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Applications of sequential shifts to an interpolation problem

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Abstract

In the present paper initial operators for a right invertible operator, which are induced by sequential shifts and have the property $c(R)$ (cf. [23]) are constructed. An application to the Lagrange type interpolation problem is given. Moreover, an example with the Pommiez operator is studied.

§ **0.** Let X be a linear space over the field $\mathbb{C}$ of the complex numbers. Denote by $L(X)$ the set of all linear operators with domains and ranges in X and by $L_0(X)$ the set of those operators from $L(X)$ which are defined on the whole space X. We denote by $R(X)$ the set of all right invertible operators belonging to $L(X)$, by $\mathcal{R}_D$ – the set of all right inverses of a $D \in R(X)$ and by $\mathcal{F}_D$ – the set of all initial operators for D, i.e.

$$\mathcal{R}_D := \{R \in L_0(X): DR = I\},$$
$$\mathcal{F}_D := \{F \in L_0(X): F^2 = F, FX = \ker D \text{ and } \exists\, R \in \mathcal{R}_D: FR = 0\}.$$

In the sequel, we shall assume that $\dim \ker D > 0$, i.e. D is right invertible but not invertible. The theory of right invertible operators and its applications can be found in the book of D. Przeworska-Rolewicz [17].

Here and in the sequel we admit that $0^0 := 1$. We also write: $\mathbb{N}$ for the set of all positive integers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

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For a given operator $D \in R(X)$ we shall write (cf. [17]):

$$S := \bigcup_{i=1}^{\infty} \ker D^i. \tag{0.1}$$

If $R \in \mathcal{R}_D$ then the set S is equal to the linear span $P(R)$ of all D-monomials, i.e.

$$S = P(R) := \text{lin}\,\{R^k z : z \in \ker D, k \in \mathbb{N}_0\}. \tag{0.2}$$

Evidently, the set $P(R)$ is independent of the choice of the right inverse R.

§ **1.** Suppose that $Y = (s)$ is the set of all sequences $a = \{a_n\}$, where $a_n \in \mathbb{C}$ $(n \in \mathbb{N}_0)$.

In the sequel, a non-empty set $\Omega \subseteq \mathbb{C}$ containing a number different from zero and a sequence $a = \{a_n\} \in Y$ are arbitrarily fixed.

DEFINITION 1.1. Suppose that $D \in R(X)$ and dim ker $D > 0$. We say that $T_{a,\Omega} = \{T_{a,h}\}_{h\in\Omega} \subset L_0(X)$ is a family of *sequential shifts* for the operator D induced by the sequence a if the following conditions are satisfies:

$$T_{a,h} = \sum_{n=0}^{\infty} a_n h^n D^n \quad \text{on the set } S,$$

for all $h \in \Omega; k \in \mathbb{N}_0$, where S is defined by Formula (0.2).

We should point out that by definition of the set S, the last sum has only a finite number of members different from zero.

The listed properties and other information about shifts for right invertible operators can be found in the author's papers [1]-[11] (cf. also works of D. Przeworska-Rolewicz [16]-[20], [22]).

Theorem 1.1 (cf. [5])

Suppose that $D \in R(X)$ and dim ker $D > 0$, *F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and a family $T_\Omega = \{T_h\}_{h\in\Omega} \subset L_0(X)$. Then the following two conditions are equivalent:*

a) *T_Ω is a family of sequential shifts for the operator D induced by the sequence $a = \{a_n\}$,*

b) $T_h R^k F = \sum_{j=0}^{k} a_j h^j R^{k-j} F$ *for all* $h \in \Omega;\ k \in \mathbb{N}_0$.

Proposition 1.1 (cf. [5])

Suppose that $D \in R(X)$, $\dim \ker D > 0$ and $T_{a,\Omega} = \{T_{a,h}\}_{h\in\Omega}$ is a family of sequential shifts for the operator D induced by the sequence $a = \{a_n\}$. Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Then

(i) For all $h \in \Omega :\ z \in \ker D;\ k \in \mathbb{N}_0$

$$T_{a,h}R^k z = \sum_{j=0}^{k} a_j h^j R^{k-j} z\,. \tag{1.1}$$

(ii) The operators $T_{a,h}$ $(h \in \Omega)$ are uniquely determined on the set S.

