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On weak topology of Orlicz spaces*

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ABSTRACT

This paper presents some properties of singular functionals on Orlicz spaces, from which, criteria for weak convergence and weak compactness in such spaces are obtained.

In [1], T. Ando shows that every linear bounded functional can be decomposed into a function part and a singular part, and the last part is represented by some class of finite additive set functions. Since very few properties of such set functions are known, most problems concerning weak topology in Orlicz spaces are left open; for instance, even T. Ando himself in [2], leaving the singular functionals aside, discusses only the L_N-weak convergence and L_N-weak compactness. In this paper, we first give criteria for a singular functional on an Orlicz space to be norm attainable and to be an extreme point of the unit ball of the dual space, then, applying Rainwater's Theorem, we obtain criteria for weak convergence and weak compactness in the space.

Throughout this paper, we denote by $M: \mathbb{R} \to \mathbb{R}^+$ an Orlicz function, i.e., it is even, continuous, convex and satisfies $M(u) = 0$ iff $u = 0$, and $\frac{M(u)}{u} \to 0$ as $u \to 0$, $\frac{M(u)}{u} \to \infty$ as $u \to \infty$. If M is an Orlicz function, then its complemented

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Orlicz function N is defined by

$$N(v) = \sup_{u \in \mathbb{R}} \{uv - M(u)\}.$$

Let (G, Σ, μ) be a non-atomic, σ-finite and complete measurable space. For each μ-measurable function $x(t)$ on G, we define

$$\begin{aligned} \rho(x) = \rho_M(x) &= \int_G M(x(t))d\mu \\ L_M &= \{x: \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\} \\ E_M &= \{x: \rho_M(\lambda x) < \infty \text{ for all } \lambda > 0\} \\ \|x\| &= \inf\left\{\lambda > 0: \rho_M\left(\frac{x}{\lambda}\right) \leq 1\right\}, \end{aligned}$$

then the Orlicz space L_M and its subspace E_M are Banach spaces with the Luxemburg norm $\|\cdot\|$.

The following four lemmas can be found in [1].

Lemma 1

For any $f \in L_M^*$, there exist unique $v \in E_M^* = L_N$ (the Orlicz space generated by the complemented Orlicz function N) and $\varphi \in S = \{f \in L_M^*: f(E_M) = \{0\}\}$ such that $f = v + \varphi$. Moreover, $\|f\| = \|v\| + \|\varphi\|$, where $\|v\|$ and $\|\varphi\|$ are norms of v and φ as functionals on L_M respectively.

Let $L_M^+ = \{|x| = (|x(t)|): x = (x(t)) \in L_M\}$. If $\varphi \in S$ is nonnegative on L_M^+, then we say φ is positive. For any $\varphi \in S$, $x \in L_M^+$, let

$$\begin{aligned} \varphi^+(x) &= \sup\{\varphi(y): 0 \leq y(t) \leq x(t), t \in G\} \\ \varphi^-(x) &= -\inf\{\varphi(y): 0 \leq y(t) \leq x(t), t \in G\}. \end{aligned}$$

When $x \in L_M$ is arbitrary, we denote $x^+ = \frac{(|x|+x)}{2}$ and $x^- = x - x^+$, and define $\varphi^\pm(x) = \varphi^\pm(x^+) - \varphi^\pm(-x^-)$, then both $\varphi^\pm$ are positive and $\varphi = \varphi^+ - \varphi^-$.

Lemma 2

For any $\varphi \in S$, we have $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\|$.

Lemma 3

$\|\varphi + \psi\| = \|\varphi\| + \|\psi\|$ for all positive $\varphi, \psi \in S$.

Lemma 4

If $\varphi \in S$ is positive, then there exists $x \in L_M^+$ with $\|x\| = 1$ such that $\varphi(x) = \|\varphi\|$, i.e., every positive $\varphi \in S$ is norm attainable.

Theorem 5

Let $f \in S$, then for any disjoint subsets N', N'' of G, we have $\|f|_{N' \cup N''}\| = \|f|_{N'}\| + \|f|_{N''}\|$, where for any subset A of G, $x \in L_M$, $f|_A(x) = f(x|_A)$, and where $x|_A(t) = x(t)$, when $t \in A$ and $= 0$, when $t \in G \setminus A$.

