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On the Maximal Operator associated to a Convex Body in $\mathbb{R}^n$

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Abstract

In this paper we study the behavior of the constants appearing in weak type (1,1) inequalities for the dyadic maximal operator associated to a convex body. We show that for "sufficiently" rapidly increasing sequences these constants are uniformly bounded, independently of dimension and the convex body. From this result we easily recover a theorem of Stein and Strömberg. A simple argument shows that in the case of radial functions, the constants for the full maximal operator are indeed uniformly bounded.

Introduction and statement of results

Let B be an arbitrary open and symmetric convex body in $\mathbb{R}^n$, of unit Lebesgue measure ($|B| = 1$). Given $r > 0$ and $x \in \mathbb{R}^n$, we denote by $B_r(x)$ the dilation by r and the translation by x of B.

If $q = \{q_k\}_{k\in\mathbb{Z}}$ is a lacunary sequence of positive numbers (say, $q_{k+1}/q_k \geq a_q > 1$, for every k), we define the maximal operator

$$M_q f(x) = M_{q,B} f(x) := \sup_{k\in\mathbb{Z}} \frac{1}{(q_k)^n} \int_{B_k(x)} |f(y)|\, dy$$

for every L^1-function f, where now B_k denotes dilation of B by q_k.

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We also define the full maximal operator

$$Mf(x) = M_B f(x) = \sup_{r\in\mathbb{R}_+} \frac{1}{r^n} \int_{B_r(x)} |f(y)|\,dy.$$

In the last years, considerable attention has been given to the study of the boundedness of these operators on Lorenz spaces, specially from the point of view of Harmonic Analysis in $\mathbb{R}^n$, for "large n". This simply means that the study has been directed toward the analysis of the constants of the boundedness of them and their possible growth as n tends to ∞. The central point of this program would be the making of a "reasonable" Harmonic Analysis for functions with infinite variables, defined say in Banach space.

It was E. Stein [6] the first to realize that when B is the unit ball in $\mathbb{R}^n$ then the operator M is bounded in each L^p, $1 < p \leq \infty$, with a constant which is independent of dimension. Later on, Bourgain [1] proved this result for arbitrary symmetric convex bodies whenever $p \geq 2$. The range of p's for which the result holds has finally been extended to $p > 3/2$ independently by Carbery [3] and Bourgain [2].

Stein and Strömberg in [7] considered the problem of the constants appearing in the weak type (1,1) estimates. If B is the ordinary euclidean unit ball, an argument involving the method of rotations shows that such constant grows no faster than $O(n)$. For a general convex body they also showed that the constant grows no faster than $O(n \log(n))$.

The aim of this paper is to show that this theorem of Stein and Strömberg is simply a reflection of the following principle: "There exist $n\log(n)$ maximal operators, each one satisfying a uniform weak type (1,1) estimate independent of dimension and which control the operator M_B".

For radial functions it is possible to show that the constant for the full maximal operator is indeed uniformly bounded.

We first state the following

Theorem 1

There exists a universal constant C_1 such that if q is a lacunary sequence as above, then

$$\left|\{x \in \mathbb{R}^n : M_q f(x) > \lambda\}\right| \leq \frac{C_1}{\lambda}\left(1 + \frac{\log n}{\log a_q}\right)\|f\|_1,$$

for every function $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$.

We also give the following technical result:

Theorem 2

There exist a universal constant C_2 and n lacunary sequences q^j, $j = 1, \ldots, n$, with $a_{q^j} = 2$, such that

$$Mf(x) \leq C_2 \max_{1 \leq j \leq n} M_{q^j} f(x),$$

for every $f \in L^1(\mathbb{R}^n)$.

Combining Theorems 1 and 2 we obtain

Corollary (Stein-Strömberg)

There exists a constant C_3 independent of dimension such that

$$\left|\left\{x \in \mathbb{R}^n : Mf(x) > \lambda\right\}\right| \leq \frac{C_3}{\lambda} n(1 + \log n) \|f\|_1,$$

for every function $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$.

Finally, we present

Theorem 3

Let M be the maximal operator defined over centered balls, and let f be a radial function on $\mathbb{R}^n$. Then

$$\left|\left\{x \in \mathbb{R}^n : Mf(x) > \lambda\right\}\right| \leq \frac{4}{\lambda} \int_{\mathbb{R}^n} |f(y)| \, dy.$$

Proof of results

We first show Theorem 2. For $j = 1, 2, \ldots, n$ we define

$$q^j = \left\{2^k + \frac{j 2^k}{n}\right\}_{k \in \mathbb{Z}}$$

Thus q^j is a lacunary sequence with $a_{q^j} = 2$.

