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On totally umbilical submanifolds of a locally Minkowski manifold

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Abstract

We study totally-umbilical submanifolds of Finslerian manifolds. Any complete totally-umbilical real hypersurface (whose induced and intrinsic connections coincide) of a locally Minkowski manifold is shown to have diameter $\leq \pi/a_0$.

1. Introduction

It is a classical result (cf. e.g. [26, vol II, p. 30]) that the only totally-umbilical real hypersurfaces of the Euclidean space are the (open pieces of) hyperplanes and hyperspheres. The reason is that the norm f of the mean curvature vector is a solution of the Codazzi equations (and then $f = f_0 =$ const); the case of the plane (sphere) occurs as $f_0 = 0$ ($f_0 \neq 0$). As to the corresponding statement in Finslerian geometry, only partial results are known, cf. O. Varga [37], M. Matsumoto [27].

Let $(M^n, L(x,y))$ be a real hypersurface of a locally Minkowski manifold M^{n+1}. Consider the following system of first order linear PDE's:

$$\frac{\partial f}{\partial x^i} - N_i^j(x,y)\frac{\partial f}{\partial y^j} = 0, \qquad 1 \leq i \leq n \tag{1.1}$$

where N_j^i are the coefficients of the nonlinear connection of the induced connection of M^n. If the induced and intrinsic (Cartan) connections of M^n coincide then N_j^i

are specified by:

$$N^i_j = \frac{1}{2}\frac{\partial}{\partial y^j}\left|{i \atop 0\,0}\right|, \qquad \left|{i \atop 0\,0}\right| = \left|{i \atop j\,k}\right| y^j y^k$$

$$\left|{i \atop j\,k}\right| = g^{im}|jk,m|, \qquad |ij,k| = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

where $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\,\partial y^j}$ and $g^{ij}g_{jk} = \delta^i_k$. When M^n is totally-umbilical in M^{n+1}, the norm of the mean curvature vector of the given immersion is shown to satisfy (1.1).

In [29] the authors find all solutions of (1.1) which are positive homogeneous of degree r, provided that M^n is a Finsler space of scalar curvature $K \neq 0$, an assumption which amounts to a prescribed form (cf. e.g. (26.4) in [28, p. 168]) of the obstruction:

$$R^i_{jk} = \frac{\partial N^i_j}{\partial x^k} - \frac{\partial N^i_k}{\partial x^j} + N^m_j\frac{\partial N^i_k}{\partial y^m} - N^m_k\frac{\partial N^i_j}{\partial y^m}$$

towards the complete integrability of the Pfaffian system:

$$dy^i + N^i_j(x,y)dx^j = 0.$$

As to the case of a totally-umbilical hypersurface, the R^i_{jk} torsion is given by (4.10). Section 2 reviews the material we need on induced bundles, Finslerian metrics, nonlinear connections and the Cartan connection of a Finslerian manifold. Section 3 reviews the imbedding Gauss-Codazzi equations of (M^n, L) in an ambient Finslerian manifold $(M^{n+p}, \overline{L})$. In Section 4 we exploit the structure of the horizontal (cf. (4.21)) and mixed (cf. (4.31)) Codazzi equations to show that the (norm of the) normal curvature is given by:

$$\|N_0\| = a_0 L^2$$

$a_0 = \text{const} > 0$ (and thus the mean curvature of a totally-umbilical hypersurface is constant). In particular, we prove that any totally-umbilical surface M^2 whose induced and intrinsic connections coincide is either totally-geodesic (and then locally Minkowski) or a Finsler space of negative scalar curvature $-a_0^2$. Along the way, we obtain a result on the topology of totally-umbilical hypersurfaces of a locally Minkowski manifold. There, the main ingredient is a theorem of F. Moalla, [31], on complete Finslerian manifolds with Ricci curvature $\geq e^2 > 0$. In Section 5 we study totally-umbilical CR submanifolds (in the sense of [18]) and extend a result of A. Bejancu, [7].

2. Finslerian manifolds and the Cartan connection

Let M^n be a real n-dimensional C^∞ differentiable manifold. Denote by $T(M^n) \to M^n$ the tangent bundle over M^n. Set $V(M^n) = T(M^n) - j(M^n)$, where $j: M^n \to T(M^n)$ denotes the natural imbedding of M^n in the total space of its tangent bundle, as the zero cross-section (i.e. $j(x) = 0_x \in T_x(M^n)$, $x \in M^n$). Let $\pi: V(M^n) \to M^n$ be the natural projection. Note that $V(M^n)$ is an open submanifold of $T(M^n)$.

