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Barrelledness of $L_b(\lambda_p, X)$

M. Angeles Miñarro

Dpto. Matemática Aplicada, E.T.S.I. Agrónomos, Univ. de Córdoba, 14004 Córdoba, Spain

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Abstract

Given a Banach space X and a Köthe sequence space λ_p, with $1 \leq p < \infty$ or $p = 0$, it is known that $L_b(\lambda_p, X)$ is barrelled if λ_p satisfies Heinrich's density condition. In this note we show that if $p > 1$ and if λ_p does not satisfy the density condition then $L_b(\lambda_p, X)$ is barrelled if and only if every continuous linear operator from λ_p into X is compact. As a consequence we get a characterization of the distinguished spaces of type $\ell_\infty \hat{\otimes}_\pi \lambda_p$.

Let E and X denote a Fréchet and a Banach space respectively and let $L_b(E, X)$ be the space of all continuous linear maps from E into X endowed with the topology of uniform convergence on the bounded sets of E. The space $L_b(E, X)$ admits a fundamental sequence of bounded sets but in general it is not a (DF)-space, i.e., it may happen that $L_b(E, X)$ is not $\aleph_0$-quasibarrelled; see [15] for a counterexample and [8] for more information. The following results concerning barrelledness of $L_b(E, X)$ are already known.

Theorem 1

(a) [3, 2.9] Let us assume that $L_b(E, X)$ is a (DF)-space and that E satisfies the density condition (i.e., the bounded sets of E'_b are metrizable, [2]), then $L_b(E, X)$ is barrelled.

(b) [4] Let λ_p be a Köthe sequence space with $1 \leq p < \infty$ or $p = 0$. Then $L_b(\lambda_p, X)$ is a (DF)-space. By (a) it is barrelled if λ_p satisfies the density condition.

(c) [2] If $L_b(E, \ell_\infty)$ is barrelled then E satisfies the density condition.

Moreover the barrelled spaces of type $L_b(\lambda_p, \ell_q)$, with $1 \le p < \infty$, $1 \le q < \infty$, have been characterized in [4], also see [5]. However a characterization of barrelled spaces of type $L_b(\lambda_p, X)$ for a general Banach space X remained still open. This note is devoted to solve this case. As a consequence we can also give a description of the distinguished Fréchet spaces of type $\lambda_p \hat{\otimes}_\pi \ell_\infty$.

We refer the reader to [11] and [13] for notation and general theory. The Banach space c_0 will also be denoted by ℓ_0.

DEFINITION. Let I be a countable index set and let A be a Köthe matrix on I, i.e., $A = \big(a_k(i)\big)_{k\in\mathbb{N}, i\in I}$ with $0 < a_k(i) \le a_{k+1}(i)$, for every $k \in \mathbb{N}$, $i \in I$. Given $1 \le p < \infty$ or $p = 0$ the Köthe sequence space of order p is defined as

$$\lambda_p(I, A) = \Big\{(x_i)_{i\in I} : \big\|(x_i)\big\|_k := \Big(\sum_{i\in I}|x_i|^p a_k(i)\Big)^{1/p} < \infty,\ k \in \mathbb{N}\Big\} \quad \text{if } 1 \le p < \infty,$$

$$\lambda_0(I, A) = \Big\{(x_i)_{i\in I} : \lim_{i\in I} x_i a_k(i) = 0\Big\}, \qquad \big\|(x_i)\big\|_k := \sup|x_i|a_k(i),\ k \in \mathbb{N}.$$

We write $\lambda_p(A)$ or even λ_p if there is no chance of confusion.

DEFINITION. Let (G, τ) be a locally convex space. A Schauder decomposition of G is a sequence of continuous operators $P_n : G \to G$, $n \in \mathbb{N}$, such that:

(i) $P_i \circ P_j = \delta_{ij} P_j$, $\forall i, j \in \mathbb{N}$.

(ii) $x = \sum_{j=1}^{\infty} P_j(x)$, $\forall x \in G$, where the series converges in τ.

