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Parametrization and Schur Algorithm for the Integral Representation of Hankel Forms in $\mathbb{T}^2$

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Abstract

The parametrization problem of the minimal unitary extensions of an isometric operator allows its application, through the spectral theorem, to the case of the Fourier representations of a bounded Hankel form with respect to the norms $\left(\int |f|^2 d\mu_1\right)^{1/2}$ and $\left(\int |f|^2 d\mu_2\right)^{1/2}$ where $\mu_1, \mu_2 \geq 0$ are finite measures in $\mathbb{T} \sim [0, 2\pi)$. In this work we develop a similar procedure for the two-parametric case, where $\mu_1, \mu_2 \geq 0$ are measures defined in $\mathbb{T}^2 \sim [0, 2\pi) \times [0, 2\pi)$. With this purpose, we define the generalized Toeplitz forms on the space of two-variable trigonometric polynomials and use the lifting existence theorems of Cotlar and Sadosky. We provide a parametrization formula which is also valid in the special case of the Nehari problem and gives rise to a Schur-type algorithm for this problem.

Introduction

In the theory of trigonometric moments, the problem of characterizing the finite and non-negative in $\mathbb{T} \sim [0, 2\pi)$ measures μ whose Fourier coefficients $\widehat{\mu}$ defined by $\widehat{\mu}(n) = \int_0^{2\pi} \exp(-int)\, d\mu(t)$ are given by any sequence $\{s_n\}_{n\geq 0}$, i.e. $\widehat{\mu}(n) = s_n$, for all $n \geq 0$, is a basic representation problem.

The Herglotz-Bochner theorem provides the answer for this problem; it says that a necessary and sufficient condition for the existence of a solution μ is that the

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Toeplitz kernel $K : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ defined by $K(m,n) = s_{m-n}$ is positive definite (we write $s_{-n} = \overline{s}_n$ whenever necessary). Moreover, if a solution exists, it is unique.

There are some variants and generalizations of this problem, such as the Nehari problem [10] and the generalized Bochner problem, whose solutions are not unique and we can state the problem of their parametrization, i.e., to provide a formula for obtaining all the solutions.

In this way the theorem due to Adamjan, Arov and Krein parametrizes the set of all the solutions in the Nehari problem and the results of Arocena and Katznelson parametrize with differents techniques, the general case of Cotlar and Sadosky. In [1] we give another constructive way for the parametrization.

The Nehari problem can be extended to the matricial cases; it was first solved by Adamjan, Arov and Krein [2]. They obtained a parametrization of the solution set in terms of a linear fractional transform. A more general version was considered by Fritzsche and Kirstein [7] and their approach also gives rise to the study of the generalized Toeplitz kernels. Katznelson [8] gave a parametrization for the matricial generalized Bochner theorem by means of the so-called Potapov's fundamental matricial inequality.

Another way in which the Nehari problem can be generalized is the two-parametric case, where the sequences are double and the associated kernels are defined in $\mathbb{Z}^2 \times \mathbb{Z}^2$. Recently, Cotlar and Sadosky [4] have proved the two-parametric version of the generalized Bochner theorem but they have not considered the parametrization problem.

In this work, and following the constructive method started in [1], we give a parametrization formula for the generalized Bochner theorem in both matricial and two-parametric cases. Moreover we build a Schur-type algorithm to solve the Nehari problem generating the solutions through solving problems in which only one coefficient is not zero. To this end, we combine the method used in the scalar case with other parametrization formulas for restricted interpolation problems considered by Dym, de Branges, Fedchina and others (see [6]). The relations among these methods, the ones developed by Adamjan, Arov and Krein, Fritzsche and Kirstein, Dym, Rovnyak and de Branges and the more recent one by Katznelson inspired in the theory of Potapov are not studied here but they would be interesting.

In this paper the following definitions and results will be useful.

If V is a linear space, W_1, W_2 two linear subspaces of V and $\sigma, \tau : V \to V$ two linear isomorphisms satisfying *i)* $\sigma W_1 \subset W_1$, $\sigma^{-1} W_2 \subset W_2$, *ii)* $\tau W_1 \subset W_1$, $\tau^{-1} W_2 \subset W_2$, and *iii)* $\sigma\tau = \tau\sigma$, then the set $(V, W_1, W_2, \sigma, \tau)$ is called a discrete two-parametric *algebraic scattering system*, which we abreviate a.s.s. In particular, if $\sigma = I$, we have a discrete one-parametric a.s.s.

In an a.s.s. $(V, W_1, W_2, \sigma, \tau)$, a sesquilinear form $B: V \times V \to \mathbb{C}$ is called *Toeplitz* if $B(\sigma f, \sigma g) = B(\tau f, \tau g) = B(f,g)$, $\forall (f,g) \in V \times V$. A sesquilinear form $B_0: W_1 \times W_2 \to \mathbb{C}$ is called *Hankel* if $B_0(\sigma f, g) = B_0(f, \sigma^{-1} g)$, $B_0(\tau f, g) = B_0(f, \tau^{-1} g)$, for all $(f,g) \in W_1 \times W_2$. So the restrictions of Toeplitz forms to $W_1 \times W_2$ are Hankel.

If B_0, B_1, B_2 are Toeplitz forms, we say that B_0 is *bounded* by B_1 and B_2, and write $B_0 \leq (B_1, B_2)$, if $B_1, B_2 \geq 0$ and $|B_0(f,g)|^2 \leq B_1(f,f) \cdot B_2(g,g)$, $\forall (f,g) \in V \times V$. If this inequality is valid for $(f,g) \in W_1 \times W_2$, we say that B_0 is *weakly bounded* with respect to B_1 and B_2, and write $B_0 \prec (B_1, B_2)$.

If $B_0 \prec (B_1, B_2)$, we define the matrix $(B_{\alpha\beta})_{\alpha,\beta=1,2}$ where $B_{\alpha\alpha} = B_\alpha$ $(\alpha = 1,2)$, $B_{12} = B_0$, $B_{21} = B_0^*$, and say that the form $B: V \times V \to \mathbb{C}$ such that $B(f,g) = B_{\alpha\beta}(f,g)$, for $(f,g) \in W_\alpha \times W_\beta$ $(\alpha, \beta = 1,2)$, is a *generalized Toeplitz form*, GTF.

The general lifting theorem due to Cotlar and Sadosky states that, given an a.s.s. $(V, W_1, W_2, \tau, \sigma)$ such that, for $j = 1$ or 2, $\tau^n W_j \subset W_j$ or $\sigma^n W_j \subset W_j$, $\forall n \in \mathbb{Z}$, if B_0, B_1, B_2 are three Toeplitz forms in $V \times V$ and $B_0 \prec (B_1, B_2)$, then there exists a Toeplitz form $B_0' : V \times V \to \mathbb{C}$ such that $B_0' \leq (B_1, B_2)$ and $B_0'|_{W_1 \times W_2} = B_0$.

In the classical example where V is the space of trigonometric polynomials in two variables, a parametrization of all liftings of a GTF can be obtained. To do so, in section 1 a parametrization formula for the unitary extensions of a special class of isometric operators is provided. This formula is applied in section 2 to parametrize the positive liftings of a operator valued weakly positive measure matrix. In section 3 a Schur-type algorithm for the reduced matricial Nehari problem is provided. This procedure and the parametrization formula for the two-parametric lifting theorem deduced in section 4 are used in section 5 for the algorithm of the two-parametric case of this problem.

1. Characterization of the Unitary Extensions of a Class of Isometric Operators

In this section we consider the following class of isometric operators. Let $\mathcal{H}$ be an abstract Hilbert space, $U: \mathcal{H} \to \mathcal{H}$ be an isometric operator with domain $\mathcal{D}$ and range Δ which are closed subspaces of $\mathcal{H}$ and let $\mathcal{M} = \mathcal{H} \ominus \mathcal{D}$, $\mathcal{N} = \mathcal{H} \ominus \Delta$ denote the deficiency subspaces of $\mathcal{H}$. Suppose that there exists $\{e_{0j}\}_{1 \leq j \leq n} \in \mathcal{D}$, $\{e_{-1,i}\}_{1 \leq i \leq m} \subset \Delta$ such that

(1) $\quad U^k e_{0j} \in \mathcal{D}, \quad 1 \leq j \leq n, \qquad U^{-k} e_{-1,i} \in \Delta, \quad 1 \leq i \leq m, \qquad \forall k \geq 0,$

and $\mathcal{H}$ is spanned by the sets

(2) $\quad \left\{ U^k e_{0j} : k \geq 0,\ 1 \leq j \leq n \right\}, \qquad \left\{ U^k e_{-1,i} : k \leq 0,\ 1 \leq i \leq m \right\}.$

This implies that the deficiency indices of U are $\dim \mathcal{M} \leq m$, $\dim \mathcal{N} \leq n$. We consider here the case that $\dim \mathcal{M} = m$, $\dim \mathcal{N} = n$ and $m, n \neq 0$ (thus, U has infinite unitary extensions and $e_{0j} \notin \Delta$, $e_{-1,i} \notin \mathcal{D}$, for any i, j).