(iii) If X is a complete linear metric space, $\overline{S} = X$ and $T_{a,h}$ are continuous for $h \in \Omega$ then $T_{a,h}$ are uniquely determined on the whole space.

(iv) For all $h \in \Omega$ the operator $T_{a,h}$ commute on the set S with the operator D.

Proposition 1.2

Suppose that all assumptions of Proposition 1.1 are satisfied and $a_m \neq 0$ for a number $m \in \mathbb{N}$. For an arbitrary fixed $h \in \Omega \setminus \{0\}$ we define the operator

$$F_{m,h} := \alpha(h) F T_{a,h} R^m, \tag{1.2}$$

where

$$\alpha(h) := h^{-m} a_m^{-1}\,. \tag{1.3}$$

Let an operator $A \in L_0(X)$ be arbitrary fixed. Then

(i) The operator $F_{m,h}$ is an initial operator for D corresponding to the right inverse

$$R_{m,h} := R - F_{m,h}R\,. \tag{1.4}$$

(ii) The operator

$$D_{m,h} := D + AF_{m,h} \tag{1.5}$$

is right invertible and $R_{m,h} \in \mathcal{R}_{D_{m,h}}$.

Proof. (i) Theorem 1.1. and the equality $FR = 0$ together imply

$$\begin{aligned} F_{m,h}^2 &= [\alpha_m(h) F T_{a,h} R^m][\alpha_m(h) F T_{a,h} R^m] \\ &= \alpha_m^2(h) F T_{a,h} R^m F [T_{a,h} R^m] = \alpha_m^2(h) F \sum_{j=0}^{m} a_{m-j} h^{m-j} R^j F [T_{a,h} R^m] \\ &= \alpha_m^2(h) a_m h^m F^2 T_{a,h} R^m = \alpha_m(h) F T_{a,h} R^m = F_{m,h}\,. \end{aligned}$$

Moreover, the operator $F_{m,h}$ is a projection onto $\ker D$. Indeed, for all $z \in \ker D$ by Formula (1.1) we have

$$\begin{aligned} F_{m,h}z &= \alpha_m(h)FT_{a,h}R^m z = \alpha_m(h)F\sum_{j=0}^{m} a_{m-j}h^{m-j}R^j z \\ &= \alpha_m(h)a_m h^m Fz = Fz = z\,. \end{aligned}$$

The operator $F_{m,h}$ is an initial operator for D corresponding to the right inverse determined by Formula (1.4) (cf. [17]).

(ii) Consider the operator $D_{m,h} := D + AF_{m,h}$. Point (i) and the definition together implies

$$F_{m,h}R_{m,h} = 0, \quad DF_{m,h} = 0\,.$$

This yields that on X

$$\begin{aligned} D_{m,h}R_{m,h} &= [D + AF_{m,h}]R_{m,h} = DR_{m,h} + AF_{m,h}R_{m,h} \\ &= DR_{m,h} = I, \end{aligned}$$

i.e.

$$R_{m,h} \in \mathcal{R}_D \cap \mathcal{R}_{D_{m,h}}\,. \square$$

Following [23], an initial operator F_0 for D has the property $c(R)$ for an $R \in \mathcal{R}_D$ if there exist scalars c_k such that

$$F_0 R^k z = \frac{c_k}{k!} z \quad \text{for all} \quad z \in \ker D;\ k \in \mathbb{N} \tag{1.6}$$

and $c_k = 0$ for all $k \in \mathbb{N}$ if $F_0 = F$. We shall write: $F_0 \in c(R)$. A set $\mathcal{F}_D^0 \subseteq \mathcal{F}_D$ has the property (c) if for every $F_0 \in \mathcal{F}_D^0$ there exists an $R \in \mathcal{R}_D$ such that $F_0 \in c(R)$.

The set $\mathcal{F}_D$ of all initial operators has the property (c) if and only if $\dim \ker D = 1$ (cf. [23]).