E.g., if (G, τ) has a Schauder basis $(e_n)_{n\in\mathbb{N}}$ and $(e_n^*)_{n\in\mathbb{N}}$ denotes the associated sequence of biorthogonal functionals then $(e_n^* \otimes e_n)_{n\in\mathbb{N}}$ is a finite dimensional decomposition of G. (We omit the word Schauder from now on.)

Let $(P_n)_{n\in\mathbb{N}}$ be a decomposition in a locally convex space G. $(P_n)_{n\in\mathbb{N}}$ is said to be shrinking if the sequence of dual operators $(P_n')_{n\in\mathbb{N}}$ is a decomposition of the strong dual of G. E.g., the canonical basis of λ_p induces a shrinking decomposition if $1 < p < \infty$ or $p = 0$, moreover every decomposition in a reflexive Fréchet space is shrinking ([10]). Let us denote $Q_j := \sum_{i=1}^{j} P_i$, $j \in \mathbb{N}$. The decomposition is said to be equicontinuous if the sequence $(Q_n)_{n\in\mathbb{N}}$ is equicontinuous. Every decomposition of a barrelled space is equicontinuous. A sequence $(z_n)_{n\in\mathbb{N}}$ in G is said to be a block sequence if it has the form $z_j = \big(Q_{n_j} - Q_{n_{j-1}}\big)(z_j)$ for every $j \in \mathbb{N}$ and some increasing sequence $0 = n_0 < n_1 < n_2 < \dots$

The following characterization is straightforward (also see the proof of Lemma 2.(a) below).

Lemma 1

Let $(P_n)_{n\in\mathbb{N}}$ be an equicontinuous decomposition of a locally convex space G. $(P_n)_{n\in\mathbb{N}}$ is shrinking if and only if every bounded block sequence is weakly convergent to 0.

Let $K(E,X)$ denote the subspace of $L_b(E,X)$ of all compact operators. Given a decomposition $(P_n)_{n\in\mathbb{N}}$ in the Fréchet space E we define continuous linear operators $\circ P_n\colon K(E,X) \to K(E,X)$, $f \to f\circ P_n$, $n\in\mathbb{N}$.

Lemma 2

(a) Let E be a Fréchet space with a shrinking decomposition $(P_n)_{n\in\mathbb{N}}$ and let X be a Banach space. Then $\left(\circ P_n\right)_{n\in\mathbb{N}}$ is an equicontinuous decomposition of $K(E,X)$.

(b) Let G be a (DF)-space having an equicontinuous decomposition $(P_n)_{n\in\mathbb{N}}$ such that $P_n(G)$ is quasibarrelled for every $n\in\mathbb{N}$. Then G is quasibarrelled.

Proof. *(a)* The condition *(i)* of decompositions is clear. To prove *(ii)* we have to check the following equality for any $f\in K(E,X)$ and every bounded set B in E,

$$\lim_{j\to\infty}\sup_{z\in B}\left\|(f - f\circ Q_j)(z)\right\| = \lim_{j\to\infty}\sup_{z\in B}\left\|f\big((\mathrm{id} - Q_j)(z)\big)\right\| = 0.$$

Let us assume on the contrary that this condition does not hold for some bounded set B and some f in $K(E,X)$. By induction we can select sequences $(z_n)_{n\in\mathbb{N}}\subset B$ and $j(1) < j(2) < \ldots$ such that

$$\left\|f\Big(\big(Q_{j(n+1)} - Q_{j(n)}\big)(z_n)\Big)\right\| > \varepsilon, \quad n\in\mathbb{N},$$

for some $\varepsilon > 0$. Now the sequence $\big((Q_{j(n+1)} - Q_{j(n)})(z_n)\big)_{n\in\mathbb{N}}$ is a bounded block sequence by the equicontinuity of $(Q_n)_{n\in\mathbb{N}}$, thus it is weakly null by Lemma 1 and the hypothesis that $(P_n)_{n\in\mathbb{N}}$ is shrinking. Since f is compact we have that $\Big(f\big((Q_{j(n+1)} - Q_{j(n)})\big)(z_n)\Big)_{n\in\mathbb{N}}$ is a null sequence. A contradiction.

