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Locally nearly uniformly smooth Banach spaces*

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ABSTRACT

The aim of this paper is to study the relationships between the concepts of local near uniform smoothness and the properties H and H^*.

1. Introduction

The aim of this paper is to study relationships between the concepts of local near uniform smoothness and the properties H and H^*. These notions play very significant role in some recent trends of the geometric theory of Banach spaces. These trends depend upon the study of classical notions of the geometry of Banach spaces from the view point of compactness conditions (cf. [1,2,7,10,11,16,17,18,19], for example).

Let us mention that such an approach in the geometric theory of Banach spaces was initiated by the papers of Huff [11], Partington [17] and Goebel and Sękowski [10]. In these papers the authors introduced an interesting generalization of the classical Clarkson's notion of uniform convexity in Banach spaces [5]. The generalization of such a type was realized with help of the notion of a measure of noncompactness.

After the papers [10,11,17] there have appeared a lot of papers (cf. the papers cited before) devoted to the study of other notions and properties of Banach spaces which can be formulated with help of compactness conditions. The fairly recent state of this theory is presented in the papers [2,3], for instance.

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The investigations of this paper are continuation of the study from the paper [1], where the concept of local near uniform smoothness was introduced and some properties of this concept were derived.

2. Notation, definitions and auxiliary facts

Let E be a given real Banach space with the norm denoted by $\|\cdot\|_E$ or $\|\cdot\|$ and the zero element θ. The dual space of E will be denoted by E^* and the second dual by E^{**}.

Let $B(x,r)$ denote the closed ball in E centered at x and with radius r. Moreover, we write $B = B_E = B(\theta,1), B^* = B_{E^*}, B^{**} = B_{E^{**}}$. The symbol $S = S_E$ stands for the unit sphere in E while $S^* = S_{E^*}, S^{**} = S_{E^{**}}$.

The canonical embedding from E into E^{**} is denoted by κ.

For a bounded and nonempty subset X of E by the symbol $\alpha(X)$ we will denote the Kuratowski measure of noncompactness of X:

$$\alpha(X) = \inf \big[\varepsilon > 0{:}X \text{ can be covered by a finite family of sets having diameters smaller than } \varepsilon\big] .$$

For the properties of the function α we refer to [4].

In what follows assume that $f \in S^*$ is arbitrary fixed. For $\varepsilon \in [0,1]$ denote by $F(f,\varepsilon)$ the slice defined in the following way

$$F(f,\varepsilon) = \{x \in B{:}\, f(x) \geq 1-\varepsilon\} .$$

Similarly, for $x \in S$ we define the slice $F^*(x,\varepsilon)$ in the space E^* as

$$F^*(x,\varepsilon) = \{f \in B^*{:}\, f(x) \geq 1-\varepsilon\}.$$

Now we recall two definitions which are important for our purposes (cf. [1,16]).

DEFINITION 1. We say that a Banach space E is referred to as locally nearly uniformly convex (LNUC) if $\lim\limits_{\varepsilon\to 0} \alpha(F(f,\varepsilon)) = 0$ for every $f \in S^*$.

In other words, E is LNUC if for any $f \in S^*$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\alpha\big(F(f,\delta)\big) \leq \varepsilon.$$

It is worthwhile to mention that E is LNUC if any only if the norm $\|\cdot\|_E$ has the so-called drop property [16, 18].

DEFINITION 2. A Banach space E is called locally nearly uniformly smooth (LNUS) if for any $x \in S$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\alpha\big(F^*(x,\delta)\big) \le \varepsilon.$$

Thus, E is LNUS if and only if $\lim_{\varepsilon \to 0} \alpha(F^*(x,\varepsilon)) = 0$ for every $x \in S$.

The basic relationship between the concepts of LNUC and LNUS spaces was established in [1]. It is contained in the theorem given below.

Theorem 1

A Banach space is LNUC if and only if E^* is LNUS.

Observe that in the light of Corollary 2 from [3], the proof of this theorem given in [1] is entirely correct. Thus Remark formulated in [2] is not true.

3. Main results

The following two definitions will be essential for our further considerations.

