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Every nonreflexive Banach lattice has the packing constant equal to 1/2

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Abstract

Kottman [9] has proved that any P-convex Banach space X is reflexive. In the case when X is a Banach lattice our result says more. It says that any Banach lattice X with $\Lambda(X) < 1/2$ is reflexive. This result generalizes the results of Berezhnoi [2] who proved that $\Lambda(\Lambda(\varphi)) = \Lambda(M(\varphi)) = 1/2$ for nonreflexive Lorentz space $\Lambda(\varphi)$ and Marcinkiewicz space $M(\varphi)$. It is proved also that for any Banach lattice X such that its subspace X_a of order continuous elements is nontrivial we have $\Lambda(X) = \Lambda(X_a)$. It is noted also that Orlicz sequence space l^Φ is reflexive iff $\Lambda(l^\Phi) < 1/2$.

In the sequel $\mathbb{N}, \mathbb{R}, \mathbb{R}_+$ denote respectively the sets of natural numbers, of reals and of nonnegative reals. If X is a real Banach space, $S(X)$ and $B(X)$ denote its unit sphere and unit ball, respectively. The packing constant of X is defined by the formula (see [15] and [16]):

$$\Lambda(X) = \sup\Big\{r > 0 : \exists (x_n)_{n=1}^\infty \text{ in } X \text{ s.t. } \|x_n\| \le 1-r, \text{ and } \|x_m - x_n\| \ge 2r \text{ if } m \ne n\Big\}.$$

Kottman [9] has proved that for any infinite dimensional Banach space X, we have

$$\Lambda(X) = D(X)/(2 + D(X)),$$

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where

$$D(X) = \sup\left\{ \inf_{m\neq n} \|x_m - x_n\| : (x_n)_{n=1}^{\infty} \text{ contained in } S(X)\right\}.$$

Recall that a Banach space X is said to be P-convex (see [10]) if $P(n,X) < 1/2$ for some $n \in \mathbb{N}, n \geq 2$, where

$$P(n,X) = \sup\left\{ r > 0 : \exists (x_i)_{i=1}^{n} \text{ s.t. } \|x_i\| \leq 1 - r \text{ and } \|x_i - x_j\| \geq 2r \text{ for } i \neq j\right\}.$$

Kottman [10] has proved that any P-convex Banach space is reflexive.

A map $\Phi: \mathbb{R} \to \mathbb{R}_+$ is said to be an Orlicz function if $\Phi(0) = 0, \Phi$ is even, convex, and $\Phi(u) \to +\infty$. By l^0 we denote the space of all real sequences. Given any Orlicz function Φ define on l^0 a convex functional $I_\Phi(x\lambda) \leq 1$ by

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi(x(i)) \qquad \left(\forall x = (x(i))_{i=1}^{\infty} \in l^0\right).$$

The Orlicz sequence space l^Φ is then defined to be the set of these x in l^0 for which $I_\Phi(\lambda x) < +\infty$ for some $\lambda > 0$ depending on x. The space l^Φ equipped with the Luxemburg norm

$$\|x\|_\Phi = \inf\left\{\lambda > 0 : I_\Phi(x/\lambda) \leq 1\right\} \quad \left(\forall x \in l^\Phi\right)$$

is a Banach space (see [9], [11], [12], [13] and [14]).

Recall that an Orlicz function Φ satisfies the Δ_2-condition at 0 if there exist $K > 0$ and $u_0 > 0$ such that $0 < \Phi(u_0) < +\infty$ and $\Phi(2u) \leq K\Phi(u)$ whenever $|u| \leq u_0$.

Let X, Y be Banach spaces. We say that X contains an isomorphic (almost isometric) copy of Y if for some (for every) $\varepsilon > 0$ there exists a linear operator $P: Y \to X$ such that the inequality

$$(*) \qquad \|y\|_Y \leq \|Py\|_X \leq (1+\varepsilon)\|y\|_Y$$

holds for every $y \in Y$.

For the theory of Banach lattices see [1], [4] and [9]. We start with the following result.

Proposition 1

Let X and Y be Banach spaces and assume that X contains an almost isometric copy of Y. Then $\Lambda(X) \geq \Lambda(Y)$.

Proof. Let $\varepsilon > 0$ be arbitrary and take an arbitrary sequence (y_n) in $S(Y)$. Let P be a linear operator from Y into X satisfying condition $(*)$. Define the sequence $x_n = Py_n/\|Py_n\|_X$ in X. In virtue of $(*)$, we have

$$(1)\qquad \begin{aligned} \|x_m - x_n\|_X &\leq \frac{1}{\|Py_m\|_X}\|Py_m - Py_n\|_X \\ &\quad + \|Py_n\|_X \left| \frac{1}{\|Py_m\|_X} - \frac{1}{\|Py_n\|_X} \right| \\ &\leq (1+\varepsilon)\|y_m - y_n\|_Y + (1+\varepsilon) \left| \frac{1}{\|Py_m\|_X} - \frac{1}{\|Py_n\|_X} \right| \\ &\leq (1+\varepsilon)\|y_m - y_n\|_Y + \varepsilon. \end{aligned}$$

On the other hand,

$$(2)\qquad \left\| \frac{y_m}{\|Py_m\|_X} - \frac{y_n}{\|Py_n\|_X} \right\|_Y \leq \left\| P\left(\frac{y_m}{\|Py_m\|_X}\right) - P\left(\frac{y_n}{\|Py_n\|_X}\right) \right\|_X$$

and

$$(3)\qquad \left| \frac{1}{\|Py_m\|_X} - \frac{1}{\|Py_n\|_X} \right| \leq 1 - \frac{1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon} < \varepsilon.$$

