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A Fourier inequality with A_p and weak-L^1 weight

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Abstract

The object of this note is to generalize some Fourier inequalities.

The following weighted Fourier norm inequality is known:

Theorem A ([1], [2]).

Suppose w is a radial weight function on $\mathbb{R}^n$ and as radial function non-decreasing on $(0,\infty)$. Let $1 < p \le q \le p' < \infty$, then there is a constant $C > 0$ such that

$$\left\{ \int_{\mathbb{R}^n} |\hat{f}(x)|^q |x|^{-n(1-q/p')} w\Big(\frac{1}{|x|}\Big)^{q/p} dx \right\}^{1/q} \le C \left\{ \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right\}^{1/p} \qquad (1)$$

holds, if and only if $w \in A_p$.

Here $\hat{f}$ denotes the Fourier transform of f, defined by

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{-ixy} f(y) dy, \qquad x \in \mathbb{R}^n$$

whenever the integral converges. The Muckenhoupt weight class A_p consists of all non-negative measurable functions w for which

$$\sup_{Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w(x) dx \right] \left[\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right]^{p-1} < \infty,$$

where Q denotes a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, $|Q|$ its Lebesgue measure and $p' = \frac{p}{p-1}$ is the conjugate index of p.

A function φ belongs to weak L^1 (i.e. $\varphi \in L^1_{\text{weak}}$) if there is a constant $C > 0$ such that for all $\lambda > 0, \lambda m(\{x \in \mathbb{R}^n : |\varphi(x)| > \lambda\}) \leq C$, or equivalently $y\varphi^*(y) \leq C$, $y > 0$, where $\varphi^*(y) = \inf\{\lambda > 0 : m(\{x \in \mathbb{R}^n : |\varphi(x)| > \lambda\}) \leq y\}$ is the equimeasurable decreasing rearrangement of φ.

Since $|x|^{-n} \in L^1_{\text{weak}}$ one might expect that the term $|x|^{-n}$ occurring in (1) can be replaced by any $\varphi \in L^1_{\text{weak}}$. The object of this note it to prove this is indeed the case.

Theorem 1

Suppose w is a radial weight function in A_p and as radial function non-decreasing in $(0, \infty)$. If $1 < p \leq q \leq p' < \infty$ and $\varphi \in L^1_{\text{weak}}$, then there is a constant $C > 0$, such that

$$\left\{ \int_{\mathbb{R}^n} |\hat{f}(x)|^q w\Big(\frac{1}{|x|}\Big)^{q/p} \varphi(x)^{1-q/p'} dx \right\}^{1/q} \leq C \left\{ \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right\}^{1/p}. \qquad (2)$$

Note that the case $q = p'$ may be found in [2] while the case $w(x) = 1$ yields Corollary 1.6 of [4].

Proof. The hypotheses of Theorem 1 imply that inequality (1) holds. Writing $u(x) = |x|^{n(\frac{1}{p'} - \frac{1}{q})} w(\frac{1}{|x|})^{\frac{1}{p}}$ and $v(x) = w(x)^{\frac{1}{p}}$ then u and v are radial and as radial functions decreasing on $(0, \infty)$. Hence with this change (1) implies by [3, Theorem 3.1] that

$$\sup_{s>0} \left\{ \int_0^{\kappa} u(t)^q t^{n-1} dt \right\}^{1/q} \left\{ \int_0^{\overline{\kappa}} v(t)^{-p'} t^{n-1} dt \right\}^{1/p'} < \infty$$

where $t = |x|, \kappa = s^{-2} \theta_n^{\frac{-1}{n}}, \overline{\kappa} = s^2 \theta_n^{\frac{-1}{n}}$, and θ_n is the measure of the unit n-sphere. Writing $\overline{w}(t) = w(\frac{1}{t})$ the supremum takes the form

$$\sup_{s>0} \left[\int_0^{\kappa} t^{n([1/p' - 1/q]q+1)-1} \overline{w}(t)^{q/p} dt \right]^{1/q} \left[\int_0^{\overline{\kappa}} \Big(\frac{1}{w}\Big)(t)^{p'/p} t^{n-1} dt \right]^{1/p'} < \infty$$

and the change of variable $t = y^{\frac{1}{n}} \theta_n^{\frac{-1}{n}}$ shows that this implies

$$\sup_{s>0} \left[\int_0^{s^{-2n}} \overline{w}(y^{1/n} \theta_n^{-1/n})^{q/p} y^{q/p'-1} dy \right]^{1/q} \left[\int_0^{s^{2n}} \Big(\frac{1}{w}\Big)(y^{1/n} \theta_n^{-1/n})^{p'/p} dy \right]^{1/p'} < \infty. \qquad (3)$$

But $\overline{w}$ and $\frac{1}{w}$ are decreasing as radial functions and so equal to their radially decreasing rearrangements. Now the equimeasurable rearrangement of a function g, defined by

$$g^*(y) = \inf\left\{\lambda > 0: m(\{x: |g(x)| > \lambda\}) \leq y\right\},$$

is related to its radially decreasing rearrangement $g^{\otimes}$ by $g^*(y) = g^{\otimes}(y^{\frac{1}{n}}\theta_n^{\frac{-1}{n}})$ (cf. [3]). Hence, with $\lambda = s^{2n}$, (3) takes the form

$$\sup_{\lambda>0}\left[\int_0^{1/\lambda} \overline{w}^*(y)^{q/p} y^{q/p'-1} dy\right]^{1/q}\left[\int_0^{\lambda}\left(\frac{1}{w}\right)^*(y)^{p'/p} dy\right]^{1/p'} < \infty .$$

But since $\varphi \in L^1_{\text{weak}}$, $\varphi^*(y) \leq \frac{C}{y}, y > 0$, so this implies

$$\sup_{\lambda>0}\left[\int_0^{1/\lambda} \overline{w}^*(y)^{q/p} \varphi^*(y)^{1-q/p'} dy\right]^{1/q}\left[\int_0^{\lambda}\left(\frac{1}{w}\right)^*(y)^{p'-1} dy\right]^{1/p'} < \infty .$$

Since powers and rearrangements commute, i.e. $(g^\alpha)^* = (g^*)^\alpha$ and since for any h and g, $h^*(y)g^*(y) \geq (hg)^*(2y)$, then after a change of variable the last supremum inequality implies

$$\sup_{\lambda>0}\left[\int_0^{1/\lambda} (\overline{w}^{q/p}\varphi^{1-q/p'})^*(y) dy\right]^{1/q}\left[\int_0^{\lambda}\left(\frac{1}{w}\right)^*(y)^{p'-1} dy\right]^{1/p'} < \infty .$$

But this (cf. [5]) implies the inequality (2). □

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