DEFINITION 3 [6,8]. We say that the norm $\|\cdot\|$ in a Banach space E has property H (this property is also known as the Kadec-Klee property or the Radon-Riesz property) whenever for any sequence (x_n) in E converging weakly to some $x \in E$ with $\lim_{n\to\infty} \|x_n\| = \|x\|$ we have that (x_n) converges to x in norm.

The dual property to H is formulated in the next definition which is taken from [1].

DEFINITION 4. We say that the norm $\|\cdot\|$ in a Banach space E has property H^* whenever for any sequence $(f_n) \subset E^*$ converging weakly star to $f \in E^*$ with $\|f_n\|_{E^*} \to \|f\|_{E^*}$ we have that (f_n) converges to f in the norm of E^* (i.e. $\|f_n - f\|_{E^*} \to 0$).

The fundamental result obtained in [1] asserts that under the assumption of reflexivity of a space E we have that E is LNUS if and only if the norm $\|\cdot\|$ in E has the property H^*.

Analyzing the proof of this result given in [1] it has been observed that the implication "if the norm in E has property H^* then E is LNUS" is true without the assumption on reflexivity of E. Unfortunately, this observation was suggested by the false theorem given in the books [13, 15]. That "theorem" says that the unit ball B^* in the space E^* is weakly star sequentially compact (cf. [13], p.3 and [15], p. 8). Obviously, such an assertion is generally not true [9]. The first author would like to thank professor L. Vesely for indicating this error [20].

Now let us notice that the proof given in [1] is correct under an additional assumption. More precisely, we have the following result being improved version of Corollary 1 from [1].

Theorem 2

Let E be a Banach space with the norm having H^* property and such that the ball B^* in E^* is weakly star sequentially compact. Then E is LNUS.

Let us point out some classes of spaces having the dual ball weakly star sequentially compact.

Recall [9] that the Banach space E is said to be *weakly compactly generated* (WCG) whenever there exists a weakly compact set $K \subset E$ such that the linear span of K is dense in E.

For example, all separable or reflexive Banach spaces are WCG [9].

It can be shown [9] that if E is WCG Banach space then B^* is weakly star sequentially compact. Thus the spaces c_0, c and l^1 have the dual ball weakly star sequentially compact. On the other hand the space l^∞ has no longer this property [9].

In order to illustrate our considerations let us pay attention to the below given examples.

EXAMPLE 1: It was shown in [2] that the space c_0 is NUS so it is obviously LNUS. On the other hand, using the same argumentation as in [2] we can easily seen that c is not LNUS.

This means, in the light of Theorem 2, that c does not have property H^*. Hence we can infer that c_0 does not have property H^*.

Indeed, suppose the contrary. Then, for any sequence $(f_n) \subset S_{(c_0)^*} = S_{l^1}$ and for $f \in S_{l^1}$ such that (f_n) is weakly star convergent to f i.e.

$$f_n(x) \to f(x) \quad \text{as } n \to \infty \tag{1}$$

for any $x \in c_0$, we have that $f_n \to f$ in the norm of l^1.
Now, let us assume that

$$f_n(x) \to f(x) \quad \text{as } n \to \infty \tag{2}$$

for every $x \in c$. Since (2) implies (1) this yields that $f_n \to f$ in the norm of l^1. This means that c has the property H^*.

Thus we get a contradiction.

EXAMPLE 2: It is well known that the space l^1 has property H.

But on the other hand this space does not have property H^*. Indeed, let us consider the sequence $(f_n) \subset S_{(l^1)^*} = S_{l^\infty}$ such that

$$f_n = (1, 1, \dots, 1, 0, 1, 1, \dots) \quad (0 \text{ on } n\text{-th place}).$$

Further, let $f = (1, 1, \dots) \in S_{l^\infty}$. Taking an arbitrary $x = (x_1, x_2, \dots) \in l^1$ we have

$$f_n(x) = -x_n + \sum_{k=1}^{\infty} x_k,$$

$$f(x) = \sum_{k=1}^{\infty} x_k.$$

Thus $f_n(x) - f(x) = -x_n \to 0$ as $n \to \infty$. Hence we infer that (f_n) is weakly star convergent to f. On the other hand we have $\|f_n - f\| = 1 (n = 1, 2, \dots)$ in the norm of l^∞. It allows us to deduce that l^1 has not property H^*.