Proof. For any $\varepsilon > 0$, since f is singular, we can find x, y in L_M with their supports in N' and N'' respectively such that $\rho(x) \leq \frac{1}{2}$, $\rho(y) \leq \frac{1}{2}$ and such that

$$f(x) = f|_{N'}(x) \geq \|f|_{N'}\| - \varepsilon, \quad f(y) = f|_{N''}(y) \geq \|f|_{N''}\| - \varepsilon.$$

Let $u = x + y$, then $\rho(u) = \rho(x) + \rho(y) \leq 1$ and hence,

$$\begin{aligned} \|f|_{N'}\| + \|f|_{N''}\| \geq \|f|_{N'} + f|_{N''}\| = \|f|_{N' \cup N''}\| \geq f(u) \\ = f|_{N'}(x) + f|_{N''}(y) \geq \|f|_{N'}\| + \|f|_{N''}\| - 2\varepsilon\,. \ \square \end{aligned}$$

Theorem 6

For any $f \in S$, if there exists $x \in L_M$, $\|x\| = 1$, such that $f(x) = \|f\|$, then for any subset A of G, we have $f(x|_A) = \|f|_A\|$.

Proof. Let $B = G \setminus A$, then by Theorem 5,

$$\|f\| = \|f|_A\| + \|f|_B\| \geq f|_A(x) + f|_B(x) = \|f\|.$$

Hence, we must have $f(x|_A) = f|_A(x) = \|f|_A\|$ and $f(x|_B) = \|f|_B\|$. $\square$

Theorem 7

$f \in S$ is norm attainable iff there exists a subset A of G such that $f^+ = f|_A$ and $f^- = -f|_B$, where $B = G \setminus A$.

Proof. Sufficiency. According to Lemma 4, there exist $x, y \in L_M$ such that $\rho(x) \leq \frac{1}{2}, \rho(y) \leq \frac{1}{2}$ and $f|_A(x) = f^+(x) = \|x\|$, $-f|_B(y) = f^-(y) = \|f^-\|$. Obviously, we may assume $x = x|_A$ and $y = y|_B$, hence, if we define $u = x - y$, then $\rho(u) \leq 1$ and thus,

$$\|f\| = \|f^+\| + \|f^-\| = f|_A(x) - f|_B(y) = f(u).$$

Necessity. Choose $x \in L_M$ with $\rho(x) \leq 1$ such that $f(x) = \|f\|$ and let $A = \{t \in G: x(t) \geq 0\}$, $B = G \setminus A$. It follows from the definition of f^+ and Theorem 6 that $\|f^+|_A\| \geq f^+|_A(x) \geq f|_A(x) = \|f|_A\|$. Hence, by Lemma 3

$$\|f^+\| = \|f^+|_A\| + \|f^+|_B\| \geq \|f|_A\| + \|f^+|_B\|.$$

Similarly, we have $\|f^-\| \geq \|f|_B\| + \|f^-|_A\|$. Therefore

$$\|f\| = \|f^+\| + \|f^-\| \geq \|f|_A\| + \|f|_B\| + \|f^+|_B\| + \|f^-|_A\|.$$

It follows from Theorem 5 that $f^+|_B = f^-|_A = 0$. Thus, for any $u \in L_M$,

$$f^+(u) = f^+(y|_A) - f^-(u|_A) = f(u|_A) = f|_A(u).$$

In the same way, we have $f^-(u) = -f|_B(u)$. □

Theorem 8

The set of all norm attainable singular functionals is dense in S.