If (U, x^i) is a local coordinate system on M^n, let $(\pi^{-1}(U), x^i, y^i)$ be the induced local coordinates on $V(M^n)$. Then x^i (respectively y^i) are referred to as the positional arguments (respectively directional arguments).

A Finsler energy E on M^n is a function $E: T(M^n) \to [0, +\infty)$ so that *i)* $E(u) = 0 \iff u \in j(M^n)$, *ii)* $E \in C^1(T(M^n))$, $E \in C^\infty(V(M^n))$, *iii)* $E(\lambda u) = \lambda^2 E(u)$ for any $\lambda > 0$, $u \in V(M^n)$, i.e. E is positive-homogeneous of degree 2, and *iv)* if $g_{ij} = \frac{1}{2}\frac{\partial^2 E}{\partial y^i \partial y^j}$, then $g_{ij}(u)\xi^i\xi^j$ is a positive-definite quadratic form, for any $u \in \pi^{-1}(U)$. A pair (M^n, E) is a Finslerian manifold. Its (fundamental) Lagrangian function is given by $L = E^{1/2}$, cf. H. Rund, [34]. For practical purposes, several violations of the axioms *i)*-*iv)* (in the definition of the concept of Finsler energy) are tacitly admitted. For instance, let (M^n, a) be a Riemannian manifold, $a \in \Gamma^\infty(S^2(T^*M^n))$, and $b \in \Gamma^\infty(T^*M^n)$, a given 1-form in M^n. We define the Randers metric $L: T(M^n) \to \mathbb{R}$ by $L(u) = a_x(u,u)^{1/2} + b_x(u)$, for any $u \in T_x(M^n)$, $x \in M^n$. Then (M^n, L^2) is a Finslerian manifold. Yet axiom *iv)* is not fully satisfied as $g_{ij}(u)$ has Lorenzian signature, cf. G. Randers, [33].

Let $\pi^{-1}TM^n \to V(M^n)$ be the pullback of $T(M^n)$ by π. One has a commutative diagram:

$$\begin{array}{ccc} \pi^{-1}TM^n & \longrightarrow & V(M^n) \\ \downarrow \hat{\pi} & & \downarrow \pi \\ T(M^n) & \longrightarrow & M^n \end{array}$$

Here $\hat{\pi}$ denotes the restriction to $\pi^{-1}TM^n$ of the natural projection $V(M^n) \times T(M^n) \to T(M^n)$. Cross-sections in $\pi^{-1}TM^n$ are Finsler vector fields on M^n. The Liouville vector is the Finsler vector field $v \in \Gamma^\infty(\pi^{-1}TM^n)$ defined by $v(u) = (u, u)$, for any $u \in V(M^n)$. Any tangent vector field $X: M^n \to T(M^n)$ admits a natural lift to a Finsler vector field $\overline{X}: V(M^n) \to \pi^{-1}TM^n$ given by $\overline{X}(u) = (u, X(\pi(u)))$, for any $u \in V(M^n)$. If (U, x^i) is a local coordinate system on M^n, let X_i denote the natural lifts of the (local) tangent vector fields $\frac{\partial}{\partial x^i}$ on U. Then $v = y^i X_i$ on $\pi^{-1}(U)$.

The induced bundle $\pi^{-1}TM^n \to V(M^n)$ of a Finslerian manifold (M^n, E) carries a Riemannian (bundle) metric g naturally associated with E. Indeed, let $u \in V(M^n)$. Set $x = \pi(u)$. Let (U, x^i) be a local coordinate neighborhood of x. Set

$g_u(X,Y) = g_{ij}(u)\xi^i\eta^i$, for any $X,Y \in \pi_u^{-1}TM^n$ where $X = \xi^i X_i(u)$, $Y = \eta^j X_j(u)$. Also $\pi_u^{-1}TM^n = \{u\} \times T_x(M^n)$ is the fibre over u in $\pi^{-1}TM^n$. The definition of $g_u(X,Y)$ does not depend on the choice of local coordinates around x.