A similar result to the one obtained in the scalar case is the following.

Proposition 1.1

Under the hypotheses (1) and (2), every minimal unitary extension $\widetilde{U}: \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ of $U: \mathcal{H} \to \mathcal{H}$ is uniquely determined (up to unitary equivalences) by the matrix

$$\text{(3)} \qquad \left[\langle R_z e_{-1,i}, e_{0j}\rangle\right]_{1 \leq i \leq m;\, 1 \leq j \leq n} \qquad \textit{with } |z| < 1,$$

where R_z is the generalized resolvent of U.

Proof. Each extension $\widetilde{U}$ is determined by $\{\langle \widetilde{U}^p f, g\rangle : f, g \in \mathcal{H},\, p \in \mathbb{Z}\}$. By virtue of (2), it is enough to know the sets $A_1 = \{\langle \widetilde{U}^p e_{0j}, e_{0k}\rangle : 1 \leq j, k \leq n,\, p \in \mathbb{Z}\}$, $A_2 = \{\langle \widetilde{U}^p e_{0j}, e_{-1,i}\rangle : 1 \leq j \leq n,\, 1 \leq i \leq m,\, p \in \mathbb{Z}\}$, $A_3 = \{\langle \widetilde{U}^p e_{-1,i}, e_{0j}\rangle : 1 \leq i \leq m,\, 1 \leq j \leq n,\, p \in \mathbb{Z}\}$, $A_4 = \{\langle \widetilde{U}^p e_{-1,i}, e_{-1,k}\rangle : 1 \leq i, k \leq m,\, p \in \mathbb{Z}\}$.

Since, for $p \geq 0$, $U^p e_{0j} \in \mathcal{D}$, $U^{-p} e_{-1,i} \in \mathcal{D}$ and $\widetilde{U}^{-1} = \widetilde{U}^*$, we have $\widetilde{U}^p e_{0j} = U^p e_{0j}$ and $\widetilde{U}^{-p} e_{-1,i} = U^{-p} e_{-1,i}$, for $p \geq 0$. Then A_1 and A_4 are determined uniquely by U, and $A_2 = \overline{A_3}$. Thus, all the extensions $\widetilde{U}$ are obtained from $\{\langle \widetilde{U}^p e_{-1,i}, e_{0j}\rangle : 1 \leq i \leq m,\, 1 \leq j \leq n,\, p \geq 0\}$.

Remembering that the generalized resolvent of U is, for $|z| < 1$,

$$R_z = P_{\mathcal{H}} \widetilde{R}_z = P_{\mathcal{H}} \sum_{p \geq 0} z^p \widetilde{U}^p = \sum_{p \geq 0} z^p P_{\mathcal{H}} \widetilde{U}^p,$$

where $\widetilde{R}_z$ is the resolvent of $\widetilde{U}$, and $P_{\mathcal{H}}$ the projection of $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$, it can be deduced that $\widetilde{U}$ is determined by the matrix (3). □

In order to obtain a parametrization formula for the matrix (3) in terms of two matricial sequences $\{C^k\} \equiv \{c^k_{ir}\}_{k \geq 0}$ and $\{D^k\} \equiv \{d^k_{ir}\}_{k \geq 0}$ and a polynomial family $\{P^k(\Phi)\} \equiv \{P^k_1(\Phi), \ldots, P^k_m(\Phi)\}^T$, for $k \geq 0$, with m components in $m \times n$ variables, we will use the characterization of all generalized resolvents R_z of any isometric operator U due to Chumakin [5]. The formula he obtained is

$$\text{(4)} \qquad R_z = (I - zT_z)^{-1}, \qquad \text{for } |z| < 1,$$

where $T_z = U \oplus \Phi_z$ and Φ_z is an analytic function and, for each z, $\Phi_z: \mathcal{M} \to \mathcal{N}$ is a contractive operator.

Taking into account that $\dim \mathcal{M} = m$, $\dim \mathcal{N} = n$, we can consider the orthonormal bases $\{u_1, \dots, u_m\}$, $\{u_{01}, \dots, u_{0n}\}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively. Since $\mathcal{H} = \mathcal{D} \oplus \mathcal{M}$, we can write

$$e_{-1,i} = \sum_{j=1}^{m} c_{ij}^{0} u_j + v_{0i} : v_{0i} \in \mathcal{D}, \qquad 1 \le i \le m;$$
$$u_{0j} = \sum_{i=1}^{m} d_{ji}^{0} u_i + w_{0j} : w_{0j} \in \mathcal{D}, \qquad 1 \le j \le n.$$

If we denote by $\mathbf{u_0} = (u_{01}, \dots, u_{0n})^T$, $\mathbf{u} = (u_1, \dots, u_m)^T$, $\mathbf{e_{-1}} = (e_{-1,1}, \dots, e_{-1,m})^T$, $\mathbf{v_0} = (v_{01}, \dots, v_{0m})^T$, $\mathbf{w_0} = (w_{01}, \dots, w_{0n})^T$, the previous identities can be expressed in a matricial form by:

$$\mathbf{e_{-1}} = C^0 \mathbf{u} + \mathbf{v_0}; \qquad C^0 \in \mathbb{C}_{m \times m},$$
$$\mathbf{u_0} = D^0 \mathbf{u} + \mathbf{w_0}; \qquad D^0 \in \mathbb{C}_{n \times m}.$$

By recurrence, we define the matricial sequences $\{C^k\}_{k \ge 0}$ and $\{D^k\}_{k \ge 0}$ and the vectorial sequences $\mathbf{v_k} = (v_{k1}, \dots, v_{km})^T, \mathbf{w_k} = (w_{k1}, \dots, w_{kn})^T$ as follows:

(5)
$$U\mathbf{v_k} = C^{k+1}\mathbf{u} + \mathbf{v_{k+1}}, \qquad C^{k+1} \in \mathbb{C}_{m \times m},$$
$$U\mathbf{w_k} = D^{k+1}\mathbf{u} + \mathbf{w_{k+1}}, \qquad D^{k+1} \in \mathbb{C}_{n \times m}.$$

On the other hand, since $\Phi_z : \mathcal{M} \to \mathcal{N}$ has norm less than or equal to one, we can write:

$$\Phi_z \mathbf{u} = \begin{pmatrix} \Phi_z u_1 \\ \vdots \\ \Phi_z u_m \end{pmatrix} = \begin{pmatrix} \varphi_{11}(z) & \dots & \varphi_{1n}(z) \\ \dots & \dots & \dots \\ \varphi_{m1}(z) & \dots & \varphi_{mn}(z) \end{pmatrix} \cdot \begin{pmatrix} u_{01} \\ \vdots \\ u_{0n} \end{pmatrix} \equiv \Phi(z)\, \mathbf{u_0},$$

where, for any $z \in \mathbb{D}$, $\Phi(z) \in \mathbb{C}_{m \times n}$ and $\|\Phi(z)\| \le 1$.

The next polynomial family $\{\mathbf{P_i^k}\}_{k \ge 0}$, for $1 \le i \le m$, can also be defined:

(6)
$$\begin{cases} \mathbf{P_i^0}(\Phi) = (c_{i1}^0, \dots, c_{im}^0) \equiv \mathbf{c_i^0}, \\ \mathbf{P_i^k}(\Phi) = \sum_{r=1}^{k} \mathbf{P_i^{r-1}}(\Phi)\Phi D^{k-r} + \mathbf{c_i^k}, \quad \text{if } k \ge 1, \end{cases}$$

where $\mathbf{c_i^k}$ is the i-th row of C^k.

Hence the following parametrization formula for the generalized resolvent holds.