In what follows let us observe that the assertion being partially converse to that from Theorem 2 is no longer true. More precisely, if we assume that E is LNUS Banach space with B^* being weakly star sequentially compact then the norm $\|\cdot\|_E$ is not in general H^*.

In fact, let us take the space c_0. Then in virtue of Example 1 we see that the norm in c_0 has not property H^* although this space is LNUS and $B_{(c_0)^*} = B_{l^1}$ is weakly star sequentially compact.

Thus the assumption on reflexivity in the result quoted immediately after Definition 4, is essential.

Now let us recall [1] that if we assume that the norm in E^* has property H^* then the norm in E has property H. The converse assertion is true under the additional assumption on reflexivity, for example [1].

In the sequel we are going to study some further connections between the properties H and H^*.

We start with the following theorem.

Theorem 3

If the norm in E has property H^* then the norm in E^* has property H.

Proof. Take a sequence $(f_n) \subset S^*$ and $f \in S^*$ such that $f_n \to f$ weakly. Then $f_n \to f$ weakly star what, in view of our assumption, allows us to conclude that $f_n \to f$ in the norm of E^*. □

Observe that the converse theorem is not true. In fact, putting $E = c_0$ we see (cf. Example 1) that the norm in E^* has property H but the norm in E has not property H^*.

Nevertheless, we can prove that at least partially converse assertions to that from Theorem 3 are valid.

Theorem 4

Let E be a Grothendieck space such that the norm in E^* has property H. Then the norm in E has property H^*.

Let us recall [9] that a Banach space E is called a Grothendieck space whenever the weak star and weak convergence of sequences in E^* are the same.

Proof. Let us take an arbitrary sequence $(f_n) \subset S^*$ and $f \in S^*$ such that $f_n \to f$ weakly star. Then, by the assumption, the sequence (f_n) converges weakly to f. Since the norm in E^* has property H we can infer that (f_n) converges to f in norm and the proof is complete. □

Because every reflexive Banach space is Grothendieck, thus as an immediate consequence of the above result we derive the following theorem.

Theorem 5

Assume that E is a reflexive Banach space and that the norm in E^* has property H. Then the norm in E has property H^*.

Our next results give the criterion for the existence of a predual space.

Theorem 6

Assume that E is LNUS space. Then E has a predual space if and only if E is reflexive.

Proof. Assume that E has a predual space $F, F^* = E$. Then, by Theorem 1 we have that F is LNUC. Hence, by a result from [16] we deduce that F is a reflexive space. The converse implication is obvious. □

As an immediate consequence of the above theorem we obtain the following corollary.

Corollary 1

The spaces l^1 and l^∞ are not LNUS.

Observe now that from Theorem 1 and 2 we can obtain the following criterion for the space E to be LNUC.

Theorem 7

E is LNUC if and only if the norm in E^* has property H^* and the ball B^{**} is weakly star sequentially compact in E^{**}.

Further on we shall use the result due to Klee [12]. We express this result in the terminology accepted before.

Lemma 1

Let E be a Banach space with a norm $\|\cdot\|$ such that E^* is separable. Then there exists a norm $\|\cdot\|_1$ equivalent to the norm $\|\cdot\|$ which has the property H^*.

We have mentioned before that every separable Banach space has the property that B^* is weakly star sequentially compact. Keeping in mind this result and Theorem 2 we can derive the following theorem.

Theorem 8

Let E be a Banach space with a norm $\|\cdot\|$. Assume that E^* is separable. Then there is a norm $\|\cdot\|_1$ equivalent to the norm $\|\cdot\|$ such that the space $(E, \|\cdot\|_1)$ is LNUS.

4. Remarks concerning product spaces

In this section we are going to discuss briefly the properties introduced before in the so-called l^p product of a sequence of Banach spaces.