Now, given $r > 0$, we consider the unique $k \in \mathbb{Z}$ and $j \in \{1, 2, \ldots, n\}$ such that

$$2^k + (j-1)\frac{2^k}{n} < r \leq 2^k + j\frac{2^k}{n}.$$

Then,

$$\frac{1}{r^n}\int_{B_r(x)}|f(y)|\,dy \le \left(\frac{q_{k,j}}{r}\right)^n \frac{1}{(q_{k,j})^n}\int_{B_{k,j}(x)}|f(y)|\,dy$$
$$\le \left(\frac{q_{k,j}}{r}\right)^n M_{q^j}f(x),$$

where $q_{k,j} = 2^k + j2^k/n$ and $B_{k,j}$ denotes dilation by $q_{k,j}$.

On the other hand,

$$\left(\frac{q_{k,j}}{r}\right)^n \le \left(\frac{q_{k,j}}{q_{k,j-1}}\right)^n = \left(1 + \frac{1}{n+j-1}\right)^n \le \left(1+\frac{1}{n}\right)^n$$

and Theorem 2 follows with $C_2 = e$. □

In order to prove Theorem 1, we will state the following particular case which is interesting by itself:

Lemma 4

If $q = \{q_k\}_{k\in\mathbb{Z}}$ is a sequence of positive numbers satisfying $q_k \ge nq_{k-1}$ (i.e., $a_q \ge n$), then

$$\left|\{x \in \mathbb{R}^n : M_q f(x) > \lambda\}\right| \le \frac{C_0}{\lambda}\|f\|_1,$$

with C_0 independent of dimension, $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$.

From this Lemma, Theorem 1 easily follows, for if $q = \{q_k\}_{k\in\mathbb{Z}}$ is a general lacunary sequence and m is a natural number so that

$$(a_q)^m \ge n$$

(e.g., $m = [\log n/\log a_q] + 1$), then each of the sequences

$$\{q_{km+j}\}_{k\in\mathbb{Z}}, \qquad j = 1, \dots, m$$

satisfies the conditions of the Lemma and the corresponding maximal operators (the supremum of them) majorize M_q. □

We prove now Lemma 4.

We recall that $B_k(x)$ denotes dilation by q_k of $B(x)$. We will denote by $B'_k(x)$, $\hat{B}_k(x)$ and $B^*_k(x)$ the dilations of $B(x)$ by q_k/n, $q_k(1+1/n)$ and $q_k(1+2/n)$, respectively.

Setting

$$E_\lambda = \{x \in \mathbb{R}^n : M_q f(x) > \lambda\},$$

we may observe then

$$E_\lambda \subset \bigcup \left\{ B' : \frac{1}{|B|} \int_B |f| > \lambda \right\} =: \bigcup_{\alpha \in A} B'_\alpha,$$

for some set of indices A. By a standard limiting argument, it suffices to show

$$\left| \bigcup_{\alpha \in A_0} B'_\alpha \right| \le \frac{C_0}{\lambda} \|f\|_1,$$

for every finite subset A_0 of A. Let $\{B_j\}_{j \in A_0}$ be an ordering of that family such that, if $i < j$, we have $|B_i| \ge |B_j|$, and define

$$D_1 = B'_1, \quad \ldots \quad D_j = B'_j \setminus \bigcup_{k<j} B'_k.$$

Let us set

$$g_j(x) = \frac{|D_j|}{|B_j|} \chi_{B_j}(x)$$

and

$$G_{q_k}(x) = \sum_{\{j : |B_j| = (q_k)^n\}} g_j(x).$$

We may observe that for every x, $G_{q_k}(x) \le e$, for if $x \in B_j$, then $B'_j \subset x + q_k(1 + 1/n)B = \hat{B}_{q_k}(x)$.

Hence,

$$\begin{aligned} G_{q_k}(x) &\le \frac{1}{(q_k)^n} \sum_{\{j : B'_j \subset \hat{B}_{q_k}(x)\}} |D_j| \\ &\le \frac{1}{(q_k)^n} |\hat{B}_{q_k}(x)| = (1 + 1/n)^n \le e. \end{aligned}$$

We select now a subfamily $\{A_j\}$ of $\{B_j\}_{j \in A_0}$ as follows: We first take $A_1 = B_1$. Having selected $A_1, \ldots, A_t$, we inductively select A_{t+1} so that $|A_{t+1}|$ is maximal among those B_α satisfying

$$\sum_{j=1}^{t} g_j(x) + g_\alpha(x) \le 1 + e$$

for every x.

