\big(1 + I_\Phi(kx)\big) \tag{3}$$

(cf. [9], [10]). The set of all $k$'s at which the infimum is attained (for a fixed $x \in L^\Phi$) will be denoted by $K(x)$. In particular, the set $K(x)$ can be empty. To simplify the notation, by $<a, b>$ we shall denote the interval with the endpoints $a$ and $b$, i.e., $<a, b> = \{\lambda a + (1-\lambda)b : 0 \le \lambda \le 1\}$.

In the following, the set of all extreme points of the unit ball $B(X)$ will be denoted by $\operatorname{Ext} B(X)$.

**Theorem 1**

*Let $\Phi$ be an Orlicz function and let $\mu$ be an arbitrary non-negative complete and $\sigma$-finite measure (not necessarily atomless). If $z \in \operatorname{Ext} B(L^\Phi, \|\cdot\|_\Phi^0)$ and $\operatorname{supp} z$ does not reduce to an atom, then the set $K(z)$ consists of exactly one element.*

First, we prove an auxiliary lemma.

**Lemma 1**

*Under the assumptions of Theorem 1, the set $K(z)$ is not empty.*

*Proof.* a) Assume that $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$ and let $z \in L^\Phi \setminus \{0\}$. Then there exists $\varepsilon > 0$ such that $\mu(A_\varepsilon) > 0$, where

$$A_\varepsilon = \{t \in T : |z(t)| > \varepsilon\}.$$

Thus

$$\frac{1}{k} I_\Phi(kz) \ge \frac{1}{k} I_\Phi(kz\chi_{A_\varepsilon}) \ge \frac{1}{k}\Phi(k\varepsilon)\mu(A_\varepsilon) \xrightarrow[k\to\infty]{} \infty,$$

so

$$k_2 := \max\Big\{k \in (0,\infty) : \frac{1}{k}\big(1 + I_\Phi(kz)\big) \le 2\|z\|_\Phi^0\Big\} < \infty.$$

Since

$$\frac{1}{k}\big(1 + I_\Phi(kz)\big) \geq \frac{1}{k} > 2\|z\|^0_\Phi$$

provided $k < k_1 := (2\|z\|^0_\Phi)^{-1}$, $K(z) \subset [k_1, k_2]$. Since, moreover, the function

$$k \mapsto \frac{1}{k}\big(1 + I_\Phi(kz)\big)$$

is continuous on $[k_1, k_2]$, we infer that $K(z) \neq \emptyset$.

b) Let $g = \lim_{u\to\infty} \Phi(u)/u$ and let us assume that $0 < g < \infty$. Since $\operatorname{supp} z$ does not reduce to an atom, there exists $\varepsilon > 0$ such that the set $C = \{t \in T : |z(t)| > \varepsilon\}$ also does not reduce to an atom. Let $A$,$B$ be disjoint subsets of $C$ such that $0 < \mu(A), \mu(B) < \infty$. Without loss of generality we can assume that $\int_A |z(t)|d\mu \leq \int_B |z(t)|d\mu$. Let $\lambda \in (0,1]$ be a number such that $\int_A |z(t)|d\mu = \lambda \int_B |z(t)|d\mu$ and define

$$x = z + z\chi_A - \lambda z\chi_B, \quad y = z - z\chi_A + \lambda z\chi_B .$$

Obviously, $x \neq y$ and $(x + y)/2 = z$. Let $u_n = n\varepsilon$ for $n \in \mathbb{N}$. Applying the monotonicity of $u \mapsto \Phi(u)/u$ and the convexity of $\Phi$ we have

$$\Phi(u) \leq g|u| \quad \text{for every } u \in \mathbb{R}$$

and

$$\frac{\Phi(u_n)}{u_n}|u| \leq \Phi(u) \quad \text{for every } |u| \geq u_n .$$

Thus

$$\begin{aligned}\frac{I_\Phi(2nz\chi_A) + I_\Phi(n(1-\lambda)z\chi_B)}{I_\Phi(nz\chi_{A\cup B})} &\leq \frac{gn\left(2\int_A |z(t)|d\mu + (1-\lambda)\int_B |z(t)|d\mu\right)}{\frac{\Phi(u_n)}{u_n}\cdot n\int_{A\cup B}|z(t)|d\mu} \\ &= \frac{gu_n}{\Phi(u_n)}\end{aligned}$$