Applying (3), we get

$$(4)\qquad \begin{aligned} \left\| \frac{y_m}{\|Py_m\|_X} - \frac{y_n}{\|Py_n\|_X} \right\|_Y &\geq \left\| \frac{y_m - y_n}{\|Py_m\|_X} \right\|_Y - \left| \frac{1}{\|Py_m\|_X} - \frac{1}{\|Py_n\|_X} \right| \\ &> \frac{1}{1+\varepsilon} \|y_m - y_n\|_Y - \varepsilon\ . \end{aligned}$$

Combining (2) and (4), we obtain

$$(5)\qquad \|x_m - x_n\|_X \geq \frac{1}{1+\varepsilon} \|y_m - y_n\|_Y - \varepsilon.$$

Inequalities (1) and (5) imply our result. □

Theorem 1

Every nonreflexive Banach lattice X has the packing constant equal to $1/2$.

Proof. By the assumption, X contains an isomorphic copy of c_0 or l^1 (see [1], [4] and [9]). So, by the James theorem (see [8]) X contains an almost isometric copy of c_0 or l^1, respectively. By Proposition 1, we get $\Lambda(X) \geq \Lambda(c_0)$ or $\Lambda(X) \geq \Lambda(l^1)$, respectively. However, $\Lambda(c_0) = \Lambda(l^1) = 1/2$ (see [7], [16] and [17]), whence it follows that $\Lambda(X) \geq 1/2$. Since, the inequality $\Lambda(X) \leq 1/2$ is always true, we get $\Lambda(X) = 1/2$. □

Remark 1. Since $\Lambda(X) \leq P(n, X)$ for any $n \in \mathbb{N} (n \geq 2)$, Theorem 1 gives stronger result for Banach lattices than Kottman theorem which says that any P-convex Banach space is reflexive.

Remark 2. There exists a reflexive Banach lattice X with $\Lambda(X) = 1/2$.

In fact, define X to be the Hilbertian direct sum $\bigoplus l^{p_i}$, where $1 < p_i \searrow 1$. Every $l^{p_i}, i = 1, 2 \ldots$, is isometrically embedded into X, so we get for any $i \in \mathbb{N}$:

$$\Lambda(X) \geq \Lambda(l^{p_i}) = 1/(2 + 2^{1-1/p_i}) \nearrow 1/2$$

(for the inside equality see [16] and [17]). Since we always have $\Lambda(X) \leq 1/2$, it follows that $\Lambda(X) = 1/2$. Clearly, X is a Banach lattice and as the Hilbertian direct sum of reflexive Banach lattices $l^{p_i} (1 < p_i < +\infty)$, X is reflexive.

Theorem 2

Let X be an arbitrary Banach lattice and X_a be its subspace of order continuous elements. If $X_a \neq \{0\}$, then $\Lambda(X_a) = \Lambda(X)$.

Proof. Assume first that X has an order continuous norm, i.e. $X_a = X$. Then the equality is obvious. Assume now that $X_a \neq X$ and $X_a \neq \{0\}$. Then X_a contains an isomorphic copy of c_0 (see [1], [4] and [9]), and so, by the James theorem (see [7]), X_a contains almost isometric copy of c_0. Hence, $\Lambda(X) \geq \Lambda(X_a) = 1/2$, and consequently $\Lambda(X) = \Lambda(X_a) = 1/2$. □

Yining Ye, He Miaohong and Ryszard Płuciennik [18] have proved that Orlicz function space L^Φ as well as Orlicz sequence space l^Φ equipped with the Luxemburg norm is P-convex iff it is reflexive, i.e. both Φ and Φ^* (the complementary function to Φ in the sense of Young) satisfy the suitable Δ_2-condition (i.e. the Δ_2-condition at zero in the sequence case). We will prove now an analogous result for l^Φ in terms of $\Lambda(l^\Phi)$.

Theorem 3

An Orlicz sequence space l^Φ is reflexive if and only if $\Lambda(l^\Phi) < 1/2$.

Proof. If $\Lambda(l^\Phi) < 1/2$ then l^Φ is reflexive by Theorem 1. Assume that l^Φ is reflexive, i.e. Φ and Φ^* satisfy the Δ_2-condition at zero. Then (see [15] and [17]),

$$D(l^\Phi) = \sup_{x\in S(l^\Phi)} \left\{c_x > 0: I_\Phi(x/c_x) = \frac{1}{2}\right\}. \tag{6}$$

To get this formula only the Δ_2-condition at zero for Φ is important. Since Φ^* also satisfies the Δ_2-condition at zero, we have for $a > 0$ such that $\Phi(a) = 1$:

$$\exists p > 1\ \forall\lambda \in (0,1)\ \forall|u| \le a\colon\ \Phi(\lambda u) \le \lambda^p\Phi(u). \tag{7}$$

Hence, taking into account that, in view of the Δ_2-condition at zero for Φ, the equality $I_\Phi(x) = 1$ holds for any $x \in S(l^\Phi)$, we get for any $x \in S(l^\Phi)$:

$$I_\Phi\left(\frac{x}{2^{1/p}}\right) \le \frac{1}{2} I_\Phi(x) = \frac{1}{2}.$$

Hence, it follows that $c_x \le 2^{1/p}$ for any $x \in S(l^\Phi)$. Therefore, $D(l^\Phi) \le 2^{1/p}$, and consequently $\Lambda(l^\Phi) \le 1/(2 + 2^{1-1/p}) < 1/2$. □

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