Proof. Given any $\varphi \in S$ and $\varepsilon > 0$, by the Bishop-Phelps Theorem, we can find a norm attainable functional $f \in L_M^*$ such that $\|f - \varphi\| < \varepsilon$. By Lemma 1, $f = v + \psi$ for some $v \in L_N$ and $\psi \in S$. Choose $x \in L_M$ with $\|x\| = 1$ such that

$$\|v\| + \|\psi\| = \|f\| = \langle v, x\rangle + \langle \psi, x\rangle$$

then $\langle v, x\rangle = \|v\|$ and $\langle \psi, x\rangle = \|\psi\|$, hence, ψ is norm attainable and

$$\|\varphi - \psi\| = \|\varphi - f\| - \|v\| < \varepsilon. \ \square$$

Lemma 9

Suppose that $f \in S$, $x, y \in L_M$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and A is a subset of G, then $x(t)y(t) \geq 0$, $t \in A$ implies $f(y|_A) \geq f(x) - \|f\|$; $x(t)y(t) \leq 0$, $t \in A$ implies $f(y|_A) \leq \|f\| - f(x)$.

Proof. If $x(t)y(t) \geq 0,\ t \in A$, then

$$\rho(x - y|_A) \leq \rho(x) + \rho(y) < \infty.$$

Since f is singular, we have $f(x - y|_A) \leq \|f\|$. By the same way, if $x(t)y(t) \leq 0$ on $t \in A$, we have $f(y|_A) - f(x) \leq \|f\|$. □

Theorem 10

Let $f \in S$ and $\|f\| = 1$. Then f is an extreme point of the unit ball $B(L_M^*)$ of L_M^* iff for any subset A of G, $\|f|_A\| \cdot \|f|_{G\setminus A}\| = 0$.

Proof. The "only if" part. If there exists a subset A of G such that $\|f|_A\| > 0$ and $\|f|_B\| > 0$, where $B = G \setminus A$, then $\varphi := f|_A/\|f|_A\|$, $\psi := f|_B/\|f|_B\| \in B(L_M^*)$, and $f = \|f|_A\|\varphi + \|f|_B\|\psi$, which contradicts the condition that f is an extreme point since $\|f|_A\| + \|f|_B\| = \|f\| = 1$.

The "if" part. We first point out that $\|f^+\| \cdot \|f^-\| = 0$. To show this, by Theorem 8, we may assume that f is norm attainable. It follows from Theorem 7 that $\|f^+\| \cdot \|f^-\| = 0$. Thus, without loss of generality, we may assume that f is positive (otherwise, we consider the positive functional $-f = f^-$). It follows from Lemma 4 that there exists $x \in L_M^+$ such that $f(x) = \|f\| = \|x\| = 1$.

Suppose $f_1, f_2 \in B(L_M^*)$ satisfying $f_1 + f_2 = 2f$, we have to show that $f_1 = f_2$. First, by Lemma 1, we can easily deduce that $f_1, f_2 \in S$. For each $y \in L_M$ satisfying $f(y) = 0$, define $A = \{t \in G: x(t)y(t) \geq 0\}$ and $B = G \setminus A$, then, without loss of generality, we may assume that $f|_A = 0$. Hence,

$$f_1(x|_B) + f_2(x|_B) = 2f(x) = 2$$

and therefore, $f_i(x|_B) = \|f_i\| = 1,\ i = 1, 2$, which indicates $f_1|_A = f_2|_A = 0$. It follows from Lemma 9 that

$$f_i(y|_B) \leq \|f_i\| - f_i(x) = 0, \quad i = 1, 2.$$

Hence, $f_i(y) = f_i(y|_B) \leq 0,\ i = 1, 2$. Since $y \in \ker(f)$ is arbitrary, we have $f_i(y) = 0$, i.e., $\ker(f_i)$ contains $\ker(f)$, which shows that $f = \alpha_i f_i$ for some $\alpha_i \in \mathbb{R}$, and so, we must have $f = f_i,\ i = 1, 2$. □

For each $x \in L_M$, let $\theta(x) = \inf\{\alpha > 0: \rho(\frac{x}{\alpha}) < \infty\}$, then

Lemma 11 ([1])