Theorem 1.2

If U is an isometric operator that satisfies the hypotheses (1) and (2) the (i,j)-term of the matrix (3) is parametrized by

$$(7) \qquad \langle R_z e_{-1,i}, e_{0j}\rangle = \sum_{p\geq 1} z^p \left(\sum_{k=1}^{p} \mathbf{P}_\mathbf{i}^{\mathbf{k-1}}(\mathbf{\Phi})\Phi\langle \mathbf{w}_{\mathbf{p-k}}, e_{0j}\rangle\right) + \sum_{p\geq 0} z^p \langle v_{pi}, e_{0j}\rangle,$$

where $\|\Phi\| \leq 1$, $\mathbf{P}_\mathbf{i}^{\mathbf{k-1}}$ *is defined in (6),* v_{pi} *is the i-th component of* $\mathbf{v_p}$ *and* $\langle \mathbf{w}_{\mathbf{p-k}}, e_{0j}\rangle = \left(\langle w_{p-k,1}, e_{0j}\rangle, \ldots, \langle w_{p-k,n}, e_{0j}\rangle\right)^T$.

Proof. Applying the formula $T_z = U \oplus \Phi_z$ with $\Phi_z \mathbf{u} = \Phi(z)\mathbf{u_0}$ and $\|\Phi\| \leq 1$, we get

$$\begin{aligned} T_z e_{-1,i} &= \left(U \oplus \Phi_z\right)\left(\sum_{j=1}^{m} c_{ij}^0 u_j + v_{0i}\right) \\ &= U v_{0i} + \sum_{j=1}^{m} c_{ij}^0 \Phi_z u_j \\ &= \mathbf{c_i^1 u} + v_{1i} + \mathbf{c_i^0 \Phi u_0} \\ &= \mathbf{c_i^1 u} + v_{1i} + \mathbf{c_i^0}\Phi(D^0\mathbf{u} + \mathbf{w_0}) \\ &= \mathbf{P_i^1(\Phi)u} + \mathbf{P_i^0(\Phi)\Phi w_0} + v_{1i}. \end{aligned}$$

An inductive procedure shows that

$$T_z^p e_{-1,i} = \mathbf{P_i^p(\Phi)u} + \sum_{k=1}^{p} \mathbf{P_i^{k-1}(\Phi)\Phi w_{p-k}} + v_{pi}.$$

Therefore, applying the formula (4) and taking into account that $\langle u_i, e_{0j}\rangle = 0$, it turns out the desired result. □

Remark. Formula (7) is similar to the one obtained in the particular case of deficiency indices $(1,1)$. and we can also derive a procedure in order to obtain all the positive liftings of an operator-valued measure matrix, without much effort.

2. Parametrization of the Positive Liftings of an Operator-valued Weakly Positive Measure Matrix

In this section, we will apply the previous results to the problem of the positive liftings of an operator-valued weakly positive measure matrix as an extension of the scalar case. To this end, we need to define the corresponding concepts for this subject.

In this section, $\mathcal{N}_1$ and $\mathcal{N}_2$ will be two arbitrary complex euclidean spaces and $\dim \mathcal{N}_1 = n$, $\dim \mathcal{N}_2 = m$; let $\mathcal{P}(\mathcal{N}_1, \mathcal{N}_2) = \{f(t) = \sum_{r=-N}^{N} e_r(t)\widehat{f}(r) : \widehat{f}(r) \in \mathcal{N}_\alpha$ if $r \in \mathbb{Z}_\alpha$, $\alpha = 1, 2$, $t \in \mathbb{T}\}$ denotes the space of the generalized trigonometric polynomials and $\mathcal{P}(\mathcal{N}_1) = \{f(t) \in \mathcal{P}(\mathcal{N}_1, \mathcal{N}_2) : \widehat{f}(r) = 0 \text{ if } r < 0\}$, $\mathcal{P}(\mathcal{N}_2) = \{f(t) \in \mathcal{P}(\mathcal{N}_1, \mathcal{N}_2) : \widehat{f}(r) = 0 \text{ if } r \geq 0\}$ denote the subspaces of the analytic and conjugate analytic generalized trigonometric polynomials, respectively.

So the set $(\mathcal{P}(\mathcal{N}_1, \mathcal{N}_2), \mathcal{P}(\mathcal{N}_1), \mathcal{P}(\mathcal{N}_2), \tau)$, where $\tau f(t) = e^{it} f(t)$ for all $f(t) \in \mathcal{P}(\mathcal{N}_1, \mathcal{N}_2)$, is a discrete 1-parametric a.s.s.

Definition 2.1. A $L(\mathcal{N}_1, \mathcal{N}_2)$-valued measure μ is that one which maps every Borel set Δ in $\mathbb{T}$ over an operator $\mu(\Delta) \in L(\mathcal{N}_1, \mathcal{N}_2)$ such that, for any $\xi \in \mathcal{N}_1$, $\eta \in \mathcal{N}_2$, $\langle \mu(\Delta)\xi, \eta \rangle$ is a complex measure in $\mathbb{T}$.

A $L(\mathcal{N}_1 \oplus \mathcal{N}_2)$-valued measure M can be written as a 2×2 matrix $M = (\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ where $\mu_{\alpha\beta}(\Delta) \in L(\mathcal{N}_\alpha, \mathcal{N}_\beta)$, $(\alpha, \beta = 1, 2)$, and we say that $M = (\mu_{\alpha\beta})$ is *positive*, and write $(\mu_{\alpha\beta}) \geq 0$, if $M(\Delta)$ is a non-negative operator in $L(\mathcal{N}_1 \oplus \mathcal{N}_2)$, i.e.,

$$M(f_1, f_2) \equiv \sum_{\alpha,\beta} \sum_{r,s} \left\langle \mu_{\alpha\beta}(e_{r-s})\widehat{f}_\alpha(r), \widehat{f}_\beta(s) \right\rangle \geq 0, \qquad \forall f_1, f_2 \in \mathcal{P}(\mathcal{N}_1, \mathcal{N}_2).$$

If $M(f_1, f_2) \geq 0$ for each $(f_1, f_2) \in \mathcal{P}(\mathcal{N}_1) \times \mathcal{P}(\mathcal{N}_2)$, M is said to be *weakly positive*, and write $(\mu_{\alpha\beta}) \succ 0$.

In this context, the following lifting theorem holds (see [3]).

Theorem 2.2 (Lifting of weakly positive measure matrix)

Given the matrix $M = (\mu_{\alpha\beta}) \succ 0$, then there exists $M' = (\mu'_{\alpha\beta}) \geq 0$ such that $M(f_1, f_2) = M'(f_1, f_2)$, $\forall (f_1, f_2) \in \mathcal{P}(\mathcal{N}_1) \times \mathcal{P}(\mathcal{N}_2)$.

As a consequence of this we can deduce that there exists $h \in H^1(\mathbb{T})$ such that

$$\begin{aligned} \mu'_{11} &= \mu_{11}, & d\mu'_{12} &= d\mu_{12} + h^* dt \\ d\mu'_{21} &= d\mu_{21} + h dt, & \mu'_{22} &= \mu_{22}. \end{aligned} \tag{8}$$

Now, the problem of parametrizing all the positive liftings of M can be related to the problem of parametrizing the unitary extensions of certain isometric operator, as follows:

Assume that B is the form associated to $M \succ 0$, B defined by

$$B(f,g) = \int_0^{2\pi} f(t)\overline{g(t)}\, d\mu_{\alpha\beta}(t) \qquad \text{for } (f,g) \in \mathcal{P}(\mathcal{N}_\alpha) \times \mathcal{P}(\mathcal{N}_\beta),\ \alpha,\beta = 1,2.$$

This form defines in $\mathcal{P}(\mathcal{N}_1, \mathcal{N}_2)$ an eventually degenerated inner product by $\langle f,g\rangle = B(f,g)$. Thus, we obtain a Hilbert space $\mathcal{H}$ such that $\mathcal{P}(\mathcal{N}_1, \mathcal{N}_2)$, or its quotient, is a dense subspace.

Let $\mathcal{H}_{-1}$ and $\mathcal{H}_0$ denote the closed subspaces of $\mathcal{H}$ spanned by $\{e_r\eta : r \neq -1,\ \eta \in \mathcal{N}_\alpha \text{ if } r \in \mathbb{Z}_\alpha\}$ and $\{e_s\eta : s \neq 0,\ \eta \in \mathcal{N}_\alpha \text{ if } s \in \mathbb{Z}_\alpha\}$ respectively; we define the isometric operator $U{:}\mathcal{H}_{-1} \to \mathcal{H}_0$ by $U(e_r\eta) = e_{r+1}\eta$. Since $\dim \mathcal{N}_1 = n$ and $\dim \mathcal{N}_2 = m$, we can choose two bases $\{\eta_1, \ldots, \eta_n\}$ and $\{\xi_1, \ldots, \xi_m\}$ of $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. We adopt, for reason of simplicity, the notation $e_{0j} = e_0\eta_j$, for $1 \leq j \leq n$, and $e_{-1,i} = e_{-1}\xi_i$, for $1 \leq i \leq m$. The operator U satisfies the hypotheses (1) and (2) because

$$\begin{aligned} U^k e_{0j} &= e_k\eta_j \in \mathcal{H}_{-1}, && \text{for } k \geq 0,\ 1 \leq j \leq n, \\ U^k e_{-1,i} &= e_{-1}\xi_i \in \mathcal{H}_0, && \text{for } k \leq 0,\ 1 \leq i \leq m, \end{aligned}$$

and we say that U is the operator associated with M.