and

$$\frac{I_\Phi(n(1+\lambda)z\chi_B)}{I_\Phi(nz\chi_{A\cup B})} \leq \frac{gn(1+\lambda)\int_B |z(t)|d\mu}{\frac{\Phi(u_n)}{u_n} n\int_{A\cup B}|z(t)|d\mu} = \frac{gu_n}{\Phi(u_n)}.$$

Now, suppose that $K(z) = \emptyset$. Then

$$1 = \|z\|^0_\Phi = \inf_{k>0}\frac{1}{k}\big(1 + I_\Phi(kz)\big) = \lim_{k\to\infty}\frac{1}{k}\big(1 + I_\Phi(kz)\big) = \lim_{k\to\infty}\frac{1}{k}I_\Phi(kz).$$

Thus

$$\begin{aligned}\frac{1}{n}I_\Phi(nx) &= \frac{1}{n}\left[I_\Phi(2nz\chi_A) + I_\Phi(n(1-\lambda)z\chi_B) + I_\Phi(nz\chi_{T\setminus(A\cup B)})\right]\\ &\le \frac{gu_n}{n\Phi(u_n)}I_\Phi(nz\chi_{A\cup B}) + \frac{1}{n}I_\Phi(nz\chi_{T\setminus(A\cup B)})\\ &\le \frac{gu_n}{n\Phi(u_n)}I_\Phi(nz) \xrightarrow[n\to\infty]{} 1,\end{aligned}$$

so $\|x\|^0_\Phi \le 1$. Analogously,

$$\begin{aligned}\frac{1}{n}I_\Phi(ny) &= \frac{1}{n}\left[I_\Phi(n(1+\lambda)z\chi_B) + I_\Phi(nz\chi_{T\setminus(A\cup B)})\right]\\ &\le \frac{gu_n}{n\Phi(u_n)}I_\Phi(nz) \xrightarrow[n\to\infty]{} 1,\end{aligned}$$

so $\|y\|^0_\Phi \le 1$ as well. Thus z is not an extreme point of $B(L^\Phi, \|\cdot\|^0_\Phi)$ - and we arrived at a contradiction which ends the proof. □

The assumption "supp $z$ does not reduce to an atom" cannot be omitted. Indeed, consider the sequence space $\ell^1$ and the sequence $z = (1, 0, \ldots)$. Obviously, $z$ is an extreme point of $B(\ell^1)$. Since

$$\frac{1}{k}\Big(1 + \sum_{i=1}^{\infty}|kz_i|\Big) = \frac{1}{k} + 1 > 1$$

for every $k \neq 0$ the set $K(z)$ is empty.

*Proof of Theorem 1.* Suppose that $K(z)$ is not a one element set and let $k_2 > k_1$ be such that $k_1, k_2 \in K(z)$. We have

$$\begin{aligned}\|z\|^0_\Phi &\le \frac{k_1+k_2}{2k_1k_2}\left(1 + I_\Phi\left(\frac{2k_1k_2}{k_1+k_2}z\right)\right)\\ &= \frac{k_1+k_2}{2k_1k_2}\left(1 + I_\Phi\left(\frac{k_2}{k_1+k_2}k_1z + \frac{k_1}{k_1+k_2}k_2z\right)\right)\\ &\le \frac{1}{2}\left[\frac{1}{k_1}(1 + I_\Phi(k_1z)) + \frac{1}{k_2}(1 + I_\Phi(k_2z))\right] = \|z\|^0_\Phi.\end{aligned}$$

Thus the numbers $k_1z(t)$, $\frac{2k_1k_2}{k_1+k_2}z(t)$, $k_2z(t)$ belong to the same interval on which $\Phi$ is affine and this fact holds true for $\mu$-a.e. $t$ in $T$.

