Collect. Math. **44** (1993), 41–46

On infinitely smooth almost-wavelets with compact support

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Abstract

In 1985 Y. Meyer has constructed the infinitely smooth function $\psi(t), t \in \mathbb{R}$, with compact spectrum such that the system of functions $2^{\frac{j}{2}}\psi(2^j t - k)$, $j, k \in \mathbb{Z}$, forms an orthonormal basis for $L_2(\mathbb{R})$ [1]. Now such systems are called wavelets. There are known wavelets with exponential decay on infinity [2,3,4] and wavelets with compact support [5]. But these functions have finite smoothness. It is known that there does not exist infinitely differentiable compactly supported wavelets

In the article we present the system of infinitely smooth functions $\psi = \{\varphi_{0k}, \psi_{jk}, j = 0, 1, 2, \dots; k \in \mathbb{Z}\}$ with the following properties:

1) ψ forms an orthonormal basis for $L_2(\mathbb{R})$;
2) $\varphi_{0k}(t) = \varphi_{00}(t - k); \psi_{jk}(t) = \psi_{j0}(t - k2^{-j}); t \in \mathbb{R}, j = 0, 1, 2, \dots; k \in \mathbb{Z}$;
3) $\operatorname{supp} \varphi_{00} = (-3, 0); \operatorname{supp} \psi_{j0} = [-(j+3)2^{-j}, j2^{-j}]$.

In contrast to wavelets, the system ψ is not generated by dilations and translations of a single function. However, the measure of ψ_{jk}-supports converges to zero if j tends to infinity and under the fixed j the functions $\psi_{jk}, k \neq 0$, are obtained by translations of ψ_{j0}. These properties allow to call the system ψ as almost-wavelets.

Construction of ψ. Let $N = 1, 2, 3, \ldots$. Define the function

$$m_N(\xi) := \Big(\frac{1+e^{i\xi}}{2}\Big)^N Q_N(e^{i\xi}), \xi \in \mathbb{R},$$

where the polynomial

$$Q_N(\xi) = \sum_{l=0}^{N-1} q_N(l)\xi^l, \quad q_N(0) \neq 0,$$

satisfies the identity

$$|Q_N(e^{i\xi})|^2 = \sum_{l=0}^{n-1} \binom{N-1+l}{l} \Big(\sin^2(\frac{\xi}{2})\Big)^l,$$

and is determined as in [5]. The functions $m_N(\xi)$ meet the identity

$$|m_N(\xi)|^2 + |m_N(\xi+\pi)|^2 = 1, \quad \xi \in \mathbb{R}. \tag{1}$$

As usual $F\varphi$ denotes the Fourier transform of φ and $F^{-1}\varphi$ its inverse, respectively.

One can prove that infinite products

$$G_j(\xi) := \prod_{N=j+1}^{\infty} m_N(2^{-N}\xi), \quad j = 0, 1, 2, \ldots$$

converge absolutely for any $\xi \in \mathbb{R}$, and the convergence is uniform on any bounded set. And what is more the following inequality is true

$$|G_j(\xi)| \leq \begin{cases} 1, & |\xi| \leq 2^j\pi; \\ |\xi|^{-\alpha \ln |\xi|}, & |\xi| > 2^j\pi; \end{cases} \tag{2}$$

where α is a positive constant depending only on j. Define

$$\varphi_j(t) := (2\pi)^{-1/2} 2^{-j/2} (F^{-1}G_j)(t), \ \ j \in \{0, 1, 2, \ldots\}.$$

By virtue of (2), the definition is correct. Let

$$\varphi_{jk}(t) := \varphi_j(t - k2^{-j}), \quad k \in \mathbb{Z}.$$

The functions $\psi_j(t), j = 0, 1, 2, \ldots$ are determined by their Fourier transforms

$$F\psi_j(\xi) = 2^{1/2} e^{-i(2^{-j-1}\xi+\pi)} m_{j+1}(2^{-j-1}\xi + \pi) F\varphi_{j+1}(\xi), \ \ \xi \in \mathbb{R}.$$

By virtue of (2-3), the definition is correct. Consider the functions

$$\psi_{jk}(t) = \psi_j(t - k2^{-j}), \ \ j = 0, 1, \ldots; k \in \mathbb{Z};$$

and denote

$$\psi = \{\varphi_{0k}, \psi_{jk}, \ j = 0, 1, 2, \ldots; \ \ k \in \mathbb{Z}\}.$$

Properties of ψ. From (1-2) it follows:

Lemma 1

Functions φ_{jk} and ψ_{jk}, $j = 0, 1, \ldots$; $k \in \mathbb{Z}$ are infinitely smooth.

Lemma 2

For any $j = 0, 1, 2, \ldots; k, k' \in \mathbb{Z}$

$$\begin{aligned}&(\varphi_{jk}, \varphi_{jk'}) = \delta_{kk'};\\&(\psi_{jk}, \psi_{jk'}) = \delta_{kk'};\ (\varphi_{jk}, \psi_{jk'}) = 0\,,\end{aligned} \tag{3}$$

where $\delta_{kk'}$ is the Kronecker symbol, (f, g) denotes inner product in $L_2(\mathbb{R})$.

