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On a q-deformed harmonic oscillator with variable linear momentum

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ABSTRACT

A quantum mechanical model is introduced which includes variable momentum. This may be associated notionally with the variable moment of inertia model and is applied to give exact solutions to a q-deformed harmonic oscillator. The eigenfunctions are given in terms of a class of q-Hermite polynomials. When the base $q \to 1$, the classical case is recovered.

1. Introduction

Recently, a great deal of interest has been manifested in various q-deformed quantum systems, in particular the q-deformed harmonic oscillator in relation to the quantum group $SU_q(2)$. See, for example, [6]. Connections of the same group with the variable moment of inertia model have also been indicated by Bonatsos, Argyres, Drenska, Raychev and Rousev in [1]. These suggested the possibility of considering variable linear momentum models which might in certain cases yield exactly soluble quantum systems. In this study, a q-analogue of the harmonic oscillator is discussed and an analytic solution arises quite naturally.

For this purpose, the momentum operator in dimensionless form is replaced by

$$P_q = \left\{1 - \alpha x^2 (1 - q^2)\right\} \underset{q,x}{\mathrm{D}}, \tag{1.1}$$

in which the q-differential operator $\underset{q,x}{\mathrm{B}}$ is given by

$$\underset{q,x}{\mathrm{B}}\, y(x) = \frac{y(qx) - y(x)}{x(q-1)}. \tag{1.2}$$

F.H. Jackson was the first author to introduce a coherent notation on the subject of q-functions (see [4]). This has been developed by several mathematicians including Jain ([5]) and Exton ([3]) to which the reader is referred. It must be pointed out, however, that in this field, the notation is far from standardised.

The q-analogue of the governing equation of the harmonic oscillator considered here is

$$\left\{P_q^2 - \left(\lambda - \alpha x^2\right)\right\}\psi(x) = 0. \tag{1.3}$$

As is usual with the theory of q-functions, the corresponding classical form of any expression is recovered on putting $q = 1$. Hence, as expected, (1.2) then reduces to the ordinary differential operator and (1.3) to the usual form of the equation associated with the classical harmonic oscillator.

2. The solution of (1.3)

If (1.3) is expanded, bearing the rules of manipulation of q-derivatives ([3]), we have, after a little algebra,

$$\begin{aligned} &\left\{1 - \alpha x^2(1-q)\right\}\left\{1 - \alpha q^2 x^2(1-q)\right\} \underset{q,x}{\mathrm{B}}{}^2\, \psi(x) \\ &\quad - \alpha x(1-q^2)\left\{1 - \alpha x^2(1-q)\right\} \underset{q,x}{\mathrm{B}}\, \psi(x) = \left(\alpha x^2 - \lambda\right)\psi(x). \end{aligned} \tag{2.1}$$

The classical technique of making an exponential substitution in order to solve the differential equation governing the ordinary harmonic oscillator suggests that a similar approach using a suitable q-analogue of the exponential function should be made here. Hence, put

$$\psi(x) = E_{1/q^2}\left(\frac{-\alpha x^2}{1+q}\right)u(x), \tag{2.2}$$

where

$$E_{1/q}(x) = \sum_{m=0}^{\infty} \frac{x^m}{[m;q]!}\, q^{m(m-1)/2}, \tag{2.3}$$

and

$$[a;q] = [a] = \left(1 - q^a\right) / \left(1-q\right), \qquad [n;q]! = [1]\,[2]\cdots[n]. \tag{2.4}$$

The series (2.3) is convergent for all values of x if $|q| \leq 1$. After some manipulation, the left-hand member of (2.1) becomes

$$E_{1/q^2}\Big(\frac{-\alpha x^2}{1+q}\Big)\big\{\mathrm{B}^2 u - \alpha(1+q)x\,\mathrm{B}\,u - \alpha(1-\alpha x^2)u\big\} \tag{2.5}$$

and (2.1) then takes the form

$$\mathrm{B}^2 u(x) - \alpha(1+q)x\,\mathrm{B}\,u(x) = \big(\alpha - \lambda\big)u(x). \tag{2.6}$$

A power series solution of (2.6) is then possible (see [3, Chapter 2]). The even solution is found to be the q-Hermite function

$$u_1 = \sum_{r=0}^{\infty} \frac{\big[-\nu/2; q^2, r\big]\big(\alpha q^{\nu} x^2\big)^r}{\big[1/2; q^2, r\big]\big[r; q^2\big]!}, \tag{2.7}$$

where, for convenience, we have put $\lambda = \alpha\big\{1 + (1+q)[\nu; q]\big\}$. The q-Pochhammer symbol is given by

$$\big[a; q, n\big] = \big[a; q\big]\big[a+1; q\big]\big[a+2; q\big]\cdots\big[a+n-1; q\big], \qquad \big[a; q, 0\big] = 1. \tag{2.8}$$

The q-Hermite equations and its solutions have been discussed elsewhere (see [2] for example). If T_r is the r^{th} term of (2.7), then

$$T_{r+1} \,/\, T_r = \frac{\alpha q^{\nu} x^2\big[r - \nu/2; q^2\big]}{\big[r+1/2; q^2\big]\big[1+r; q^2\big]} = \frac{\alpha q^{\nu} x^2\big(1-q^{2r-\nu}\big)\big(1-q^2\big)}{\big(1-q^{2r+1}\big)\big(1-q^{2r+2}\big)}. \tag{2.9}$$

When $|q| < 1$,

$$\lim_{r\to\infty}\big(T_{r+1} \,/\, T_r\big) = \alpha q^{\nu} x^2\big(1-q^2\big) \tag{2.10}$$

and the series (2.7) then converges if

$$\big|x^2\big| < \frac{1}{\alpha q^{\nu} x^2\big(1-q^2\big)}. \tag{2.11}$$

When $|q| \geq 1$, (2.7) converges for all values of x.

Following the usual classical procedure, the boundary conditions require that the series representation of the eigenfunction must terminate, so that for the even solution, ν must be an even non-negative integer. Similarly, in the case of the odd solution, ν must be an odd positive integer. Hence, the eigenvalues $\{\lambda\}$ are given by

$$\lambda = \lambda_0\big\{1 + \big[2; q\big]\big[N; q\big]\big\}, \qquad N = 0, 1, 2, \ldots \tag{2.12}$$

which is an exact q-analogue of the classical result.

3. Conclusion

Unless $q = 1$, the classical case, the eigenvalues as given by (2.12) are not evenly spaced. The base q may assume any value, real or complex, but in the present context, it will be taken that q is real. If $|q| < 1$, the eigenvalues become successively more closed spaced, and reach a limiting value of

$$\lambda_0\left(1 + \frac{1+q}{1-q}\right). \tag{3.1}$$

When $|q| > 1$, the eigenvalues become progressively less closely spaced, and the above analysis remains substantially the same, except that $E_{1/q}(x)$ must be replaced by $1 \;/\; E_q(-x)$ for reasons of convergence (see [4]).

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