In order to simplify the notation, put $k_0 = \frac{2k_1k_2}{k_1+k_2}$ and denote by $\mathrm{SC}_\Phi$ the set of all $u \in \mathbb{R}$ for which $(u, \Phi(u))$ is a point of strict convexity of the epigraph of $\Phi$. Then there exist sequences $(a_n)$, $(b_n)$ of numbers, $b_n > a_n$ for every $n \in \mathbb{N}$, such that

$$<k_1z(t), k_2z(t)> \subset \mathbb{R} \setminus \mathrm{SC}_\Phi = \bigcup_n (a_n, b_n) \tag{4}$$

for $\mu$-a.e. $t$ in $T$ ($\Phi$ is affine on each interval $[a_n, b_n]$). Therefore

$$\mu\big(\{t \in T : <k_1z(t), k_2z(t)> \subset [a_n, b_n]\}\big) > 0$$

for some, fixed from now on, $n \in \mathbb{N}$. Denote

$$C = \{t \in T : <k_1z(t), k_2z(t)> \subset [a_n, b_n]\}.$$

If C reduces to an atom then, by assumptions and by (4), there exists $p \neq n$ such that $\mu(D) > 0$, where

$$D = \{t \in T : <k_1z(t), k_2z(t)> \subset [a_p, b_p]\}.$$

Let us define the sets $A_1$, $A_2$ and the numbers $\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2$ in the following manner:
- if $C$ reduces to an atom: put $A_1 = C$, $\alpha_1 = a_n$, $\beta_1 = b_n$, $\alpha_2 = a_p$, $\beta_2 = b_p$ and take $A_2 \subset D$ with $0 < \mu(A_2) < \infty$;
- in the other case: let $A_1$, $A_2 \subset C$ be disjoint sets such that $0 < \mu(A_1)$, $\mu(A_2) < \infty$ and put $\alpha_1 = \alpha_2 = a_n$, $\beta_1 = \beta_2 = b_n$.

Since $\Phi$ is affine on the intervals $[\alpha_i, \beta_i]$, $\Phi(u) = m_iu + p_i$ for every $u \in [\alpha_i, \beta_i]$ and some $m_i, p_i \in \mathbb{R}$ $(i = 1, 2)$.

Our first claim is that both $m_1$ and $m_2$ are different from zero. Suppose $m_1$ is equal to zero. Take $\lambda = \frac{k_2-k_1}{2k_2}$ and put

$$x = z + \lambda z\chi_{A_1}, \qquad y = z - \lambda z\chi_{A_1}.$$

Obviously, $x \neq y$ and $(x + y)/2 = z$. Further, since $k_2 > k_1$,

$$\begin{aligned} k_0 \cdot \max\{1-\lambda, 1+\lambda\} &\le \frac{2k_1k_2}{k_1+k_2} \cdot \max\left\{1 - \frac{k_2-k_1}{2k_2}, 1 + \frac{k_2-k_1}{2k_1}\right\} \\ &= \max\{k_1, k_2\} = k_2. \end{aligned}$$

Moreover, $I_\Phi(k_2x(t)\chi_{A_1}) = 0$, so

$$\begin{aligned} I_\Phi(k_0x) &= I_\Phi(k_0z\chi_{T\setminus A_1}) + I_\Phi(k_0(1+\lambda)z\chi_{A_1}) \\ &\le I_\Phi(k_0z\chi_{T\setminus A_1}) + I_\Phi(k_2z(t)\chi_{A_1}) = I_\Phi(k_0z\chi_{T\setminus A_1}) \le I_\Phi(k_0z). \end{aligned}$$

Thus $\|x\|_\Phi^0 \le 1$ and, analogously, $\|y\|_\Phi^0 \le 1$ – a contradiction.