Proposition 2.3

If $M = (\mu_{\alpha\beta})$ is a weakly positive measure matrix, there exists a bijection between the set of all positive liftings of M and the set of the minimal unitary extensions $\widetilde{U}$ of the isometry U associated with M.

Taking into account that

$$\langle e_r\eta_j, e_s\xi_i\rangle = \langle \mu_{\alpha\beta}(e_{r-s})\eta_j, \xi_i\rangle \qquad \text{if } (r,s) \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta,$$

it is plain that μ_{11} and μ_{22} are uniquely determined. However μ_{12} is defined only in $\mathcal{P}(\mathcal{N}_1)$ and μ_{21} in $\mathcal{P}(\mathcal{N}_2)$:

$$\begin{aligned}
\langle \mu_{12}(e_r)\eta_j, \xi_i \rangle &= \langle e_0\eta_j, e_{-r}\xi_i \rangle \\
&= \langle e_0\eta_j, U^{-r+1}e_{-1}\xi_i \rangle = \langle U^{r-1}e_{0j}, e_{-1,i} \rangle, && \text{if } r > 0, \\
\langle \mu_{21}(e_r)\xi_i, \eta_j \rangle &= \langle e_r\xi_i, e_0\eta_j \rangle = \langle U^{r+1}e_{-1,i}, e_{0j} \rangle, && \text{if } r < 0.
\end{aligned}$$

So the set of all minimal unitary extensions of U is determined by means of $\langle \widetilde{U}^{r+1}e_{-1,i}, e_{0j} \rangle$, $1 \leq i \leq m$, $1 \leq j \leq n$, $r \geq 0$, and this is equivalent to the determination of $\langle \mu'_{21}(e_r)\xi_i, \eta_j \rangle$, for $r \geq 0$.

By the same way as the scalar case, if $\{\widetilde{E}_t\}$ is the spectral function of $\widetilde{U}$, we can prove that $\langle \mu'_{21}(\Delta)\xi_i, \eta_j \rangle = \langle \widetilde{E}(\Delta)e_{-1,i}, e_{0j} \rangle$. Applying theorems (1.1) and (1.2), the parametrization is given by (7), which can also be determined throughout the Stieltjes transform of μ'_{21} and μ_{21}.

Summing up, we can obtain the following parametrization.

Theorem 2.4

Let $M = (\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ be a $L(\mathcal{N}_1 \oplus \mathcal{N}_2)$-valued weakly positive measure matrix with more than one positive lifting; then there exist two matricial sequences $\{C^k\}$ and $\{D^k\}$ and one polynomial family $\{\mathbf{P}_\mathbf{i}^\mathbf{k}\}$ defined in (5) and (6), respectively, such that all positive liftings $M' = (\mu'_{\alpha\beta})$ of M have the following general form

$$\mu'_{11} = \mu_{11}; \quad \mu'_{22} = \mu_{22}; \quad \mu'_{12} = {\mu'_{21}}^*; \quad \mu'_{21} = \mu_{21} + h\,dt;$$

$$(9) \qquad (h(z))_{i,j} = (h_0(z))_{i,j} + \sum_{p\geq 1} z^p \left(\sum_{k=1}^{p} \mathbf{P}_\mathbf{i}^{\mathbf{k}-1}(\mathbf{\Phi})\Phi\langle \mathbf{w}_{\mathbf{p}-\mathbf{k}}, e_{0j} \rangle \right),$$

where $(h(z))_{i,j}$ are the components of the matricial function $h(z)$, for $1 \leq i \leq m$, $1 \leq j \leq n$, Φ belongs to the unit ball of all the $m \times n$-matrix valued functions of class H^∞, and h_0 is a fixed particular solution.

3. Schur-type Algorithm for the Matricial Nehari Problem

The parametrization formula (9) also gives the general solution for the following matricial Nehari problem and, consequently, for the N-reduced problem associated to it. The same notation as in section 2 will be used here.

Theorem 3.1

Given an arbitrary sequence $s\colon \mathbb{Z} \to L(\mathcal{N}_2, \mathcal{N}_1)$*, the next statements are equivalent:*

i) There exists $F\colon \mathbb{T} \to L(\mathcal{N}_2, \mathcal{N}_1)$*, such that* $I - F(t)F(t)^* \geq 0$, $\widehat{F}(k) = s(k)$*, for* $k < 0$.

ii) The sesquilinear form

$$B(e_m\eta_j, e_n\xi_i) \quad \begin{cases} \langle \delta_{m-n}\eta_j, \xi_i \rangle & \text{for } m, n \text{ of the same sign,} \\ \langle s(m-n)\eta_j, \xi_i \rangle & \text{if } m < 0,\ n \geq 0, \\ \overline{\langle s(n-m)\xi_i, \eta_j \rangle} & \text{if } m \geq 0,\ n < 0, \end{cases}$$

is a generalized Toeplitz form.

We now provide an algorithm for which the general solution could be constructed using only solutions of the 1-reduced problem, and generating in a recurrent form a sequence of associated parameters in the same way as the scalar case.

In the first step, if we fix the matrix $s(-N) = \sigma_0$, all the solutions F are given by:

$$(10) \qquad (z^N F(z))_{i,j} \quad \langle R_z e_{-1,i}, e_{0,j} \rangle \\ - \sum_{p \geq 1} z^p \left(\sum_{k=1}^{p} \mathbf{P}_{\mathbf{i}}^{\mathbf{k-1}}(\Phi)\Phi \langle \mathbf{w}_{\mathbf{p-k}}, e_{0j} \rangle \right) + \sum_{p \geq 0} z^p \langle v_{pi}, e_{0j} \rangle,$$

where Φ belongs to the operator-valued Schur class $\mathcal{S}$, $\mathbf{P}_{\mathbf{i}}^{\mathbf{k-1}}$ depends on $s(-N)$ and $\langle v_{0i}, e_{0j} \rangle = \left(s(-N)\right)_{i,j}$.

For the second step, given $\left\{s(-N), s(-N+1)\right\}$, in order for $\left(s(-N+1)\right)_{i,j}$ to be the coefficient in z of $\left(z^N F(z)\right)_{i,j}$, we have to choose a sub-class $\mathcal{S}_1 \subset \mathcal{S}$. Such coefficient is $\mathbf{c}_{\mathbf{i}}^{\mathbf{0}}\Phi(0)\langle \mathbf{w}_{\mathbf{0}}, e_{0j} \rangle + \langle v_{1i}, e_{0j} \rangle = (s(-N+1))_{i,j}$, so we have to take all the Φ for which $\mathbf{c}_{\mathbf{i}}^{\mathbf{0}}\Phi(0)\langle \mathbf{w}_{\mathbf{0}}, e_{0j} \rangle = (s(-N+1))_{i,j} - \langle v_{1i}, e_{0j} \rangle \equiv \mathbf{c}_{\mathbf{i}}^{\mathbf{0}}\sigma_1 \langle \mathbf{w}_{\mathbf{0}}, e_{0j} \rangle$ is given. Then we define the class

$$\mathcal{S}_1 \quad \left\{ \Phi \in \mathcal{S} : \mathbf{c}_{\mathbf{i}}^{\mathbf{0}}\Phi(0)\langle \mathbf{w}_{\mathbf{0}}, e_{0j} \rangle \quad \mathbf{c}_{\mathbf{i}}^{\mathbf{0}}\sigma_1 \langle \mathbf{w}_{\mathbf{0}}, e_{0j} \rangle,\ \forall i, j \right\}.$$

This class $\mathcal{S}_1$ is completely parametrized by what Dym [6] called a bi-lateral interpolation problem. However, Dym's parametrization does not allow the algorithm to follow. Here, we will substitute that formula by a Schur-type sequence of parameters. To do so, we decompose $\mathcal{S}_1$ in union of classes $\mathcal{S}_{1\alpha}$ defined by

$$\mathcal{S}_{1\alpha} = \left\{\Phi \in \mathcal{S}_1 : \mathbf{c}_i^0 \Phi(0)\langle \mathbf{w}_0, e_{0j}\rangle = \mathbf{c}_i^0 \sigma_{1\alpha}\langle \mathbf{w}_0, e_{0j}\rangle\right\}$$

for each possible value $\sigma_{1\alpha}$.