Let $[x_l, l \in L]$ designate the closure of the linear span of $\{x_l, l \in L\}$ in L_2-norm, L-some index set. Introduce the vector spaces

$$\begin{aligned}V_j &:= [\varphi_{jk}, k \in \mathbb{Z}],\\W_j &:= [\psi_{jk}, k \in \mathbb{Z}], j = 0, 1, 2, \ldots\end{aligned}$$

By virtue of Lemma 2, systems $\{\varphi_{jk}, k \in \mathbb{Z}\}$ and $\{\psi_{jk}, k \in \mathbb{Z}\}$ are orthonormal bases of V_j and W_j, respectively, $j = 0, 1, 2, \ldots$.

Lemma 3

For any $j = 0, 1, 2, \ldots$

$$\begin{aligned}&V_j \subset V_{j+1}, V_j \ominus W_j = V_{j+1};\\&\overline{\bigcup_{j=0}^{\infty} V_j} = L_2(\mathbb{R}).\end{aligned} \tag{4}$$

Lemma 4

For any $f \in L_2(\mathbb{R})$

$$\sum_{k\in\mathbb{Z}} |(f, \varphi_{0k})|^2 + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} |(f, \psi_{jk})|^2 = \|f\|^2_{L_2}\,.$$

Let $j = 0, 1, \ldots; r = 1, 2, \ldots$. Denote

$$\eta^{(j)}_{r,0}(t) := 2^{j/2}\chi_{(-2^{-j-1}, 2^{-j-1})}(t),$$

where χ_e is an indicator function of the set e.

Define functions $\eta_{r,\nu}^{(j)}, \nu = 1, \dots, r$; by recursion

$$\eta_{r,\nu}^{(j)}(t) := \sqrt{2} \sum_{l=0}^{(2j+2r+1-2\nu)} h_{j+r+1-\nu}(l)\eta_{r,\nu-1}^{(j)}(2t + 2^{-j}l),$$

where numbers $h_N(l), l = 0, 1, \dots, 2N-1$, are determined by the identity

$$m_N(\xi) \equiv 2^{-1/2} \sum_{l=0}^{2N-1} h_N(l)e^{il\xi}.$$

Lemma 5

For any $j = 0, 1, 2, \dots$ the functions $\eta_{r,r}^{(j)}$ converge to φ_j pointwise and in $L_2(\mathbb{R})$ as $r \to \infty$.

Lemma 5 implies

Lemma 6

The function φ_j and $\psi_j, j = 0, 1, \dots$ have compact supports:

$$\begin{aligned} \text{supp}\ \varphi_j &= [-(2j+3)2^{-j}, 0], \\ \text{supp}\ \psi_j &= [-(j+3)2^{-j}, j2^{-j}]. \end{aligned}$$

Finally we have

Theorem

The system Ψ, which consists of infinitely smooth functions with compact supports, forms an orthonormal basis for $L_2(\mathbb{R})$.

REMARK

1. Let $N = 1, 2, \dots$. In [4] Ingrid Daubechies has considered the functions

$$\begin{aligned} \Phi_1^{(N)}(t) &= (2\pi)^{-1/2} \left(F^{-1}\Big(\prod_{l=1}^{\infty} m_N(2^{-l}\xi) \Big) \right)(t), \\ \Phi_2^{(N)}(t) &= (2\pi)^{1/2} \sum_{\nu \in \mathbb{Z}} g_N(\nu)\Phi_1^{(N)}(2t+\nu) \end{aligned}$$

where

$$g_N(\nu) := (-1)^\nu h_N(-\nu+1), \quad \nu \in \mathbb{Z}.$$

She has proved that the system

$$\{2^{j/2}\Phi_2^{(N)}(2^j t - k),\ j \in \mathbb{Z},\ k \in \mathbb{Z}\}$$

is an orthonormal basis for $L_2(\mathbb{R})$ and supp $\Phi_2^{(N)} = [-N+1, N]$. Besides, the smoothness of $\Phi_1^{(N)}, \Phi_2^{(N)}$ is finite and grows with increase of N.

2. Using the formula

$$\prod_{N=1}^{\infty} \cos(2^{-N}\xi) = \frac{\sin \xi}{\xi}$$

it is easy to show that for any $j = 0, 1, \dots$

$$F\varphi_j(\xi) = (2\pi)^{-1/2} 2^{-j/2} e^{i2^{-j-1}(j+2)\xi} R_j(\xi) \prod_{N=j+1}^{\infty} Q_N(2^{-N}\xi),$$

where

$$R_j(\xi) = \left(\frac{\sin 2^{-j-1}\xi}{2^{-j-1}\xi}\right)^{j+1} \prod_{N=j+2}^{\infty} \frac{\sin 2^{-N}\xi}{2^{-N}\xi}.$$

Remark that functions

$$up_j(t) = (2\pi)^{-1/2}(F^{-1}R_j)(t), \quad j = 0, 1, 2, \dots$$

were introduced and studied in [6].

Comments 1,2 show that our approach develops the ideas of [5-6].

3. The construction of functions $\psi_{jk}, j = 0, 1, \dots; k \in \mathbb{Z}$, satisfying (3-4) uses the concepts of multiresolution analysis [7].

4. It is interesting to investigate the basis properties of the almost-wavelets in Besov-Lizorkin-Triebel spaces. The authors intend to devote a separate paper to this.

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