$\theta(x)$ =dist (x, E_M) and for each $f \in S$,

$$\|f\| = \sup\left\{\frac{f(x)}{\theta(x)}: x \in L_M \setminus E_M\right\}.$$

Lemma 12

For any $x \in L_M$ and a partition $\{N_k\}_{k\le m}$ of G,

$$\max_k \{\theta(x|_{N_k})\} = \theta(x).$$

Proof. It is clear that $\theta(x|_{N_k}) \le \theta(x)$ for all $k \le m$. If $\alpha := \max_k \{\theta(x|_{N_k})\} < \theta(x)$, then for any $\beta \in (\alpha, \theta(x))$

$$\rho\Big(\frac{x}{\beta}\Big) = \sum_{k\le m} \rho\Big(\frac{x|_{N_k}}{\beta}\Big) < \infty$$

which shows $\theta(x) \le \beta < \theta(x)$, a contradiction. □

Let $\{x_n\}$ be a sequence in L_M and F a subset of L_M^*. We say $x_n \to x \in L_M$ F-weakly as $n \to \infty$, provided that $f(x_n - x) \to 0$ for all $f \in F$.

Theorem 13

The necessary and sufficient condition for $x_n \to 0$ S-weakly is that for any subsequence $\{y_k\}$ of $\{x_n\}$

$$\lim_m \theta\big(\min_{k\le m} |y_k|\big) = 0 \tag{1}$$

where $\min_{k\le m} |y_k|(t) = \min_{k\le m} |y_k(t)|,\ t \in G$.

Proof. If the condition is not sufficient, then by Rainwater's Theorem (see [3], p. 155), there exist $\varepsilon > 0$, an extreme point $f \in S$ of $B(L_M^*)$ and a subsequence $\{y_k\}$ of $\{x_n\}$ such that $f(y_k) > \varepsilon$ for all $k \in \mathbb{N}$. It follows from Theorem 10 that we may assume that f is positive. From condition (1), we can find some $m \in \mathbb{N}$ such that $\theta(\min_{k\le m} |y_k|) < \varepsilon$. Let

$$N_k = \Big\{t \in G: |y_k(t)| = \min_{k\le m} |y_k|(t)\Big\}, \quad k = 1, 2, \ldots, m$$

then by Theorem 10, there exists $k' \le m$ such that $f|_{N_{k'}} = f$. Hence

$$f(|y_{k'}|\big|_{N_{k'}}) \ge f(y_{k'}|_{N_{k'}}) = f(y_{k'}) > \varepsilon.$$

On the other hand, by Lemma 11

$$f(|y_{k'}|\big|_{N_{k'}}) \le \theta(|y_{k'}|\big|_{N_{k'}})\|f\| \le \theta\big(\min_{k\le m} |y_k|\big) < \varepsilon$$

a contradiction.

If the condition is not necessary, then there exist $\varepsilon > 0$ and a subsequence $\{y_k\}$ of $\{x_n\}$ such that

$$\theta\big(\min_{k\le m}|y_k|\big) > \varepsilon, \quad m \in \mathbb{N}.$$

Let $N_1(1) = \{t \in G: y_1(t) \ge 0\}$, $N_1(2) = G \setminus N_1(1)$. If $N_k(s)$ has been found for $s = 1, 2, \dots, 2^k$, $k = 1, 2, \dots, m$, then let

$$N_{m+1}(2s-1) = \{t \in N_m(s): y_{m+1}(t) \ge 0\},$$
$$N_{m+1}(2s) = \frac{N_m(s)}{N_{m+1}(2s-1)},$$

$s = 1, 2, \dots, 2^{m+1}$. By induction, for any $k \in \mathbb{N}$, we find a partition $\{N_k(s): s \le 2^k\}$ of G satisfying for any $m \ge k$, $y_k(t)$ is nonnegative or nonpositive on $N_m(s)$, $s = 1, 2, \dots, 2^m$. By Lemma 12, there is some $s_m \le 2^m$ such that

$$\theta\Big(\big(\min_{k\le m}|y_k|\big)|_{N_m(s_m)}\Big) = \theta\big(\min_{k\le m}|y_k|\big) > \varepsilon.$$

Hence,by Lemma 11 and the Hahn-Banach Theorem, we can find $f_m \in S$ with $\|f_m\| = 1$ such that

$$f_m\Big(\big(\min_{k\le m}|y_k|\big)|_{N_m(s_m)}\Big) = \theta\Big(\big(\min_{k\le m}|y_k|\big)|_{N_m(s_m)}\Big) > \varepsilon, \quad m \in \mathbb{N}.$$