Proposition 1.3

Suppose that all assumptions of Proposition 1.1 are satisfied. Let the operator $F_{m,h}$ be defined by Formula (1.2), where $0 \neq h \in \Omega$ is arbitrarily fixed. Then $F_{m,h} \in c(R)$ and the coefficients c_k have the form

$$c_k = \beta_k h^k \quad (k \in \mathbb{N}), \tag{1.7}$$

were $\beta_k = k! a_{m+k} a_m^{-1}$.

Proof. Let $z \in \ker D$; $k \in \mathbb{N}$ be arbitrary fixed. Then by Formula (1.1) we have

$$\begin{aligned} F_{m,h}R^k z &= [\alpha_m(h)FT_{a,h}R^m]R^k z = \alpha_m(h)FT_{a,h}R^{m+k}z \\ &= \alpha_m(h)F\sum_{j=0}^{m+k} a_{m+k-j}h^{m+k-j}R^j z = \alpha_m(h)a_{m+k}h^{m+k}Fz \\ &= \frac{a_{m+k}}{a_m}h^k z. \end{aligned}$$

By Proposition 1.2, $F_{m,h} \in \mathcal{F}_D$. □

Proposition 1.3 implies

Proposition 1.4

Suppose that $D \in R(X)$ and dim ker $D > 0, F$ *is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and let $0 \neq h \in \mathbb{C}$ be arbitrarily fixed. Then there exists $F_h \in L_0(X)$, which is an initial operator for D corresponding to a right inverse $R_h := R - F_h R$ such that*

$$F_h R^n z = h^n z \quad \text{for all } \; z \in \ker D \; (n \in \mathbb{N}). \tag{1.8}$$

The operator F_h is defined by the formula

$$F_h := F\widetilde{T}_h, \tag{1.9}$$

where $\widetilde{T}_h$ is an extension of the operator $T_h \in L_0(S)$:

$$T_h := \sum_{n=0}^{\infty} h^n D^n. \tag{1.10}$$

Proof. Consider the operator F_h determined by Formula (1.9), where $\widetilde{T}_h$ is an extension of the operator T_h defined by Formula (1.10). By Proposition 1.2, F_h is an initial operator for D corresponding R_h determined by Formula (1.4). Proposition 1.3 implies that $F_h \in c(R)$ and Formula (1.8) holds. □

We have also (cf. Proposition 2.3.-[23], Theorem 5.25.-[17]):

Proposition 1.5

Suppose that all assumptions of Proposition 1.4 are satisfied. Then there exists $F_h \in L_0(X)$, which is an initial operator for D corresponding to $R_h = R - F_h R$, such that

$$F_h R^n z = \frac{h^n}{n!} z \quad \text{for all} \quad z \in \ker D \ (n \in \mathbb{N}).$$

The operator F_h is defined by the formula

$$F_h := F\widetilde{T}_h,$$

where $\widetilde{T}_h$ is an extension of the operator $T_h \in L_0(S)$:

$$T_h := \sum_{n+0}^{\infty} \frac{h^n}{n!} D^n.$$

§ **2.** Let $D \in R(X)$ and dim ker > 0. We consider the following Lagrange type interpolation problem (cf. Przeworska-Rolewicz [23], [17], also Nguyen Van Mau [14], Tasche [24]):

Find a D-polynomial of degree $N-1$ $(N > 1)$, i.e. an element $u = \sum_{k=0}^{N-1} R^k z_k$, where $R \in \mathcal{R}_D$; $z_0, z_1, \ldots, z_{N-1} \in \ker D$ which admits, for given N different initial operators $F_0, F_1, \ldots, F_{N-1} \in \mathcal{F}_D$, the given values

$$F_j u = u_j, \quad j = 0, 1, \ldots, N-1, \tag{2.1}$$

where $u_j \in \ker D$.

Theorem 2.1 (cf. [23], Theorem 3.1)

Suppose that $D \in R(X), R \in \mathcal{R}_D$ and $F_0, F_1, \ldots, F_{N-1} \in c(R)$ such that

$$F_j R^k z = \frac{d_{jk}}{k!} z \quad \text{for} \quad j = 0, 1, \ldots, N-1, \ k \in \mathbb{N}.$$

If $V = \det(d_{jk})_{j,k=0,1,\ldots,N-1} \neq 0$ then the considered interpolation problem has a unique solution for every $u_0, u_1, \ldots, u_{N-1} \in \ker D$ of the form

$$u = \frac{1}{V} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} (-1)^{k+j} V_{jk} R^k u_j, \tag{2.2}$$

where V_{jk} is the minor determinant obtained by canceling in V the k-th column and the j-th row; $j, k = 0, 1, \ldots, N-1$.