If for each α we define $\mathcal{S}'_{1\alpha} = \{\Phi_\alpha \in \mathcal{S} : \Phi_\alpha(0) = \sigma_{1\alpha}\}$, it is easy to see that $\mathcal{S}_1 = \bigcup_\alpha \mathcal{S}'_{1\alpha}$. Each $\mathcal{S}'_{1\alpha}$ can be parametrized by the next formula, similar to (10):

$$(11) \qquad (\Phi_\alpha(z))_{i,j} = \sum_{p\geq 1} z^p \left(\sum_{k=1}^{p} \mathbf{P}_\mathbf{i}^{\mathbf{k-1}}(\Psi_\alpha)\Psi_\alpha\langle \mathbf{w}_{\mathbf{p-k}}, e_{0j}\rangle\right) + \sum_{p\geq 0} z^p \langle v_{pi}, e_{0j}\rangle,$$

where $\mathbf{P}_\mathbf{i}^{\mathbf{k-1}}$ depends on $\sigma_{1\alpha}$ and $\Psi_\alpha \in \mathcal{S}$.

Joining all these Φ_α, we obtain the desired class $\mathcal{S}_1$. Thus (10) with $\Phi \in \mathcal{S}_1$ provides all the solutions F for the second step of the algorithm, where $s(-N)$, $s(-N+1)$ are given.

If we repeat this process in the following steps, we obtain a family of parameters $\sigma_0 = s(-N)$, $\sigma_{1\alpha}$, $\sigma_{2\alpha\beta}, \ldots$ Necessary and sufficient conditions for the existence of solutions for the N-reduced problem are given by the Nehari theorem (3.1) and they are that each parameter be the value in zero of a function $\Phi \in S$.

4. Positive Liftings of Two-Parametric Toeplitz Forms

We define the space of two-variable trigonometric polynomials by:

$$V = \left\{f : \mathbb{T}^2 \to \mathbb{C} : f(s,t) = \textstyle\sum_{m,n} \widehat{f}(m,n)\, e_{m,n}(s,t),\ \widehat{f}(m,n) \text{ finitely supported}\right\}$$

where $e_{m,n}(s,t) = e^{ims}e^{int}$.

If we consider the halfplanes $\mathbb{E}_1 = \{m = (m_1, m_2) \in \mathbb{Z}^2 : m_2 \geq 0\}$ and $\mathbb{E}_2 = \{m = (m_1, m_2) \in \mathbb{Z}^2 : m_2 < 0\}$, let W_1, W_2 be the subspaces of V whose coefficients have support in $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively. In V we define the shifts $\sigma f(s,t) = e^{is} f(s,t)$ and $\tau f(s,t) = e^{it} f(s,t)$; so that $(V, W_1, W_2, \sigma, \tau)$ is an a.s.s. and satisfies the Cotlar-Sadosky theorem. Moreover a two-parametric generalized Bochner theorem is also valid.

Theorem 4.1 (Generalized Bochner Theorem) [4]

In the situation above, if B_0, B_1, B_2 are three Toeplitz forms in $V \times V$ and $B_0 \prec (B_1,\ B_2)$, then there exist μ, μ_1, μ_2 finite measures in $\mathbb{T}^2$ such that $\mu_1, \mu_2 \geq 0$, $|\mu(\Delta)|^2 \leq \mu_1(\Delta)\,\mu_2(\Delta)$, $\forall \Delta \subset \mathbb{T}^2$ and

$$B_i(f,g) = \int\!\!\int f\bar{g}\,d\mu_i, \qquad f,g \in V \quad (i = 1,2),$$
$$B_0(f,g) = \int\!\!\int f\bar{g}\,d\mu, \qquad (f,g) \in W_1 \times W_2.$$

Here μ_1, μ_2 are unique but μ is not, and the set of all the solutions can be parametrized through their Stieltjes transforms by the same method as the one-parametric case [1].

A parametrization formula describing all the liftings of the GTF (B_0, B_1, B_2) will be developed next.

In the space $\mathcal{E} = W_1 \times W_2$ we define the following inner product:

$$\Big\langle [f_1, g_1], [f_2, g_2] \Big\rangle = B_1(f_1, f_2) + B_2(g_1, g_2) + B_0(f_1, g_2) + \overline{B_0(f_2, g_1)}$$

Since $B_0 \prec (B_1, B_2)$, then $(\overline{\mathcal{E}}, \langle \cdot, \cdot \rangle)$ is a Hilbert space and we can suppose that $W_1 \sim [W_1, 0]$, $W_2 \sim [0, W_2]$ are closed subspaces of $\overline{\mathcal{E}}$.

If we define $\tau : \overline{\mathcal{E}} \to \overline{\mathcal{E}}$ by $\tau[f,g] = [\tau f, \tau g]$, it results that τ is an isometric operator with domain $\mathcal{D}_\tau = W_1 \times \tau^{-1} W_2$ and range $\Delta_\tau = \tau W_1 \times W_2$.

It is plain that $\{e_{n,0} \sim [e_{n,0}, 0] : n \in \mathbb{Z}\} \subset \mathcal{D}_\tau$, $\tau^k e_{n,0} = e_{n,k} \sim [e_{n,k}, 0] \in \mathcal{D}_\tau$, $(k \geq 0)$, $\{e_{m,-1} \sim [0, e_{m,-1}] : m \in \mathbb{Z}\} \subset \Delta_\tau$ and $\tau^k e_{m,-1} = e_{m,k-1} \sim [0, e_{m,k-1}] \in \Delta_\tau$, $(k \leq 0)$. Moreover $e_{n,0} \notin \Delta_\tau$ and $e_{m,-1} \notin \mathcal{D}_\tau$, $\forall n, m$.

Thus, $\overline{\mathcal{E}}$ is spanned by $[W_1, 0] \sim \{\tau^k e_{n,0},\ k \geq 0, n \in \mathbb{Z}\}$ and $[0, W_2] \sim \{\tau^k e_{m,-1},\ k \leq 0,\ m \in \mathbb{Z}\}$.

On the other hand, $\sigma : \overline{\mathcal{E}} \to \overline{\mathcal{E}}$, defined by $\sigma[f,g] = [\sigma f, \sigma g]$, is a unitary operator.

Our next goal is to describe the m.c.u.e. (minimal commuting unitary extensions) of (σ, τ). We say that (S, T) is a *commuting unitary extension* of (σ, τ) if there exists a Hilbert space $\mathcal{H}$ such that $\overline{\mathcal{E}} \subset \mathcal{H}$, S and T are unitary operators in $\mathcal{H}$, $S|_{\overline{\mathcal{E}}} = \sigma$, $T|_{\overline{\mathcal{E}}} = \tau$ and $ST = TS$. Such an extension is called *minimal* if $\mathcal{H} = \bigvee_{m,n} S^n T^m(\overline{\mathcal{E}})$, too.

A whole description which provides existence and unicity conditions for the minimal commuting unitary extensions (m.c.u.e.) is due to Morán [9]. In particular, a necessary and sufficient condition for the existence of such an extension is that $\langle \sigma^n \tau f, \tau f' \rangle = \langle \sigma^n f, f' \rangle$, $\forall f, f' \in \mathcal{D}_\tau$, $n = 1, 2, \ldots$ and here that condition is trivial, because σ is a unitary operator and $\sigma(\mathcal{D}_\tau) = \mathcal{D}_\tau$, $\sigma(\Delta_\tau) = \Delta_\tau$, $\sigma\tau = \tau\sigma|_{\mathcal{D}_\tau}$.

An analogous characterization to the one of section 1 is the following.

Proposition 4.2

If τ is the operator defined before, the minimal unitary extensions of τ are uniquely determined (up to unitary equivalences) by

$$\left\{\left\langle R_z e_{m,-1}, e_{n,0}\right\rangle\right\}_{m,n\in\mathbb{Z}} \qquad \text{for } |z|<1, \tag{12}$$

where R_z is the generalized resolvent of τ.