The fact that $\left(\circ P_n\right)_{n\in\mathbb{N}}$ is equicontinuous can be readily checked.

(b) Let $(B_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded sets in G. If we set $C_j := \bigcup_{i\geq 1} Q_i(B_j)$, $j\in\mathbb{N}$, then $(C_n)_{n\in\mathbb{N}}$ is also a fundamental sequence of bounded sets in G and satisfies $Q_j(C_n)\subset C_n$, for every $j,n\in\mathbb{N}$. Now the proof goes in the same way as in [7, Proposition 2]. □

Lemma 3

Let X be a Banach space and $1 < p < \infty$ or $p = 0$:
(i) A continuous linear operator $T: \ell_p \to X$ is compact if and only if the image of every bounded block sequence is a null sequence.
(ii) If there is a continuous linear operator from ℓ_p into X which is not compact then we can find a continuous linear operator $T: \ell_p \to X$ such that $\|T(e_i)\| = 1$ for every $i \in \mathbb{N}$.

Proof. (i) The "only if" part is clear since every bounded block sequence is weakly null in ℓ_p. Conversely, let us assume that T is not compact. Then we can find a weakly null sequence $(z_n)_{n\in\mathbb{N}}$ in ℓ_p such that $\|T(z_n)\| = 1$, $n \in \mathbb{N}$. By using a "gliding hump" argument we shall construct a bounded block sequence $(\bar{z}_n)_{n\in\mathbb{N}}$ such that $\|T(\bar{z}_n)\| \geq 1/2$ for every $n \in \mathbb{N}$, a contradiction with the hypothesis. In fact, let us denote $Q_j(z) := (z_1, \ldots, z_j, 0, 0, \ldots)$, with $z \in \ell_p$, $j \in \mathbb{N}$. We take $j(1) \in \mathbb{N}$ such that $\|T(Q_{j(1)}(z_1))\| \geq 1/2$ and set $\bar{z}_1 := Q_{j(1)}(z_1)$. Now the sequence $(Q_{j(1)}(z_n))_{n\in\mathbb{N}}$ converges to 0 in norm, hence can select $n(2) \in \mathbb{N}$ such that $\|T((\mathrm{id} - Q_{j(1)})(z_{n(2)}))\| > 1/2$; next we choose $j(2) > j(1)$ such that $\|T((Q_{j(2)} - Q_{j(1)})(z_{n(2)}))\| > 1/2$ and set $\bar{z}_2 := (Q_{j(2)} - Q_{j(1)})(z_{n(2)})$. We observe that $(Q_{j(2)}(z_n))_{n\in\mathbb{N}}$ converges to 0 in norm and repeat the argument above; by induction we construct the announced bounded block sequence $(\bar{z}_n)_{n\in\mathbb{N}}$.
(ii) Let $T: \ell_p \to X$ be a continuous linear operator which is not compact. By *(i)* there is a bounded block sequence $(z_n)_{n\in\mathbb{N}}$ such that $\|T(z_n)\| \geq \rho$ for some $\rho > 0$ and every $n \in \mathbb{N}$. There is a continuous linear operator $R: \ell_p \to \ell_p$ mapping e_i into $\|T(z_i)\|^{-1} z_i$ for every $i \in \mathbb{N}$ (e.g. see [12, 2.a.1]). Then $T \circ R: \ell_p \to X$ is the operator that we are looking for. □

Remark. Given a continuous linear map $T: \ell_1 \to X$ it is enough to check that $(T(e_n))_{n\in\mathbb{N}}$ is a null sequence to assure that T is compact. However this is not true if $1 < p < \infty$ or $p = 0$. We exhibit a simple example for the case $p = 2$. We define the operator $T: \ell_2 \to \ell_2$, $(x_i) \to (x_1, (x_2 + x_3)/2^{1/2}, (x_4 + x_5 + x_6)/3^{1/2}, \ldots)$. It can be readily checked that T is a continuous linear mapping and it is not compact though $(T(e_n))_{n\in\mathbb{N}}$ converges to 0.