Proposition 1.4 implies that there exist initial operators $F_0, F_1, \ldots, F_{N-1} \in \mathcal{F}_D \cap c(R)$ such that

$$F_k R^n z = h_k^n z \quad \text{for all} \quad z \in \ker D, n \in \mathbb{N}, \qquad (k = 0, 1, \ldots, N-1),$$
$$F_k := F_{h_k} = F\widetilde{T}_{h_k},$$

where $h_k \in \mathbb{C}$ are arbitrarily fixed $(0 \leq k \leq N-1)$, $\widetilde{T}_h$ is an extension of the operator $T_h \in L_0(S)$ defined by Formula (1.10). Evidently, for different $h_k, k = 0, 1, \ldots, N-1$ the determinant

$$V = \det\left(k! h_k^j\right)_{j,k=0,1,\ldots,N-1} \neq 0.$$

In particular we take $h_k = \varepsilon_k$, where $\varepsilon_k = \exp(2\pi i k/N)$, $k = 0, 1, \ldots, N-1$. Then

$$F_k = F_{\varepsilon_k} \tag{2.3}$$

and

$$F_k R^n z = \varepsilon_k^n z = \varepsilon^{nk} z,$$

where $\varepsilon := \varepsilon_1 = \exp(2\pi i/N)$. We define the vectors:

$$\mathbf{R} := \left[I, R, R^2, \ldots, R^{N-1}\right], \ \mathbf{u}^T := \left[u_0, u_1, \ldots, u_{N-1}\right],$$

(where as usually $\mathbf{A}^T$ denotes the matrix transposed to $\mathbf{A}$).

Theorem 2.2

Suppose that $D \in R(X)$ and $\dim \ker D > 0$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Then the interpolation problem with F_k $(k = 0, 1, \ldots, N-1)$ defined by Formula (2.3) has a unique solution of the form

$$u = \mathbf{R}\mathbf{B}\mathbf{u}, \tag{2.4}$$

where

$$\mathbf{B} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \varepsilon^{-1} & \varepsilon^{-2} & \ldots & \varepsilon^{-(N-1)} \\ 1 & \varepsilon^{-2} & \varepsilon^{-4} & \ldots & \varepsilon^{-2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \varepsilon^{-(N-1)} & \varepsilon^{-2(N-1)} & \ldots & \varepsilon^{-(N-1)^2} \end{bmatrix}$$

Proof. We are looking for a solution of the interpolation problem of the form $u = \mathbf{R}\mathbf{z}$, satisfying the conditions (2.1), where the vector $\mathbf{z}^T := [z_0, z_1, \ldots, z_{N-1}]$, $z_0, z_1, \ldots, z_{N-1} \in \ker D$ is to be determined, we obtain the equation

$$\mathbf{B}^{-1}\mathbf{z} = \mathbf{u},$$

where

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \varepsilon^1 & \varepsilon^2 & \ldots & \varepsilon^{(N-1)} \\ 1 & \varepsilon^2 & \varepsilon^4 & \ldots & \varepsilon^{2(N-1)} \\ . & . & . & . & . \\ 1 & \varepsilon^{(N-1)} & \varepsilon^{2(N-1)} & \ldots & \varepsilon^{(N-1)^2} \end{bmatrix}.$$

The determinant $|\mathbf{B}^{-1}| = i^{\frac{(N-1)(3N-2)}{2}} N^{\frac{N}{2}} \neq 0$ and $\mathbf{B}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$. This implies that the problem has a unique solution determined by Formula (2.4). □