By using the Chumakin formula (4), we can express the generalized resolvent R_z of τ as $R_z = \sum_{n\geq 0} z^n T_z^n$, if $|z|<1$ where $T_z = \tau \oplus \Phi_z$ and $\Phi_z : \overline{\mathcal{E}} \ominus \mathcal{D}_\tau \to \overline{\mathcal{E}} \ominus \Delta_\tau$ is, for each z in the unit circle, a contractive operator.

If $\{u_{i,-1}\}_{i\in\mathbb{Z}}$, $\{u_{j,0}\}_{j\in\mathbb{Z}}$ are two orthonormal bases of $\overline{\mathcal{E}} \ominus \mathcal{D}_\tau$ and $\overline{\mathcal{E}} \ominus \Delta_\tau$, respectively, since $\overline{\mathcal{E}} = \mathcal{D}_\tau \oplus (\overline{\mathcal{E}} \ominus \mathcal{D}_\tau)$ we can write:

$$e_{m,-1} = \sum_{i\in\mathbb{Z}} c^0_{mi} u_{i,-1} + v^0_m, \quad u_{j,0} = \sum_{i\in\mathbb{Z}} d^0_{ji} u_{i,-1} + w^0_j \qquad \text{with } v^0_m, w^0_j \in D_\tau$$

and

$$\Phi_z(u_{i,-1}) = \sum_{j\in\mathbb{Z}} \varphi_{i,j}(z) u_{j,0}, \qquad \forall i \in \mathbb{Z}.$$

If we write $\Phi = [\varphi_{ij}]_{i,j\in\mathbb{Z}}$ the associated matrix of the operator Φ_z, then $\|\Phi\| \leq 1$.

We define the sequences $\{v^p_m\}_{p\geq 0} \subset \mathcal{D}_\tau$ and $\{w^p_j\}_{p\geq 0} \subset \mathcal{D}_\tau$ by recurrence as:

$$\begin{aligned} \tau v^p_m &= \sum_{i\in\mathbb{Z}} c^{p+1}_{mi} u_{i,-1} + v^{p+1}_m \qquad (p\geq 0) \\ \tau w^p_j &= \sum_{i\in\mathbb{Z}} d^{p+1}_{ji} u_{i,-1} + v^{p+1}_j \qquad (p\geq 0) \end{aligned} \tag{13}$$

and the polynomial sequence $\{P^p_m\}$:

(14)

$$\begin{cases} P^0_j(\Phi) = (\dots, c^0_{j0}, c^0_{j1}, \dots, c^0_{jn}, \dots) \equiv c^0_j, \\ P^p_j(\Phi) = P^{p-1}_j(\Phi)\Phi D^0 + P^{p-2}_j(\Phi)\Phi D^1 + \dots + P^0_j(\Phi)\Phi D^{p-1} + c^p_j \quad (p \geq 1), \end{cases}$$

where $c^p_j \equiv (\dots, c^p_{j0}, c^p_{j1}, \dots, c^p_{jn}, \dots)$ and $D^k \equiv \left(d^k_{nj}\right)_{n,j\in\mathbb{Z}}$.

So the following general result can be stated:

Theorem 4.3

Let τ, σ be the given operators and R_z the generalized resolvent of τ. From the vectorial sequences $\{P_m^p\}$, $\{v_m^p\}$, $\{w_p\}$, the m.c.u.e. of (σ, τ) are parametrized by the matrix $\{\langle R_z e_{j,-1}, e_{n,0}\rangle;\ j, n \in \mathbb{Z}\}$, with $|z| < 1$, where

$$(15)\quad \langle R_z e_{j,-1}, e_{n,0}\rangle = \sum_{m\geq 1} z^m \left(\sum_{k=1}^{m} P_j^{k-1}(\Phi)\Phi\langle w^{m-k}, e_{n,0}\rangle\right) + \sum_{m\geq 0} z^m \langle v_j^m, e_{n,0}\rangle,$$

and $\Phi = [\varphi_{i,n}]_{i,n\in\mathbb{Z}}$ is the matrix associated to Φ_z such that

$$(16)\qquad \left\{\Phi_z : \overline{\mathcal{E}} \ominus \mathcal{D}_\tau \to \overline{\mathcal{E}} \ominus \Delta_\tau : \|\Phi_z\| \leq 1,\ \sigma\Phi_z = \Phi_z\sigma|_{\overline{\mathcal{E}}\ominus\mathcal{D}_\tau}\right\}.$$

Proof. It is easy to prove by an inductive procedure that

$$T_z^m e_{j,-1} = P_j^m(\Phi)u_{-1} + P_j^{m-1}(\Phi)\Phi w^0 + \ldots + P_j^0(\Phi)\Phi w^{p-1} + v_j^m, \quad \text{for } m \geq 0.$$

Then:

$$\begin{aligned}\langle R_z e_{j,-1}, e_{n,0}\rangle &= \sum_{m\geq 0} z^m \langle T_z^m e_{j,-1}, e_{n,0}\rangle \\ &= \sum_{m\geq 1} z^m \left(\sum_{k=1}^{m} P_j^{k-1}(\Phi)\,\Phi\langle w^{m-k}, e_{n,0}\rangle\right) + \sum_{m\geq 0} z^m \langle v_j^m, e_{n,0}\rangle\end{aligned}$$

In [9] it is shown that, if T is a minimal unitary extension of τ and Φ_z is its associated characteristic function, the set (16) parametrizes the m.c.u.e. of (σ, τ). Thus, the formula (15) also parametrizes the m.c.u.e. of (σ, τ) if Φ_z runs over the contractive operators commuting with σ. □

In order to see how these extensions produce the liftings of the given Toeplitz form, we proceed as follows:

Taking into account that $V = \bigvee_{n\in\mathbb{Z}}\{\tau^n W_1\} = \bigvee_{n\in\mathbb{Z}}\{\tau^n W_2\}$, we define the form $B' : V \times V \to \mathbb{C}$ by $B'(\tau^m w_1, \tau^n w_2) = \langle U^{m-n}[w_1, 0], [0, w_2]\rangle$ where U is a unitary extension of τ that satisfies (16). Thus it is easy to prove that B' is a τ and σ-invariant sesquilinear form and, for each $f, g \in V$, $|B'(f, g)|^2 \leq B_1(f, f)\, B_2(g, g)$. Moreover B' extends to B_0 and is uniquely determinated by U because it is sufficient to compute $B'(\tau^m e_{j,-1}, e_{k,0})$, $j, k \in \mathbb{Z}$ in order to determine the lifting.

Finally, each form B' defines a measure μ' by the formula $B'(f, g) = \int\int f\overline{g}\, d\mu'$.

5. Schur-type Algorithm for the Two-Parametric Nehari Problem

A consequence of the generalized Bochner theorem is the Nehari theorem (see [4]), and the parametrization of all the solutions is also valid in this case. We will now develop a Schur-type algorithm in order to generate the set of all the solutions of the special Nehari problem, in which is given a finitely supported sequence $s: \mathbb{Z}^2 \to \mathbb{C}$ and we want to find a function $F \in L^\infty(\mathbb{T}^2)$, such that $\|F\|_\infty \leq 1$ and $\widehat{F}(m_1, m_2) = s(m_1, m_2)$ for $-N \leq m_2 \leq -1$, but only resolving special problems of first order.

Theorem 5.1 (Two-parametric Nehari)

Given the double sequence $s: \mathbb{Z}^2 \to \mathbb{C}$, the next statements are equivalent:

(a) *There exists a bounded function $F : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ such that $\|F\|_\infty \leq 1$ and $\widehat{F}(m_1, m_2) = s(m_1, m_2)$, for $m_2 < 0$.*

(b) *For any finitely supported sequences a, b with* $\operatorname{supp} a \subset \mathbb{E}_1$, $\operatorname{supp} b \subset \mathbb{E}_2$,

$$\left| \sum_{m,n \in \mathbb{Z}^2} s(m-n)\, a(m)\, \overline{b(n)} \right|^2 \leq \sum_{m \in \mathbb{Z}^2} |a(m)|^2 \sum_{n \in \mathbb{Z}^2} |b(n)|^2.$$

(c) *The sesquilinear form $B: V \times V \to \mathbb{C}$ defined by*

$$B(e_m, e_n) \quad \begin{cases} s(m-n) & \text{if } m \in \mathbb{E}_2,\ n \in \mathbb{E}_1, \\ \overline{s}(n-m) & \text{if } m \in \mathbb{E}_1,\ n \in \mathbb{E}_2, \\ \delta(m-n) & \text{if } m, n \in \mathbb{E}_1 \text{ or } m, n \in \mathbb{E}_2, \end{cases}$$

is a generalized Toeplitz form.