Therefore $m_1 \neq 0$ and $m_2 \neq 0$. Note that $z(t)m_i > 0$ for every $t \in A_i$ $(i = 1, 2)$. Let $\lambda_1, \lambda_2 \in \left(0, \frac{k_2-k_1}{2k_2}\right)$ be numbers such that

$$\lambda_1 m_1 \int_{A_1} z(t)\,d\mu = \lambda_2 m_2 \int_{A_2} z(t)\,d\mu.$$

Observe that

$$\begin{aligned} k_1 &= k_0 \frac{k_1+k_2}{2k_2} = k_0\left(1 - \frac{k_2-k_1}{2k_2}\right) \le k_0(1-\lambda_i) \le k_0(1+\lambda_i) \\ &\le k_0\left(1 + \frac{k_2-k_1}{2k_2}\right) \le k_0\left(1 + \frac{k_2-k_1}{2k_1}\right) = k_0\frac{k_1+k_2}{2k_1} = k_2 \end{aligned}$$

for $i = 1, 2$. Now, define

$$x = z + \lambda_1 z\chi_{A_1} - \lambda_2 z\chi_{A_2}, \qquad y = z - \lambda_1 z\chi_{A_1} + \lambda_2 z\chi_{A_2}.$$

Plainly, $x \neq y$ and $(x+y)/2 = z$. Moreover,

$$\begin{aligned} I_\Phi(k_0 x) &= I_\Phi(k_0 z\chi_{T\setminus(A_1\cup A_2)}) \\ &\quad + m_1 k_0(1+\lambda_1)\int_{A_1} z(t)\,d\mu + p_1\mu(A_1) + m_2 k_0(1-\lambda_2) \\ &\quad \times \int_{A_2} z(t)\,d\mu + p_2\mu(A_2) \\ &= I_\Phi(k_0 z\chi_{T\setminus(A_1\cup A_2)}) + \int_{A_1} \big(m_1 k_0 z(t) + p_1\big)\,d\mu \\ &\quad + \int_{A_2} \big(m_2 k_0 z(t) + p_2\big)\,d\mu = I_\Phi(k_0 z). \end{aligned}$$

Thus $\|x\|_\Phi^0 \le 1$ and, analogously, $\|y\|_\Phi^0 \le 1$. This contradiction proves that the strong inequality $k_2 > k_1$ is false, i.e., $K(z)$ is a one–point set. □

**Theorem 2**

*Let $\Phi$ be an Orlicz function and let $\mu$ be an atomless measure. If $z$ is an extreme point of $\mathrm{B}(\mathrm{L}^\Phi, \|\cdot\|_\Phi^0)$ then*

*(i) the set $K(z)$ consists of one element,*

*(ii) $\big(kz(t), \Phi(kz(t))\big)$ are points of strict convexity of the epigraph of $\Phi$ for $k \in K(z)$ and $\mu$–a.e. $t$ in $T$.*

*Proof.* Condition (i) follows immediately from Theorem 1. Suppose that (ii) is not satisfied and let $k \in K(z)$. Then, there exist numbers $a, b \in \mathbb{R}$ and $\varepsilon > 0$ with $a < b$ and $\varepsilon < (b-a)/2k$ such that $\Phi$ is affine on the interval $(a, b)$, i.e., $\Phi(u) = mu + p$ for some $m, p \in \mathbb{R}$ and every $u \in (a, b)$, and, moreover, $kz(t) \in (a + k\varepsilon, b - k\varepsilon)$ on a set $A$ of positive measure. Let $B$, $C$ be two disjoint subsets of $A$ with $0 < \mu(B) = \mu(C) < \infty$. Define

$$x = (z-\varepsilon)\chi_B + (z+\varepsilon)\chi_C + z\chi_{T\setminus(B\cup C)}, \quad y = (z+\varepsilon)\chi_B + (z-\varepsilon)\chi_C + z\chi_{T\setminus(B\cup C)}.$$