Let us assume that this selection process ends with A_m. Then we have

$$\sum_{j=1}^{m} g_j(x) \le 1 + e$$

whereas, if B_α has not been choosen, then

$$G(x) = \sum_{\{j:|A_j| \ge |B_\alpha|\}} g_j(x) + g_\alpha(x) > 1 + e$$

for some x of B_α.

Now,

$$\begin{aligned} G(x) &= \sum_{\{j:|A_j|>|B_\alpha|\}} g_j(x) + \left[\sum_{\{j:|A_j|=|B_\alpha|\}} g_j(x) + g_\alpha(x) \right] \\ &= G_1(x) + G_2(x). \end{aligned}$$

But, $G_2(x) \le e$ and, therefore, $G_1(x) > 1$, for some x of B_α.

The final observation is that if $B_{\alpha'} \cap B_\alpha \neq \emptyset$ and $|B_{\alpha'}| > |B_\alpha|$, then $B_\alpha \subset B^*_{\alpha'}$.

Thus,

$$\bigcup_{B_\alpha \notin \{A_j\}} B_\alpha \subset \left\{ x : \sum_{j=1}^{m} \frac{|D_j|}{|A_j|} \chi_{A_j^*}(x) > 1 \right\} = S,$$

and one has

$$|S| \le \left(1 + \frac{2}{n}\right)^n \sum_{j=1}^{m} |D_j| \le e^2 \sum_{j=1}^{m} |D_j|.$$

Finally, we get

$$\begin{aligned} \left| \bigcup_{\alpha \in A_0} B'_\alpha(x) \right| &\le (1 + e^2) \sum_{j=1}^{m} |D_j| \\ &\le \frac{1 + e^2}{\lambda} \sum_{1 \le j \le m} \frac{|D_j|}{|A_j|} \int |f(x)| \chi_{A_j}(x)\, dx \\ &= \frac{1 + e^2}{\lambda} \int |f(x)| \sum_{j=1}^{m} g_j(x)\, dx \\ &\le (1 + e)(1 + e^2) \frac{1}{\lambda} \int |f(x)|\, dx, \end{aligned}$$

and the lemma follows with $C_0 = (1 + e^2)(1 + e)$. □

We shall now prove Theorem 3.

In order to do so, we will need certain properties of the maximal operator M_ω associated to a positive weight ω. To be more precise, let us define for $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \omega dt)$

$$M_\omega f(t) = \sup_{t \in I} \frac{1}{\omega(I)} \int_I |f(s)|\, \omega(s)\, ds,$$

where $\omega(I) = \int_I \omega(t)\, dt$, and the supremum is taken over all intervals I. Then M_ω is of weak type (1,1) with respect to ω and indeed one has

$$\omega\Big(\big\{t \in \mathbb{R} : M_\omega f(t) > \lambda\big\}\Big) \leq \frac{2}{\lambda} \int_{\mathbb{R}} |f(t)|\, \omega(t)\, dt.$$

(See [4] and [5] for a simple proof of this fact.)

We will also need the following geometric

Lemma 5

For every ball B of $\mathbb{R}^n$, there is a set Σ_B of the unit sphere S^{n-1}, $\Sigma_B \subset S^{n-1}$, and there are two functions $\varepsilon_1, \varepsilon_2 : \Sigma_B \to \mathbb{R}_+$, such that $\varepsilon_1(\alpha) \leq |x| \leq \varepsilon_2(\alpha)$ for every $\alpha \in \Sigma_B$, where x denotes the center of B. Moreover,

$$B \subset \Big\{r\alpha : \alpha \in \Sigma_B,\ \varepsilon_1(\alpha) \leq r \leq \varepsilon_2(\alpha)\Big\} =: D,$$

with $|D| \leq 2B$.

We postpone the proof of Lemma 5 and continue with the proof of Theorem 3.

Given a fixed ball $B \subset \mathbb{R}^n$, centered at the point x, according to Lemma 5, there exist $\Sigma_B \subset S^{n-1}$ and $\varepsilon_1, \varepsilon_2 : \Sigma_B \to \mathbb{R}_+$ with the stated properties. We define D as above.