Theorem 2

Let $1 < p < \infty$ or $p = 0$ and let λ_p be a Köthe sequence space without the density condition. Given any Banach space X the following are equivalent:
(i) $L(\lambda_p, X) = K(\lambda_p, X)$.
(ii) $L(\ell_p, X) = K(\ell_p, X)$.
(iii) $L_b(\lambda_p, X)$ is barrelled.

Proof. The equivalence of *(i)* and *(ii)* is clear since every continuous linear operator from λ_p into X factorizes through ℓ_p and since λ_p is not a Montel space and hence has a complemented copy of ℓ_p.

(i) $\Rightarrow$ *(iii)* Let us denote by $(P_n)_{n\in\mathbb{N}}$ the 1-dimensional shrinking decomposition of λ_p associated to the canonical basis. By Lemma 2.(a), $(\circ P_n)_{n\in\mathbb{N}}$ is an equicontinuous decomposition of $K(\lambda_p, X)$, moreover $\circ P_j(K(\lambda_p, X))$ is isomorphic to X for every $j \in \mathbb{N}$. On the other hand $L_b(\lambda_p, X)$ is a (DF)-space ([4]) and coincides with $K(\lambda_p, X)$ by hypothesis. It follows that $L_b(\lambda_p, X)$ is quasibarrelled by Lemma 2.(b) and it is barrelled since it is complete.

(iii) $\Rightarrow$ *(ii)* Assume that there is a continuous linear operator from ℓ_p into X which is not compact. From Lemma 3 there is a continuous linear operator $\varphi: \ell_p \to X$ such that $\|\varphi(e_i)\| = 1$, for every $i \in \mathbb{N}$. We have to show that $L_b(\lambda_p, X)$ is not barrelled. According to the results of [1] since λ_p does not have the density condition it contains a complemented subspace isomorphic to $\lambda_p(\mathbb{N}^2, B)$ where the matrix B satisfies:

(B1) $b_1(i,j) = 1$, $\forall i, j \in \mathbb{N}$.
(B2) $b_n(i,j) = b_1(i,j)$, $\forall i \geq n$, and $\lim_{j\to\infty} b_{n+1}(n,j) = \infty$, $\forall n \in \mathbb{N}$.

It is enough to check that $L_b(\lambda_p(B), X)$ is not barrelled. Let U_n and V denote the n-th unit ball of $\lambda_p(B)$ and the closed unit ball of X, respectively, and set $\mathcal{B}_k := \{g \in L_b(\lambda_p(B), X); g(U_k) \subset V\}$. Then $\mathcal{W} := \bigcup_{k\geq 1} \mathcal{B}_k$ is a bornivorous absolutely convex set in $L_b(\lambda_p(B), X)$. Since $L_b(\lambda_p(B), X)$ is a (DF)-space it follows from [13, 8.2.27] that $\mathcal{W}$ contains a barrel. Hence, it now suffices to prove that $\mathcal{W}$ is not a 0-neighbourhood. If it were we could find a bounded set A in $\lambda_p(B)$ such that

$$\mathcal{U} := \left\{f \in L_b(\lambda_p(B), X); f(A) \subset \|\varphi\| V\right\} \subset \tfrac{1}{2}\mathcal{W} \tag{1}$$

Let $M_i := \sup\{\|x\|_i; x \in A\}$. Given $i \in \mathbb{N}$ we use (B2) to select $n(i)$ such that $\|e_{i,n(i)}\|_{i+1} = b_{i+1}(i, n(i)) > 2^i M_{i+1}$. It is important to remark that by (B1,2) the basic sequence $(e_{i,n(i)})_{i\in\mathbb{N}}$ is equivalent to the canonical basis of ℓ_p. We denote by H the sectional subspace spanned by $(e_{i,n(i)})_{i\in\mathbb{N}}$, by ψ an isomorphism from H onto ℓ_p such that $\psi(e_{i,n(i)}) = e_i$, $i \in \mathbb{N}$ and denote by π the canonical projection from $\lambda_p(B)$ onto H. Now we define $f := \varphi \circ \psi \circ \pi \in L_b(\lambda_p(B), X)$. We first check that f belongs to $\mathcal{U}$. Let us assume $1 < p < \infty$, the case $p = 0$ is similar.