Then, obviously, $x \neq y$ and $(x+y)/2 = z$. Moreover,

$$\begin{aligned} I_\Phi(kx) &= \int_B \big(mk(z(t)-\varepsilon)+p\big)\,d\mu \\ &\quad + \int_C \big(mk(z(t)+\varepsilon)+p\big)\,d\mu + I_\Phi(z\chi_{T\setminus(B\cup C)}) \\ &= \int_{B\cup C} \big(mkz(t)+p\big)\,d\mu + I_\Phi(z\chi_{T\setminus(B\cup C)}) = I_\Phi(kz), \end{aligned}$$

so $\|x\|_\Phi^0 \leq \|z\|_\Phi^0 = 1$. Similarly, $\|y\|_\Phi^0 \leq 1$. Thus $z$ is not extreme - a contradiction. □

If the space $L^\infty$ is included in $L^\Phi$ it is interesting to establish when the extreme points of $B(L^\infty)$ are extreme in $L^\Phi$ as well.

**Theorem 3**

*Let $\mu$ be an atomless measure with $\mu(T) > 1$ and let us assume that a point $z \in L^\Phi$ satisfies the following conditions:*

*(i) $z \in \operatorname{Ext} B(L^\infty) \cap B(L^\Phi)$,*

*(ii) $K(z)$ is a one element set,*

*(iii) $(k, \Phi(k))$ is a point of strict convexity of the epigraph of $\Phi$, where $k \in K(z)$,*

*(iv) there exists $0 < \varepsilon < 2$ such that $\Phi(u) > u - 1$ for every $u > 2 - \varepsilon$.*

*Then $z$ is an extreme point of $B(L^\Phi)$.*

*Proof.* Let $z$ be an extreme point of $B(L^\infty)$. It is well-known that the absolute value of $z(t)$ must be equal to 1 for $\mu$-a.e. $t$ in $T$. Suppose that $z$ is not an extreme point of $B(L^\Phi)$, i.e., $z = (x+y)/2$ for some $x, y \in B(L^\Phi)$ with $x \neq y$.

We shall consider three cases.

$1^0$. $K(x) \neq \emptyset$ and $K(y) \neq \emptyset$. Let $a \in K(x)$ and $b \in K(y)$. Then

$$\begin{aligned}
1 = \|z\|_\Phi^0 &= \frac{1}{2}(\|x\|_\Phi^0 + \|y\|_\Phi^0) = \frac{1}{2a}(1 + I_\Phi(ax)) + \frac{1}{2b}(1 + I_\Phi(by)) \\
&= \frac{a+b}{2ab}\left(1 + \frac{b}{a+b}I_\Phi(ax) + \frac{a}{a+b}I_\Phi(by)\right) \\
&\geq \frac{a+b}{2ab}\left(1 + I_\Phi\left(\frac{2ab}{a+b}\frac{x+y}{2}\right)\right) \\
&= \frac{a+b}{2ab}\left(1 + I_\Phi\left(\frac{2ab}{a+b}z\right)\right) \geq \|z\|_\Phi^0 = 1,
\end{aligned}$$

so all the inequalities in the above formulae are, in fact, equalities. Therefore

$$\frac{a+b}{2ab} \in K(z) = \{k\},$$

$\Phi$ is affine on the intervals $<ax(t), by(t)>$ and, moreover,

$$kz(t) \in <ax(t), b(y(t)> \text{ for } \mu - \text{a.e. } t \text{ in } T.$$

Since $x \neq y$ and $\|(x+y)/2\|_\Phi^0 = 1$, $ax(t) \neq by(t)$ on a set of positive measure. Thus the epigraph of $\Phi$ is not strictly convex at $k|z(t)| = k$ and we arrive at a contradiction.