The first step is to find $F \in L^\infty$ such that

$$\widehat{F}(m_1, m_2) = \begin{cases} s(m_1, -1) & \text{if } m_1 \in \mathbb{Z},\ m_2 = -1; \\ 0 & \text{if } m_2 < -1. \end{cases}$$

By theorem (5.1), this problem has a solution if and only if the form

$$B(e_m, e_n) = \begin{cases} 1 & \text{if } m = n \\ s(m_1 - n_1, -1) & \text{if } m_2 = -1,\ n_2 = 0 \\ \overline{s}(n_1 - m_1, -1) & \text{if } m_2 = 0,\ n_2 = -1 \\ 0 & \text{otherwise,} \end{cases}$$

is a generalized Toeplitz form.

In particular, this condition implies that $\sum_{k\in\mathbb{Z}}|s(k,-1)|^2 \leq 1$. Moreover that solution will be unique if $\sum_{k\in\mathbb{Z}}|s(k,-1)|^2 = 1$.

In the sequel we will use the notation $c_0 = \sqrt{1-\sum_k |s(k,-1)|^2}$.

In the space of the two-variable trigonometric polynomials V we define the inner product $\langle e_m, e_n\rangle = B(e_m, e_n)$, and the corresponding Hilbert space $\mathcal{H}$.

The operator $\tau:\mathcal{H}\to\mathcal{H}$ defined by $\tau e_{m_1,m_2} = e_{m_1,m_2+1}$, is an isometry with domain $\mathcal{D}_\tau = \bigvee_{k\in\mathbb{Z}}\{e_{k,j} : j \neq -1\}$ and range $\Delta_\tau = \bigvee_{k\in\mathbb{Z}}\{e_{k,j} : j \neq 0\}$.

Moreover, the operator $\sigma:\mathcal{H}\to\mathcal{H}$ defined by $\sigma e_{m_1,m_2} = e_{m_1+1,m_2}$ is unitary and verifies $\sigma\tau = \tau\sigma|_{\mathcal{D}_\tau}$.

It turns out that the families $\left\{u_{k,-1} \equiv \frac{1}{c_0}\left(e_{k,-1} - \sum_{r\in\mathbb{Z}} s(r,-1)e_{k-r,0},\ k\in\mathbb{Z}\right)\right\}$ and $\left\{u_{k,0} \equiv \frac{1}{c_0}\left(e_{k,0} - \sum_{r\in\mathbb{Z}} s(r,-1)\, e_{k+r,-1},\ k\in\mathbb{Z}\right)\right\}$ are orthonormal bases of $\mathcal{H}\ominus\mathcal{D}_\tau$ and $\mathcal{H}\ominus\Delta_\tau$, respectively.

From the first one, we can decompose $e_{k,-1}$ as a sum of elements in $\mathcal{D}_\tau$ and $\mathcal{H}\ominus\mathcal{D}_\tau$, as follows:

$$e_{k,-1} = c_0\, u_{k,-1} + \sum_{r\in\mathbb{Z}} s(r,-1)\, e_{k-r,0}, \qquad k\in\mathbb{Z}.$$

In order to preserve the notation used in section 4, we call

$$v_k^0 = \sum_{r\in\mathbb{Z}} s(r,-1)e_{k-r,0}$$

and define the sequence $\{v_k^p\}_{p\geq 0}\subset\mathcal{D}_\tau$ by recurrence as:

$$v_k^p = \tau v_k^{p-1} = \tau^p v_k^0 = \sum_{r\in\mathbb{Z}} s(r,-1)e_{k-r,p} \qquad (p>0).$$

It is obvious that $\langle v_k^0, e_{n,0}\rangle = s(k-n,-1)$ and $\langle v_k^p, e_{n,0}\rangle = 0$, if $p>0$.

In the same way we obtain:

$$\begin{aligned} u_{k,0} &= \frac{1}{c_0}\left(e_{k,0} - \sum_{r\in\mathbb{Z}}\overline{s}(r,-1)\right)\left(c_0 u_{k+r,-1} + \sum_{t\in\mathbb{Z}} s(t,-1)e_{k+r-t,0}\right) \\ &= -\sum_{r\in\mathbb{Z}}\overline{s}(r,-1)u_{k+r,-1} + \frac{1}{c_0}\left(e_{k,0} - \sum_{r\in\mathbb{Z}}\sum_{t\in\mathbb{Z}}\overline{s}(r,-1)s(t,-1)e_{k+r-t,0}\right), \end{aligned}$$

and define $\{w_k^p\}_{p\geq 0} \subset \mathcal{D}_\tau$ by:

$$\begin{aligned} w_k^0 &:= \frac{1}{c_0}\Big(e_{k,0} - \sum_{r\in\mathbb{Z}}\sum_{t\in\mathbb{Z}} \overline{s}(r,-1)\, s(t,-1)\, e_{k+r-t,0}\Big), \\ w_k^p &= \tau w_k^{p-1} = \tau^p w_k^0 \\ &= \frac{1}{c_0}\Big(e_{k,p} - \sum_{r\in\mathbb{Z}}\sum_{t\in\mathbb{Z}} \overline{s}(r,-1)\, s(t,-1)\, e_{k+r-t,p}\Big), \qquad (p>0). \end{aligned}$$

Moreover if $\Phi_z: \mathcal{H} \ominus \mathcal{D}_\tau \to \mathcal{H} \ominus \Delta_\tau$ is the characteristic function associated to τ, $\Phi_z(u_{r,-1}) = \sum_{k\in\mathbb{Z}} \varphi_{r,k}(z)\, u_{k,0}$, for $r\in\mathbb{Z}$, we call $\Phi(z) := (\varphi_{rk}(z))_{r,k\in\mathbb{Z}}$ and, as is well known, $\|\Phi(z)\| \leq 1$. We also define the polynomial family in $\Phi(z)$:

$$\begin{cases} P_k^0(\Phi) = (\dots,0,\dots,0,c_0,0,\dots) \equiv \mathbf{c}_k^0, \\ P_k^p(\Phi) = P_k^{p-1}(\Phi)\Phi D^0 = \mathbf{c}_k^0(\Phi D^0)^p, \qquad (p\geq 1). \end{cases}$$

By theorem (4.3), the m.c.u.e. of (σ,τ) are parametrized by (15) which takes the particular form:

$$\begin{aligned} \langle R_\tau(z)e_{k,-1}, e_{n,0}\rangle &= \sum_{m\geq 1} z^m P_k^{m-1}(\Phi)\,\Phi\langle w^0, e_{n,0}\rangle + s(k-n,-1) \\ &= \sum_{m\geq 1} z^m c_k^0\,\Phi(D^0\Phi)^{m-1}\langle w^0, e_{n,0}\rangle + s(k-n,-1), \end{aligned}$$

for $|z|<1$, where Φ satisfies (16) and $c_k^0\Phi = c_0(\dots,\varphi_{k0},\varphi_{k1},\dots,\varphi_{kn},\dots)$,

$$\langle w^0, e_{n,0}\rangle = \frac{-1}{c_0}\Big(\dots, \textstyle\sum_r \overline{s}(r,-1)\, s(r-n,-1), \sum_r \overline{s}(r,-1) s(r-n+1,-1),\dots\Big)^T.$$

By (16) $\langle R_\tau(z)e_{k,-1}, e_{n,0}\rangle = \langle R_\tau(z)\sigma^k e_{0,-1}, \sigma^n e_{0,0}\rangle = \langle R_\tau(z)e_{0,-1}, \sigma^{n-k}e_{0,0}\rangle$; hence, the matrix $(\langle R_\tau(z)e_{0,-1}, \sigma^p e_{0,0}\rangle)_{p\in\mathbb{Z}}$ where

$$\langle R_\tau(z)e_{0,-1}, \sigma^p e_{0,0}\rangle = \sum_{m\geq 1} z^m c_0^0\,\Phi(D^0\Phi)^{m-1}\langle w^0, e_{p,0}\rangle + s(-p,-1), \qquad p\in\mathbb{Z},$$

parametrizes the m.c.u.e. of (σ,τ).

