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$*$-value functions and total subrings in $*$-fields

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Abstract

The notion of value functions and a necessary and sufficient condition for a total subring to be a valuation ring are studied in the case of a $*$-field.

1. Definitions and basic facts

Let D be a $*$-field; that is, a skew field with an involution $*$ (an anti-automorphism of order 2). For general valuation theory on skew fields one can refer to [5]. For $*$-fields, we need our valuations to also be compatible with the involution $*$. Following [2], we define a $*$-valuation on D to be a valuation ω onto an additively written ordered group with the additional property that $\omega(x^*) = \omega(x)$ for all x in $D^\bullet$ (the multiplicative group of non-zero elements of D). A subring V of D is said to be a total subring if it contains x or x^{-1} for all x in D. A total subring is called a $*$-valuation ring if it contains x^*x^{-1} for every x in $D^\bullet$. In [6], a necessary and sufficient condition is given for a total subring to be a valuation ring for a division ring. This is done by associating value functions to primes in the division ring. In this paper, we carry over these results to the case of a $*$-field, were valuations are replaced with $*$-valuations. We now recall some basic facts about $*$-valuations in $*$-fields.

Lemma 1

(i) If ω is a $*$-valuation, then the subring $V = \{x \in D \,/\, \omega(x) \geq 0\}$ is a $*$-valuation ring and $P = \{x \in D \,/\, \omega(x) > 0\}$ is a $*$-closed maximal ideal of V.

(ii) Given a $*$-valuation subring V of D then there exists a $*$-valuation ω such that V coincides with the $*$-valuation ring of ω.

Lemma 2

(i) If V is a subring of D which contains x^*x^{-1} for every x in $D^\bullet$, then V is $*$-closed and preserved under conjugation.

(ii) If V is a $*$-closed total subring which is preserved under conjugation, then V is a $*$-valuation ring.

A couple (P, R) is said to be a $*$-prime in D if the following conditions are satisfied:

(1) R is a $*$-closed subring of D.
(2) P is a $*$-closed prime ideal in R.
(3) If $x\,\mathrm{R}\,y \subset P$ with $x, y \in D$ then $x \in P$ or $y \in P$.

A $*$-prime (P, R) is called semi-restricted if and only if for all $x \in D\backslash R$ there are $a, b \in R\backslash\{0\}$ with a or $b \in P$ such that $axb \in R\backslash P$. From [6], it is known that when R is a total subring of D and P its maximal ideal, then (P, R) is semi-restricted. For any $*$-closed ideal P, define $D^P = \{x \in D \;/\; xP \subset P \text{ and } Px \subset P\}$.

Proposition 3 ([3])

A $*$-prime (P, D^P) such that D^P is preserved under conjugation, yields a $*$-valuation of D with $*$-valuation ring D^P and maximal ideal P.

2. $*$-Value Functions

Let (P, D^P) be a $*$-prime in D. Consider the sets

$$P_x = \{(b,c) \in D \times D \;/\; bxc + (bxc)^* \in P\},$$

for every $x \in D$. Define an equivalence relation on D as follows: $x \sim y \Leftrightarrow P_x = P_y$. The equivalence class of x will be denoted by $\overline{x}$.

Lemma 4

The quotient set $D/\sim$ is a partially ordered group, with respect to the multiplication induced by multiplication of D, the ordering being defined by

$$\overline{x} \le \overline{y} \Longleftrightarrow P_y \subset P_x.$$

Proof. Let $x, x', y, y' \in D$ such that $x \sim x'$ and $y \sim y'$, if $(b, c) \in D \times D$ for which, $bxyc + (bxyc)^* \in P$ then $(b, yc) \in P_x = P_{x'}$. So $bx'yc + (bx'yc)^* \in P$ and therefore $(bx', c) \in P_y = P_{y'}$ which implies $bx'y'c + (bx'y'c)^* \in P$, this proves $P_{xy} \subset P_{x'y'}$. Hence the equivalence relation is compatible with the multiplication of D.

Clearly $P_y \subset P_x$ defines a partial order relation on $D/\sim$. Also, it can be shown that $\overline{x} \le \overline{y}$ implies $\overline{xz} \le \overline{yz}$. □

Corollary 5

The map $\phi: D^\bullet \to \Gamma$ ($= D/\sim$); $x \to \overline{x} = \phi(x)$; defines a value function, from D into a partially ordered group, with the following properties:
(i) $\phi(xy) = \phi(x)\phi(y)$ for any $x, y \in D^\bullet$.
(ii) If $\phi(x) \leq \phi(z)$ and $\phi(y) \leq \phi(z)$; $x, y, z \in D^\bullet$, then $\phi(x + y) \leq \phi(z)$.