Given $x \in A$ and $i \in \mathbb{N}$ we have

$$|x_{i,n(i)}|^p 2^i M_{i+1} \leq |x_{i,n(i)}|^p \|e_{i,n(i)}\|_{i+1} \leq \|x\|_{i+1} \leq M_{i+1},$$

whence $|x_{i,n(i)}|^p \leq 2^{-i}$, therefore

$$\|\psi \circ \pi(x)\| = \Big(\sum_{i=1}^{\infty} |x_{i,n(i)}|^p\Big)^{1/p} \leq 1$$

for every $x \in A$. It follows that $\varphi \circ \psi \circ \pi(A)$ is contained in $\|\varphi\| V$ and consequently f belongs to $\mathcal{U}$. On account of (1) there are $j \in \mathbb{N}$ and $g \in \mathcal{B}_j$ such that $f = g/2$. However note that $\|f(e_{i,n(i)})\| = \|\varphi(e_i)\| = 1$, for every $i \in \mathbb{N}$. On the other hand $e_{j,n(j)}$ belongs to U_j by (B1,2) whence $\|g(e_{j,n(j)})\| \leq 1$. A contradiction. □

The following examples follow by the Theorem above; we also use [9].

Corollary 1

Let $1 < p < \infty$ or $p = 0$ and let λ_p be a Köthe sequence space without the density condition. Then:

(i) $L_b(\lambda_p, X)$ is barrelled if X has the Schur property.

(ii) If X contains a copy of c_0, $L_b(\lambda_p, X)$ is not barrelled.

(iii) $L_b(\lambda_p, L_1[0,1])$ is barrelled if and only if either $p > 2$ or $p = 0$.

(iv) $L_b(\lambda_0, X)$ is barrelled if X is reflexive.

As a further consequence we give a characterization of the Fréchet spaces of type $\ell_\infty \hat{\otimes}_\pi \lambda_p$, with $1 < p < \infty$ or $p = 0$, that are distinguished. A Fréchet space is said to be distinguished if its strong dual is barrelled (or equivalently bornological). Many authors have recently been concerned with distinguished Fréchet spaces (see [6] for a survey). In particular the Fréchet spaces of type $\ell_q \hat{\otimes}_\pi \lambda_p$ which are distinguished were characterized in [4, 7], when $1 \leq p < \infty$ or $p = 0$, $1 \leq q < \infty$ or $q = 0$. The case $q = \infty$ remained open (see [13, 13.11.3]).

Corollary 2

Let X be a Banach space and let $1 < p < \infty$ or $p = 0$:

(i) If λ_p satisfies the density condition then $X \hat{\otimes}_\pi \lambda_p$ is distinguished.

(ii) If λ_p does not satisfy the density condition then $X \hat{\otimes}_\pi \lambda_p$ is distinguished if and only if $L(\ell_p, X') = K(\ell_p, X')$; in particular, then $\ell_\infty \hat{\otimes}_\pi \lambda_p$ is distinguished if and only if either $p = 0$ or $2 < p < \infty$.

Proof. According to the results of [14] (also see [5]) the strong dual of $X \hat{\otimes}_\pi \lambda_p$ is isomorphic to $L_b(\lambda_p, X')$. Thus statement *(i)* is a particular case of [3, 1.7] and *(ii)* is a consequence of Theorem 2. To show the case $X = \ell_\infty$ one should check that $L(\ell_p, \ell'_\infty) = K(\ell_p, \ell'_\infty)$ (or equivalently $L(\ell_\infty, \ell'_p) = K(\ell_\infty, \ell'_p)$) if and only if $p = 0$ or $2 < p < \infty$ and this is already done in [9]. □

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