$2^0$. $K(x) = \emptyset$ and $K(y) = \emptyset$. Then, by the Amemiya formula (3),

$$1 = \|x\|_\Phi^0 = \lim_{n\to\infty} \frac{1}{n} I_\Phi(nx)$$

and, similarly, $\lim_{n\to\infty} \frac{1}{n} I_\Phi(ny) = 1$. Thus

$$1 = \|z\|_\Phi^0 \leq \lim_{n\to\infty} \frac{1}{n} I_\Phi(nz) \leq \frac{1}{2} \lim_{n\to\infty} \left(\frac{1}{n} I_\Phi(nx) + \frac{1}{n} I_\Phi(ny)\right) = 1.$$

On the other hand, by (iv),

$$\lim_{n\to\infty} \frac{1}{n} I_\Phi(nz) \geq \lim_{n\to\infty} \frac{1}{n}(n-1)\mu(T) > 1$$

– a contradiction.

$3^0$. $K(x) = \emptyset$ and $K(y) \neq \emptyset$. Then $\lim_{n\to\infty} \frac{1}{n} I_\Phi(nx) = 1$ and $\frac{1}{b}\big(1 + I_\Phi(by)\big) = 1$ for some $1 \le b < \infty$. For every $n \in \mathbb{N}$ sufficiently large we have

$$\begin{aligned} 2 &< \frac{n+b}{nb}\left(1 + \left(\frac{2nb}{n+b} - 1\right)\mu(T)\right) \le \frac{n+b}{nb}\left(1 + \Phi\left(\frac{2nb}{n+b}\right)\mu(T)\right) \\ &= \frac{n+b}{nb}\left(1 + I_\Phi\left(\frac{2nb}{n+b}\,\frac{x+y}{2}\right)\right) \\ &\le \frac{n+b}{nb}\left(1 + \frac{b}{n+b} I_\Phi(nx) + \frac{n}{n+b} I_\Phi(by)\right) \\ &= \frac{1}{n}\big(1 + I_\Phi(nx)\big) + \frac{1}{b}\big(1 + I_\Phi(by)\big) \xrightarrow[n\to\infty]{} 2 \end{aligned}$$

and this contradiction ends the proof. □

Now, we can apply the obtained results to the space $L^1 + L^\infty$.

**Theorem 4** (R. Grząślewicz and H. Schaefer [3], H. Schaefer [13])

*Let $\mu$ be an atomless measure. A point $z$ of the unit sphere of the space $L^1 + L^\infty$ is extreme if and only if*

*(i)* $|z(t)| \equiv 1$ *for $\mu$–a.e. $t$ in $T$,*

*(ii)* $\mu(T) > 1$.

*In other words: the set of extreme points of the unit ball of $L^1 + L^\infty$ is either empty (if $\mu(T) \le 1$) or it coincides with the set $\operatorname{Ext} B(L^\infty)$ (provided $\mu(T) > 1$).*

*Proof. Sufficiency.* Let $z^*$ be the rearrangement function of $|z|$. Then $z^*(t) \equiv 1$, so

$$\|z\|_{L^1+L^\infty} = \int_0^1 z^*(t)\, d\mu = 1,$$

i.e, $z$ belongs to the unit sphere of $L^1 + L^\infty$.

Now, let $\Phi$ be the function defined by (1). Then the set $K(z)$ consists exactly of one element and, moreover, $K(z) = \{1\}$ – this is an easy consequence of the assumption $\mu(T) > 1$ and the following equality

$$\frac{1}{k}\big(1 + I_\Phi(kz)\big) = \begin{cases} \dfrac{1}{k} \in [1, \infty) & \text{if } 0 < k \le 1, \\ \dfrac{1}{k} + \left(1 - \dfrac{1}{k}\right)\mu(T) & \text{if } 1 < k < \infty. \end{cases}$$

Obviously $(1, 0)$ is a point of strictly convexity of $\Phi$. Finally, it is evident that condition (iv) of Theorem 3 is satisfied as well. Thus $z$ is an extreme point of $B(L^1 + L^\infty)$.