We observe that

$$|D| = \int_{\Sigma_B} \Big(\int_{\varepsilon_1(\alpha)}^{\varepsilon_2(\alpha)} t^{n-1}\, dt\Big)\, d\sigma(\alpha) = \int_{\Sigma_B} \omega(I_\alpha)\, d\sigma(\alpha),$$

where $\omega(t) = t^{n-1}$, $I_\alpha = \big[\varepsilon_1(\alpha), \varepsilon_2(\alpha)\big]$, and $d\sigma$ is the standard Lebesgue measure in S^{n-1}, normalized to have total mass $\sigma(S^{n-1}) = 1$. The important fact here is that $\varepsilon_1(\alpha) \leq |x| \leq \varepsilon_2(\alpha)$ for every $\alpha \in \Sigma_B$ and, therefore, $|x| \in I_\alpha$.

If f is a radial function, we may define $f_0(|x|) = f(x)$, and we have

$$\frac{1}{|B|} \int_B |f(y)|\, dy \leq \frac{2}{|D|} \int_D |f(y)|\, dy.$$

Now, using polar coordinates, this is equal to

$$\begin{aligned}
\frac{2}{|D|}\int_{\Sigma_B}\omega(I_\alpha)\Big[\frac{1}{\omega(I_\alpha)}\int_{I_\alpha}\big|f_0(t)\big|\,\omega(t)\,dt\Big]\,d\sigma
&\le 2\sup_{|x|\in I}\frac{1}{\omega(I)}\int_I\big|f_0(t)\big|\,\omega(t)\,dt\\
&= 2M_\omega f_0\big(|x|\big),
\end{aligned}$$

where we have used the previous observation.

Hence,

$$\begin{aligned}
\Big|\big\{x\in\mathbb{R}^n : Mf(x)>\lambda\big\}\Big| &\le 2\int_{S^{n-1}}d\sigma\int_{\{r>0:\,M_\omega f_0(r)>\lambda\}} r^{n-1}\,dr\\
&= 2\omega\Big(\big\{r>0 : M_\omega f_0(r)>\lambda\big\}\Big).
\end{aligned}$$

Using the stimate for the maximal operator M_ω mentioned above, we finally have

$$\Big|\big\{x\in\mathbb{R}^n : Mf(x)>\lambda\big\}\Big| \le \frac{4}{\lambda}\int_{S^{n-1}}d\sigma\int_{\mathbb{R}}\big|f_0(t)\big|\,\omega(t)\,dt = \frac{4}{\lambda}\,\|f\|_1. \;\square$$

It remains to show then Lemma 5.

By a slight abuse of notation, let $B_R(x)$ denote the ball of radius R centered at the point x. Let us assume that $B = B_{R_0}(x_0)$.

We shall consider two cases:

If $0\in B$ (i.e., $|x_0|<R_0$), we may take $\Sigma_B = S^{n-1}$ and $D = B_{R_0}(x_0)\cup B_{|x_0|}(0)$. Also, for each direction α, we consider $\varepsilon_1(\alpha)=0$, and $\varepsilon_2(\alpha)$ as the distance between the origin and the intersection of the ray in direction α with the boundary of D. It is obvious here that $|D|\le 2|B|$.

If $0\notin B$ (i.e., $|x_0|\ge R_0$) and Γ is the region interior to the cone circumscribed to B with its center in the origin, we take first $\Sigma_B = \Gamma\cap S^{n-1}$.

For each $\alpha\in\Sigma_B$, we define $\varepsilon_1(\alpha)$ as the distance to the origin of the first point of intersection of the "ray" α with the closure of B.

We consider then

$$D = D_1\cup D_2,$$

where

$$D_1 = \big\{r\alpha : \alpha\in\Sigma_B,\ \varepsilon_1(\alpha)\le r\le |x_0|\big\} \quad\text{and}\quad D_2 = B\setminus B_{|x_0|}(0).$$

We can now define $\varepsilon_2(\alpha)$ as the distance between the origin and the "upper" point of intersection of the ray α with the closure of D.

Clearly $D_2 \subset B$. Therefore, in order to show that $|D| \leq 2|B|$, we only need to prove that D_1 is also contained in another ball of radius R_0. To see that, let z be the point on the segment $\overline{0x_0}$ with $|z|$ equal to the distance from the set $\partial\Gamma \cap B$ to the origin. Then, it is not hard to see that $D_1 \subset B_{R_0}(z)$, as we wanted. In fact, due to the symmetric properties with respect to the ray passing through x_0 of the sets considered, we can assume that we are in dimension 2, and then the computations become a simple exercise in trigonometry. □

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