EXAMPLE 2.1: Let $X = H(K)$ be the class of all functions analytic in the disk $K = \{h \in \mathbb{C}: |h| < r, r > 0\}$. We define operators D, R as follows:

$$[Dx](t) := \frac{x(t) - x(0)}{t}; [Rx](t) := tx(t); x \in X, t \in K,$$

where

$$\frac{x(t) - x(0)}{t}\Big|_{t=0} := x'(0).$$

The operators D, R are uniquely determined on the whole space X, i.e. $D, R \in L_0(X)$, $\dim \ker D = 1$, codim $RX = 1$ (cf. [13]). The operator D is called a *Pommiez operator* (cf. Pommiez [15]). We can prove (cf. [3]) that $R \in \mathcal{R}_D$,

$$[Fx](t) = [(I - RD)x] = x(0), \tag{2.5}$$

$$P(R) = \text{lin}\,\{R^k\mathbf{1}: k = 0, 1, 2, \ldots\}. \tag{2.6}$$

Evidently, $\mathcal{F}_D \subset c(R)$ and $\overline{S} = \overline{P(R)} = X$.

In order to construct the operators F_h defined in Proposition 1.4., we observe that

$$R^k\mathbf{1} = t^k, \quad R^kFx = (Fx)R^k\mathbf{1} = x(0)t^k, \quad x \in X, t \in K, \ k \in \mathbb{N}_0.$$

We take

$$T_{f,h}x := \sum_{n=0}^{\infty} h^n D^n x \quad \text{for} \quad x \in S; h \in K. \tag{2.7}$$

Clearly, $T = \{T_h\}_{h \in K}$ is a family of sequential shifts for the operator D induced by the sequence $a = \{1, 1, \ldots, 1, \ldots\}$.

Proposition 1.1 and Formula (2.5) together imply:

$$T_h R^k F x = \sum_{j=0}^{k} h^j R^{k-j} F x = x(0) \sum_{j=0}^{k} h^j t^{k-j}$$

$$= \begin{cases} x(0) \frac{h^{k+1} - t^{k+1}}{h-t} & \text{for } t \neq h \\ x(0)(k+1)h^k & \text{for } t = h. \end{cases} \tag{2.8}$$

Evidently, $T_h R^k F x \in X$.

Equality (2.6) implies that every element $x \in P(R)$ can be written in the form

$$x(t) = \sum_{k+0}^{m} b_k R^k \mathbf{1} \quad (m \in \mathbb{N}_0),$$

where b_k $(k = 0, 1, \ldots, m)$ are scalars, in one and only one manner. Let T_h be defined by Formula (2.7) for arbitrarily fixed $h \in K$ and let $x \in P(R)$. Then

$$T_h x = T_h \Big[\sum_{k=0}^{m} b_k R^k \mathbf{1} \Big] = \sum_{k=0}^{m} b_k T_h R^k \mathbf{1} = \sum_{k=0}^{m} b_k \sum_{j=0}^{k} h^j t^{k-j}$$

$$= \begin{cases} \frac{tx(t) - hx(h)}{t-h} & \text{for } t \neq h \\ \frac{d}{dt}[tx(t)]\big|_{t=h} = x(h) + hx'(h) & \text{for } t = h \end{cases}$$

This follows from Formula (2.8). We take $\widetilde{T}_h \in L_0(X)$:

$$\big[\widetilde{T}_h x\big](t) := \begin{cases} \frac{tx(t) - hx(h)}{t-h} & \text{for } t \neq h \\ x(h) + hx'(h) & \text{for } t = h \end{cases}$$

for all $x \in X, h \in K$. Hence for $h \in K, x \in X$

$$\big[F_h x\big](t) = \big[F \widetilde{T}_h x\big](t) = x(h).$$

In this case the conditions (2.1) with $F_j = F_{h_j}$, where $h_j \neq h_m$ for $j \neq m = 0, 1, \ldots, N-1$, have the form

$$\big[F_j u\big](t) = \big[F_{h_j} u\big](t) = u(h_j) = u_j \quad (j = 0, 1, \ldots, N-1),$$

where $u \in X, u_j$ are scalars. The interpolation problem has a unique solution for every scalars $u_0, u_1, \ldots, u_{N-1}$ of the form (2.2), where $V = \det(k! h_j^k) \neq 0$.

In particular, the interpolation problem with the knots $h_j = \varepsilon_j$, $\varepsilon_j = \exp(2\pi i j/N)$ $(0 \leq j \leq N-1)$ on the unit circle has a unique solution for every scalars $u_0, u_1, \ldots, u_{N-1}$ of the form (2.4).

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