*Necessity.* Let us note that if $\mu(T) \leq 1$ then the space $L^1 + L^\infty$ is isometric to $L^1$. Indeed, it is obvious that for any finite measure $\mu$, $L^\infty \subset L^1$, so $L^1 + L^\infty = L^1$. Thus any $x \in L^1 + L^\infty$ admits a decomposition $x = x + 0$, hence

$$\|x\|_{L^1+L^\infty} \leq \|x\|_1.$$

On the other hand, for any $y \in L^\infty$ we have

$$\|y\|_1 \leq \|y\|_\infty \mu(T) \leq \|y\|_\infty.$$

Thus, considering any of the decompositions $x = y + z$ of $x$, where $y \in L^1$ and $z \in L^\infty$, we have

$$\|y\|_1 + \|z\|_\infty \geq \|y\|_1 + \|z\|_1 \geq \|y + z\|_1 = \|x\|_1.$$

Hence, passing to infimum, we obtain

$$\|x\|_{L^1+L^\infty} \geq \|x\|_1,$$

i.e., $L^1 + L^\infty$ is isometric to $L^1$.

Assume that $z \in \operatorname{Ext} B(L^1 + L^\infty)$ and let the function $\Phi$ be defined by (1). Then, by Theorem 2, $K(z) = \{k\}$ for some $0 < k < \infty$ and, moreover, $k|z(t)| = 1$ for $\mu$–a.e. $t$ in $T$. Similarly as in the proof of Theorem 1, one can show that $\|z\|_{L^1+L^\infty} = 1/k$. Thus $k = 1$ and (i) is proved. Since, by the assumption, the set of extreme points is not empty, the space $L^1 + L^\infty$ can not be isometric to $L^1$, so the measure of $T$ must be grater than one. □

REMARK. Theorem 4 was given in [3] and [13] for the infinite Lebesgue measure space only.

## References

1. C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press Inc. 1988.
2. R. Grząślewicz, H. Hudzik, W. Kurc, Extreme and exposed points in Orlicz spaces, *Canad. J. Math.* **44** (1992), 505–515.
3. R. Grząślewicz, H. H. Schaefer, On the isometries of $L^1 \cap L^\infty[0,\infty)$ and $L^1 + L^\infty[0,\infty)$, *Indagationes Math.* NS **3** (2) (1992), 173–178.
4. A. Kamińska, Extreme points in Orlicz-Lorentz spaces, *Arch. Math.* **52** (1990), 1–8.

5. M. A. Krasnosel'skii, Y. B. Rutickii, *Convex Functions and Orlicz Spaces*, P. Nordhoff Ltd, Groningen 1961.
6. S. G. Krein, Yu. I. Petunin, E. M. Semenov, *Interpolation of Linear Operators*, Moskow 1978 (in Russian).
7. W. Kurc, Extreme points of the unit ball in Orlicz spaces of vector–valued functions with the Amemiya norm, *Mathematica Japonica* **38** (2) (1993), 277–288.
8. Bing Yuan Lao, Xiping Zhu, Extreme points of Orlicz spaces, *J. Zhongshan University* **2** (1983), 27–36 (in Chinese).
9. J. Musielak, Orlicz spaces and Modular Spaces, *Lecture Notes in Math.* **1034**, Springer Verlag 1983.
10. H. Nakano, Generalized modular spaces, *Studia Math.* **31** (1968), 439–449.
11. H. Nakano, *Modulared Semi–Ordered Linear Spaces*, Maruzem, Tokyo 1950.
12. W. Orlicz, Über eine gewisse Klasse von Räumen vom Typus B, *Bull. Intern. Acad. Pol. Série A, Kraków* **0** (1932), 207–220.
13. M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker Inc., New York 1991.
14. H. Schaefer, On convex hulls, *Arch. Math.* **58** (1992), 160–163.
15. Congxin Wu, Shutao Chen, Extreme points and rotundity of Musielak-Orlicz sequence spaces, *J. Math. Res. Exposition* **2** (1988), 195–200.
16. Zhuogiang Wang, Extreme points of Orlicz sequence spaces, *J. Daging Oil College* **1** (1983), 112–121 (in Chinese).