Proof. *(i)* is clear. *(ii)* Let $(b, c) \in P_z$ then $(b, c) \in P_x$ and $(b, c) \in P_y$; so $bxc + (bxc)^*$, $byc + (byc)^* \in P$. Hence $b(x + y)c + (b(x + y)c)^* \in P$. □

DEFINITION 6. We call the value function ϕ a *-value function if it satisfies the additional property that $\phi(x^*) = \phi(x)$.

From the definition it follows that the value group Γ of a *-value function is abelian. For, $\phi(x)\phi(y) = \phi(xy) = \phi\big((xy)^*\big) = \phi\big(y^* x^*\big) = \phi(y^*)\phi(x^*) = \phi(y)\phi(x)$.

Let (P, D^P) be a semi-restricted *-prime in D. Consider $\overline{D} = \big\{x \in D \;/\; \phi(x) \leq 1\big\}$, where ϕ is the value function associated with (P, D^P) and where 1 is the unit element in Γ. Clearly $\overline{D}$ is a ring which is *-closed when ϕ is a *-value function.

Lemma 7

$$\overline{D} = \bigcap_{z \in D} z\, D^P z^{-1}$$

Proof. Let $\phi(x) \leq 1$, then for $(b, c) \in P_1$ we have $bc + (bc)^* \in P$. Also $(bz, z^{-1}c) \in P_1 \subset P_x$ so that $bzxz^{-1}c + \big(bzxz^{-1}c\big)^* \in P$, for all z in D. Assume now that $zxz^{-1} \notin D^P$ for some $z \in D$. Since (P, D^P) is semi-restricted there would be elements $b, c \in D^P$ with $bc \in P$ such that $bzxz^{-1}c \notin P$. But then $bc + (bc)^* \in P$ and $bzxz^{-1}c + \big(bzxz^{-1}c\big)^* \notin P$ which is a contradiction, thus $x \in z^{-1}D^P z$ for all z in D. □

Lemma 8

The value function $\phi: D^\bullet \to \Gamma$; defines a homomorphism and its kernel is the group of units $\overline{U}$ of $\overline{D}$, i.e., $\ker(\phi) = \big\{x \;/\; \phi(x) = 1\big\} = \overline{U}$.

Proof. ϕ is a homomorphism by Corollary 5, and as in [6] we can show that x is a unit in $\overline{D}$ if and only if $\phi(x) = 1$. □

Assume now that ϕ is a $*$-value function. From $\phi(x^*) = \phi(x)$, $\phi(x^*x^{-1}) = 1$ follows, i.e., $x^*x^{-1} \in \overline{D}$ for every x in D. For every $x, y \in D$ we have

$$xyx^{-1}y^{-1} = \left[(x^*)^*(x^*)^{-1}\right]\left[(y^*x)^*(y^*x)^{-1}\right]\left[y^*y^{-1}\right].$$

Hence $\overline{D}$ contains the commutator subgroup $[D, D]$ of $D^\bullet$. Thus we have shown the

Lemma 9

If ϕ is a $*$-value function, then the ring $\overline{D}$ is preserved under conjugation.

Lemma 10

If ϕ is a $*$-value function, then D^P is a $*$-valuation ring in D.

Proof. Since ϕ is a $*$-value function then $\Gamma \sim D^\bullet/\overline{U}$ is abelian. Hence $\overline{U}$ contains the commutator subgroup $[D, D]$ of D. But, then $D^P \supset \overline{U} \supset [D, D]$ (by Lemma 7), and D^P is preserved under conjugation. In view of Proposition 3, D^P is a $*$-valuation ring in D. □

Next consider the following subgroup of Γ:

$$U_0 := \left\{\phi(x) \in \Gamma \ / \ x \in D^P \text{ and } x^{-1} \in D^P\right\},$$

i.e., the image of the group of units of D^P in Γ. As in [6] one can show that U_0 is normal in Γ if and only if it is trivial.

Let V be a total subring of D. One can show that the left ideals of V are totally ordered by inclusion. This can be extended to the set $G := \left\{xV \ / \ x \in D\right\}$ of V-submodules of D. Define $\omega: D^\bullet \to G$ to be the canonical mapping, and write $\omega(w) \geq \omega(y)$ if and only if $xV \subseteq yV$. From [2, Theorem 4.4], ω has the following properties:

(1) $\omega(x) \geq \omega(y) \Rightarrow \omega(zx) \geq \omega(zy)$ for all z in $D^\bullet$;

(2) $\omega(x+y) \geq \min(\omega(x), \omega(y))$ $(x + y \neq 0)$;

(3) ω maps onto G; and

(4) $\omega(x) > \omega(1) \Rightarrow \omega(x^*) > \omega(1)$.