If $R_\sigma(y) = (I - y\sigma)^{-1} = \sum_{j\geq 0} y^j\sigma^j$, for $|y|<1$, is the resolvent of σ, we can also write the previous parametrization, for $p\geq 0$ by:

(17a) $\quad \langle R_\tau(z)e_{0,-1}, R_\sigma(y)e_{0,0}\rangle$

$$= \sum_{p\geq 0} s(-p,-1)\,\overline{y}^p + \sum_{p\geq 0}\sum_{m\geq 1} \overline{y}^p z^m c_0^0\,\Phi(D^0\Phi)^{m-1}\langle w^0, e_{p,0}\rangle,$$

and, for $p < 0$:

(17b) $$\begin{aligned}\langle R_\tau(z)e_{0,-1}, R_\sigma^{-1}(y)e_{0,0}\rangle \\ = \sum_{p\geq 0} s(p,-1)\,\overline{y}^p + \sum_{p\geq 0}\sum_{m\geq 1} \overline{y}^p z^m c_0^0\,\Phi(D^0\Phi)^{m-1}\langle w^0, e_{-p,0}\rangle,\end{aligned}$$

where $|y| < 1$, $|z| < 1$.

Returning to the original problem of finding $F \in L^\infty(\mathbb{T}^2)$ such that $\|F\|_\infty \leq 1$ and $\widehat{F}(k,-1) = s(k,-1), k \in \mathbb{Z}$, the solution must have the form

$$F(y,z) = \sum_{k\in\mathbb{Z}} y^k z^{-1} s(k,-1) + \text{(z-analytic part)}.$$

This problem can also be stated as: *Given* $f(y,z) = \sum_{k\in\mathbb{Z}} y^k z^{-1} s(k,-1)$, *find a z-analytic function such that* $\|f+h\|_\infty \leq 1$, or as a measure matrix lifting theorem: *Given the weakly positive measure matrix* $(\mu_{\alpha\beta})$, *where* $\mu_{11} = \mu_{22} =$ *measure Lebesgue in* $\mathbb{T}^2$, $d\mu_{21}(s,t) = f(s,t)ds\,dt$, $\mu_{12} = \overline{\mu}_{21}$, *find a positive matrix* $(\mu'_{\alpha\beta})$, *such that* $d\mu'_{11} = d\mu'_{22} = ds\,dt$, $\widehat{\mu}'_{21}(m_1,m_2) = \widehat{\mu}_{21}(m_1,m_2)$, *if* $m_2 < 0$.

By the general lifting theorem, there exists a function $h(s,t)$ analytic in t, such that $d\mu'_{21} = d\mu_{21} + h(s,t)\,ds\,dt$.

In the same way as the one-parametric case, the positive lifting of $(\mu_{\alpha\beta})$ is determined knowing $\widehat{\mu}'_{21}(e_{k,j})$ for $(k,j) \in \mathbb{E}_1$ and a result equivalent of the proposition (2.3) is true.

Proposition 5.2

The positive liftings of the matrix $(\mu_{\alpha\beta})$ *are determined by means of the m.c.u.e. of* (σ,τ).

Proof. Let (S,T) be a m.c.u.e. of (σ,τ), and let $\{E_s : 0 \leq s \leq 2\pi\}$, $\{F_t : 0 \leq t \leq 2\pi\}$ be the spectral measures associated to S and T, respectively. We construct the matrix of numerical measures:

$$\begin{pmatrix} \langle F_t e_{0,0}, E_s e_{0,0}\rangle & \langle F_t e_{0,0}, E_s e_{0,-1}\rangle \\ \langle F_t e_{0,-1}, E_s e_{0,0}\rangle & \langle F_t e_{0,-1}, E_s e_{0,-1}\rangle \end{pmatrix}.$$

The measure μ_{11} is uniquely determined by (σ,τ) and it is equal to $\langle F_t e_{0,0}, E_s e_{0,0}\rangle$:

$$\mu_{11}(e_{k_1,k_2}) = \begin{cases} \langle e_{0,k_2}, e_{-k_1,0}\rangle = \langle \sigma^{k_1}\tau^{k_2} e_{0,0}, e_{0,0}\rangle & \text{if } k_2 \geq 0, \\ \langle e_{k_1,0}, e_{0,-k_2}\rangle = \langle e_{0,0}, \tau^{-k_2}\sigma^{-k_1} e_{0,0}\rangle & \text{if } k_2 < 0, \end{cases}$$

$$\begin{aligned}\int_0^{2\pi} e_{k_1,k_2}\, d\mu_{11}(s,t) &= \mu_{11}(e_{k_1,k_2}) = \left\langle S^{k_1}T^{k_2}e_{0,0}, e_{0,0}\right\rangle \\ &= \int_0^{2\pi} e^{ik_2t}\, d\left\langle F_te_{0,0}, S^{-k_1}e_{0,0}\right\rangle \\ &= \int_0^{2\pi}\int_0^{2\pi} e^{ik_1s}e^{ik_2t}\, d\left\langle F_te_{0,0}, E_se_{0,0}\right\rangle.\end{aligned}$$

In order to show that μ_{22} is uniquely determinated by (S,T), an analogous development is valid:

$$\mu_{22}(e_{k_1,k_2}) = \begin{cases} \left\langle e_{0,-1}, e_{-k_1,-k_2-1}\right\rangle = \left\langle e_{0,-1}, \tau^{-k_2}\sigma^{k_1}e_{0,-1}\right\rangle & \text{if } k_2 \geq 0, \\ \left\langle e_{k_1,k_2-1}, e_{0,-1}\right\rangle = \left\langle \sigma^{k_1}\tau^{k_2}e_{0,-1}, e_{0,-1}\right\rangle & \text{if } k_2 < 0, \end{cases}$$

and $\mu'_{22}(s,t) = \mu_{22}(s,t) = \langle F_te_{0,-1}, E_se_{0,-1}\rangle$.

On the other hand, μ_{21} is only determined if $(k_1,k_2) \in \mathbb{E}_2$:

$$\begin{aligned}\mu_{21}(e_{k_1,k_2}) &= \left\langle e_{0,-1}, e_{-k_1,-k_2-1}\right\rangle \\ &= \left\langle e_{0,-1}, \tau^{-k_2-1}\sigma^{-k_1}e_{0,0}\right\rangle \\ &= \left\langle \sigma^{k_1}\tau^{k_2+1}e_{0,-1}, e_{0,0}\right\rangle, \qquad \text{if } k_2 < 0.\end{aligned}$$

So by the spectral theorem,

$$\begin{aligned}\mu'_{21}(e_{k_1,k_2}) &= \left\langle S^{k_1}T^{k_2+1}e_{0,-1}, e_{0,0}\right\rangle \\ &= \int_0^{2\pi}\int_0^{2\pi} e^{ik_1s}e^{i(k_2+1)t}\, d\left\langle F_te_{0,-1}, E_se_{0,0}\right\rangle,\end{aligned}$$

and it can be deduced that

$$d\mu'_{21}(s,t) = e^{it}d\left\langle F_te_{0,-1}, E_se_{0,0}\right\rangle. \ \square$$

By the resolvent formula,

$$\begin{aligned}\left\langle R_\tau(z)e_{0,-1}, R_\sigma(y)e_{0,0}\right\rangle &= \int_0^{2\pi} \frac{d\langle F_te_{0,-1}, R_\sigma(y)e_{0,0}\rangle}{1-ye^{is}} \\ &= \int_0^{2\pi}\int_0^{2\pi} \frac{d\langle F_te_{0,-1}, E_se_{0,0}\rangle}{(1-ye^{is})(1-ze^{it})}.\end{aligned}$$

Since $d\mu'_{21}(s,t) = d\mu_{21}(s,t) + h(s,t)\,ds\,dt$ with $h \in H^1(\mathbb{T}^2)$,

$$\begin{aligned}\langle R_\tau(z)\,e_{0,-1}, R_\sigma(y)e_{0,0}\rangle &= \int_0^{2\pi}\int_0^{2\pi} \frac{e^{-it}f(s,t)}{(1-ye^{is})(1-ze^{it})}\,ds\,dt \\ &\quad + \int_0^{2\pi}\int_0^{2\pi} \frac{e^{-it}h(s,t)}{(1-ye^{is})(1-ze^{it})}\,ds\,dt \\ &= z\big(f(y,z) + h(y,z)\big).\end{aligned}$$

Then, by (17a), the function

$$f(y,z) + h(y,z) = \sum_{p\geq 0} s(-p,-1)\,y^{-p}\,z^{-1} + \sum_{p\geq 0}\sum_{m\geq 1} y^{-p}\,z^{m-1}\,c_0^0\,\Phi(D^0\Phi)^{m-1}\langle w^0, e_{p,0}\rangle,$$

where Φ verifies (16), is the general solution of the first step of the Schur algorithm.

In the following steps of the algorithm, we must use the same method that we applied in the matricial case. So we do not repeat the argument showed there.

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