(Observe $f_m = f_m^+ - f_m^-$, it is clear by Lemma 2 and Lemma 11 that $f_m^- = 0$, i.e. f_m is positive.) Since $B(L_M^*)$ is w^* compact, the sequence $\{f_m\}$ has a cluster point $f \in S$. It follows that for each $k \in \mathbb{N}$, there exists some $m > k$ such that

$$|f(y_k) - f_m(y_k)| < \frac{\varepsilon}{2}.$$

In view of

$$\begin{aligned}\|f_m|_{N_m(s_m)}\| &\ge \frac{f_m\Big(\big(\min_{k\le m}|y_k|\big)|_{N_m(s_m)}\Big)}{\theta\Big(\big(\min_{k\le m}|y_k|\big)|_{N_m(s_m)}\Big)}\\ &= 1 = \|f_m\|\end{aligned}$$

we find $\|f_m|_{G\setminus N_m(s_m)}\| = 0$ according to Theorem 5. Therefore

$$\begin{aligned}|f(y_k)| &\ge |f_m(y_k)| - |f(y_k) - f_m(y_k)| \ge |f_m(y_k|_{N_m(s_m)})| - \frac{\varepsilon}{2}\\ &\ge \Big|f_m\Big(\big(\min_{k\le m}|y_k|\big)|_{N_m(s_m)}\Big)\Big| - \frac{\varepsilon}{2} > \frac{\varepsilon}{2}\end{aligned}$$

contradicting the hypothesis that $x_n \to 0$ S-weakly. □

Lemma 14 ([5])

$x_n \to 0$ L_N-weakly iff

i) $\int_E x_n(t)d\mu \to 0$ *as* $n \to \infty$ *for each* $E \in \Sigma$ *and*

ii) $\lim_{\lambda\to 0} \sup_n \lambda^{-1}\rho_M(\lambda x_n) = 0$.

Lemma 15 ([2])

A subset E of L_M is L_N-weakly compact iff

$$\lim_{\lambda\to 0} \sup_{x\in E} \lambda^{-1}\rho(\lambda x) = 0 .$$

By Theorem 13 and Lemma 14, we obtain.

Theorem 16

A sequence $\{x_n\}$ in L_M converges to 0 weakly iff

a) $\lim_n \int_E x_n(t)d\mu = 0$ *for all* $E \in \Sigma$;

b) $\lim_{\lambda\to 0} \sup_n \lambda^{-1}\rho(\lambda x_n) = 0$ *and*

c) for any subsequence $\{y_k\}$ of $\{x_n\}$, we have $\lim_m \theta(\min_{k\le m} |y_k|) = 0$.

Theorem 17

A subset K of L_M is weakly compact if and only if

1) $\lim_{\lambda\to 0} \sup_{x\in E} \lambda^{-1}\rho(\lambda x) = 0$ *and*

2) $\lim_m \theta(\min_{n\le m} |x_n - x|) = 0$ *for all sequence $\{x_n\}$ in K satisfying*

$$\lim_n \int_E [x_n(t) - x(t)]d\mu = 0 \quad \text{for each} \quad E \in \Sigma . \tag{$*$}$$

Proof. Necessity. The first condition follows from Theorem 15. Now, we check the second one. Let K be a weakly compact subset of L_M, then for sequence $\{x_n\}$ in K satisfying $(*)$, we can pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ weakly convergent to some point x' in L_M. It follows from Theorem 16 that

$$0 = \lim_m \theta\big(\min_{i\le m} |x_{n_i} - x'|\big) \ge \lim_m \theta\big(\min_{n\le m} |x_n - x'|\big) \ge 0 .$$

Since $(*)$ implies that $x_n \to x$ E_N-weakly by [1], we have $x' = x$.

Sufficiency. For any sequence $\{x_n\}$ in K, by Theorem 15 and [1], it contains a subsequence, again denoted by $\{x_n\}$, L_N-weakly convergent to some $x \in L_M$. For any subsequence $\{y_j\}$ of $\{x_n\}$, by the second condition and [1],

$$\lim_m \theta\big(\min_{j \leq m} |y_j - x|\big) = 0$$

it follows from Theorem 13 that $x_n \to x$ weakly. □

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