If ω satisfies $\omega(x^*) = \omega(x)$ for all x, then the set G becomes a commutative ordered group and ω also satisfies $\omega(xy) = \omega(x) + \omega(y)$; thus ω is a $*$-valuation (see [2, Theorem 4.6]).

Theorem 11

Let V be a $*$-closed total subring of D, P its maximal ideal. If Γ is the value group associated with (P, V), define

$$\psi : \Gamma \longrightarrow G$$
$$\phi(x) \longrightarrow \omega(x)$$

then ψ is a well defined map and the following are equivalent

(i) ϕ *is a $*$-value function;*
(ii) ω *is a $*$-valuation;*
(iii) U_0 *is trivial; and*
(iv) ψ *is injective.*

Proof. To show that ψ is well defined we assume that $\phi(x) = \phi(y)$ and so $\phi(xy^{-1}) = 1$. Suppose that $y^{-1}x \in V$ but $x^{-1}y \notin V$. Then $y^{-1}x \in P$ which yields $x^{-1}xy^{-1}x + (x^{-1}xy^{-1}x)^* \in P$. From $\phi(xy^{-1}) = 1$, it follows that $(x^{-1}x) + (x^{-1}x)^* \in P$, that is, $2 \in P$ a contradiction. Thus $y^{-1}x \in V$ and $x^{-1}y \in V$ so that $x \in yV$ and $y \in xV$; and $xV = yV$ follows.

(i) $\Rightarrow$ (ii) If $\phi(x^*) = \phi(x)$ then from above $\omega(x^*) = \omega(x)$ follows and ω is a $*$-valuation.

(ii) $\Leftrightarrow$ (iii) If ω is a $*$-valuation, then V is a $*$-valuation ring and so U_0 is normal. Hence U_0 is trivial. Assume now that U_0 is trivial, so the units of V are invariant under conjugation in D. If $p \in V$ is not a unit, then $p + 1$ is a unit and so, for $d \in D$, $d(p+1)d^{-1} \in V$. From $dpd^{-1} + 1 \in V$, it follows that $dpd^{-1} \in V$ and V is closed under conjugation. Hence V is a $*$-valuation ring.

(ii) $\Rightarrow$ (iv) If $xV = yV$ then $x = yz$ and $y = xz'$ for $z, z' \in V$. Therefore $y = yzz'$ which implies $zz' = 1$ and z is a unit in V. Now, if $\phi(x) \neq \phi(y)$, then we may assume that there is an element $(b, c) \in D \times D$ such that $bxc + (bxc)^* \in P$ but $byc + (byc)^* \notin P$. So $byc \notin P$ which yields $c^{-1}y^{-1}b^{-1} \in V$ (where V is total). Next we show that $a = bxc \in P$. Indeed, from the identity

$$a(1 + a^{-1}a^*) = a + a^*$$

we get $a = (a + a^*)(1 + a^{-1}a^*)^{-1} \in P$ (where $(1 + a^{-1}a^*)^{-1} \in V$ and V is a $*$-valuation ring). Now, we have $(c^{-1}y^{-1}b^{-1})(bxc) \in P$, that is $c^{-1}zc \in P$. Similarly, from z^{-1} is a unit in V we deduce that $c^{-1}z^{-1}c \in P$, and so $1 = (c^{-1}z^{-1}c)(c^{-1}zc) \in P$ which is a contradiction.

(iv) $\Rightarrow$ (iii) Let $\phi(x) \in U_0$, then x and $x^{-1} \in D^P$ and x is a unit in V. Hence $\omega(x) = 0$. Thus $\phi(x) = 1$ (where ψ is injective), and U_0 is trivial.

(iv) $\Rightarrow$ (i) If ψ is injective then U_0 is trivial and ω is a $*$-valuation. Thus from $\omega(x^*) = \omega(x)$ it follows that $\phi(x^*) = \phi(x)$ for every x in $D^\bullet$. Hence ϕ is a $*$-value function. □

Corollary 12

A $*$-closed total subring V of a $*$-field D is a $*$-valuation ring if and only if ψ is injective.

It was known from Theorem 1.8 in [6] that for skew fields which are finite dimensional over their center, all total subrings are valuation rings. This is false as J. Gräter [1] has given an explicit counter example to that theorem. Also the $*$-field version of that theorem; which asserts that for all $*$-fields which are finite dimensional over their center, all $*$-closed total subrings are $*$-valuation rings (Theorem 9 of [3]); is false, where counter examples have been provided by P.J. Morandi and A.R. Wadsworth. In [4], they construct Baer orderings on finite dimensional $*$-fields, of every possible finite dimension, such that every corresponding order subring fails to be a $*$-valuation ring. As the order subring of a Baer ordering is known to be $*$-closed and total, these examples are counter examples to Theorem 9 of [3].

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