Assume that $(E_i, \|\ \|_i)$ $(i = 1, 2, \dots)$ is a sequence of Banach spaces. Fix a number $p \in (1, \infty)$ and consider the set $l^p(E_i) = l^p(E_1, E_2, \dots)$ consisting of all sequences $x = (x_1), x_i \in E_i$ for $i = 1, 2, \dots$, such that

$$\sum_{i=1}^{\infty} \|x_i\|_i^p < \infty.$$

It is well-known that $l^p(E_i)$ forms a Banach space with respect to the norm

$$\|x\|_p = \|(x_i)\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|_i^p \right)^{\frac{1}{p}}.$$

In the paper [14] it is shown that if the space E has property H then the product space $l^p(E, E, \dots)$ has also this property. It is not difficult to see that the same argumentation may be used to show that the space $l^p(E_i) = l^p(E_1, E_2, \dots)$ has property H whenever every space E_i has this property.

Further, let us notice that with the help of a similar reasoning we can prove the following theorem.

Theorem 9

Assume that the norm $\| \|_i$ in the space E_i has the property H^* for all $i = 1, 2, \ldots$ Then the space $l^p(E_i)$ has also the property H^*.

Moreover, we can infer also the following assertion.

Theorem 10

Let (E_i) be a sequence of Banach spaces such that the ball B_i^* in E_i^* is weakly star sequentially compact for any $i = 1, 2, \ldots$ Then the ball B_p^* in the space $(l^p(E_i))^*$ is weakly star sequentially compact.

Proof. Let us take an arbitrary sequence $(x_n) \subset B_p^*$. Since $(l^p(E_i))^* = l^q(E_i^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$ (cf. [14]), we obtain that $B_p^* = B_{l^q(E_i^*)}$ i.e. B_p^* is the unit ball in the space $l^q(E_i^*)$.
Next, let us represent any term x_n of our sequence in the form

$$x_n = (x_1^n, x_2^n, x_3^n, \ldots) = (x_k^n) \quad (k = 1, 2, \ldots).$$

Observe that for any fixed k the sequence $(x_k^n) = (x_k^1, x_k^2, x_k^3, \ldots)$ is contained in the ball B_k^*. By the assumption we infer that there exists a subsequence of the sequence (x_k^n) which is weakly star convergent to an element $x_k \in E_k^*$. Hence, applying the standard diagonal procedure we can select a subsequence (x_{j_n}) of the sequence (x_n) having the following property:

If we denote

$$x_{x_{j_n}} = (x_1^{j_n}, x_2^{j_n}, x_3^{j_n}, \ldots) = (x_k^{j_n}) \qquad (k = 1, 2, \ldots)$$

and if we fix arbitrarily $k \in \mathbb{N}$ then the sequence $(x_k^{j_n})$ $(n = 1, 2, \ldots)$ is a subsequence of (x_k^n) which is weakly star convergent to the element $x_k \in B_k^*$.
On the other hand we have that (x_{j_n}) is contained in the ball $B_p^* = B_{l^q(E_i^*)}$. Hence, applying a result from [14] we can deduce that the subsequence (x_{j_n}) is weakly star convergent to the element $x = (x_1, x_2, x_3, \ldots)$. In view of the weak star lower-semicontinuity of the norm [8] we infer that $x \in B_p^*$.

Thus the proof is complete. □

In what follows let us observe that taking into account Theorems 2, 9 and 10 we can derive the following corollary.

Corollary 2

Assume that $(E_i, \| \ \|_i)$ is a sequence of Banach spaces such that $\| \ \|_i$ has property H^* and the ball $B_{E_i^*}$ is weakly star sequentially compact (for all $i = 1, 2, \ldots$). Then the product space $l^p(E_i)$ $(1 < p < \infty)$ is LNUS.

Let us recall that the conclusion of the above corollary was obtained in [3] under the assumption that E_i is reflexive and LNUS for $i = 1, 2, \ldots$. But then we infer (in view of the result from [1] quoted before) that the norm $\| \ \|_i$ has property H^* $(i = 1, 2, \ldots)$. Moreover, reflexivity of E_i implies that the ball $B_{E_i^*}$ is weakly star sequentially compact.

Thus we conclude that the result from [3] is a particular case of Corollary 2.

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