Let $w_j(\pi^{-1}TM^n) \in H^j(V(M^n);\mathbb{Z}_2)$, $0 \le j \le n$, be the Stiefel-Whitney classes of the induced bundle $\pi^{-1}TM^n$. As v is global and nowhere vanishing it follows that $w_n(\pi^{-1}TM^n) = 1$, (cf. e.g. [30, p. 39]). Thus, in general, $\pi^*: H^n(M^n;\mathbb{Z}_2) \to H^n(V(M^n);\mathbb{Z}_2)$ is not one-to-one.

A nonlinear connection N on $V(M^n)$ is a C^∞ distribution:

$$N: u \in V(M^n) \longrightarrow N_u \subseteq T_u(V(M^n))$$

so that:

$$T_u(V(M^n)) = N_u \oplus \ker(d_u\pi)$$

for any $u \in V(M^n)$. We shall need the bundle morphism $F: T(V(M^n)) \to \pi^{-1}TM^n$ given by $F_uX = (u,(d_u\pi)X)$, for any $X \in T_u(V(M^n))$, $u \in V(M^n)$. If M^n carries a nonlinear connection N on $V(M^n)$, then $F_u: N_u \to \pi_u^{-1}TM^n$, $u \in V(M^n)$, is a $\mathbb{R}$-linear isomorphism. Let $\iota_u: N_u \to T_u(V(M^n))$, be the natural inclusion; set $\beta_u = (F_u \circ \iota_u)^{-1}$, $u \in V(M^n)$. The resulting bundle isomorphism $\beta: \pi^{-1}TM^n \to N$ is the horizontal lift associated with the nonlinear connection N. Set $\delta_i = \beta X_i$, $1 \le i \le n$. We adopt the notations $\partial_i = \partial/\partial x^i$, $\dot\partial_i = \partial/\partial y^i$. Note that there exist functions $N_j^i \in C^\infty(\pi^{-1}(U))$ so that:

$$\delta_i = \partial_i - N_i^j \dot\partial_j$$

In the more classical language of [39] a nonlinear connection N on $V(M^n)$ is therefore given by a Pfaffian system:

$$dy^i + N_j^i(x,y)dx^j = 0 \tag{2.1}$$

The n^2 functions N_j^i are referred to as the coefficients of the nonlinear connection N (with respect to $(\pi^{-1}(U), x^i, y^i)$), cf. A. Kawaguchi, [25]. A pair (∇, N) consisting of a connection ∇ in $\pi^{-1}TM^n$ and a nonlinear connection N on $V(M^n)$ is a Finsler connection on M^n. Let ∇ be a connection in $\pi^{-1}TM^n$. The following concept of torsion may be associated with ∇:

$$\hat{T}(X,Y) = \nabla_X FY - \nabla_Y FX - F[X,Y]$$

for any $X,Y \in \Gamma^\infty(T(V(M^n)))$.

Let $\gamma : \pi^{-1}TM^n \to T(V(M^n))$ be the bundle morphism given by $\gamma X_i = \dot{\partial}_i$. The definition of γ does not depend upon the choice of local coordinates. The vertical lift is the bundle isomorphism $\gamma : \pi^{-1}TM^n \to \ker(d\pi)$. The following short sequence of vector bundles and morphisms of vector bundles is exact:

$$(2.2) \qquad 0 \longrightarrow \pi^{-1}TM^n \xrightarrow{\gamma} T(V(M^n)) \xrightarrow{F} \pi^{-1}TM^n \longrightarrow 0$$

Let N be a nonlinear connection on $V(M^n)$ and $P_v : T(V(M^n)) \to \ker(d\pi)$ the natural projection. Set $K = \gamma^{-1} \circ P_v$. The resulting bundle morphism $K : T(V(M^n)) \to \pi^{-1}TM^n$ is the Dombrowski map, cf. [14]. Then

$$0 \longrightarrow \pi^{-1}TM^n \xrightarrow{\beta} T(V(M^n)) \xrightarrow{K} \pi^{-1}TM^n \longrightarrow 0$$

is a short exact sequence. Note that β is a splitting in (2.2). For more details see J. Vilms, [38]. Let (∇, N) be a Finsler connection on M^n. Another concept of torsion may be introduced as follows:

$$\hat{T}_1(X,Y) = \nabla_X KY - \nabla_Y KX - K[X,Y]$$

for any $X,Y \in \Gamma^\infty(T(V(M^n)))$. Using β, γ one decomposes the torsions $\hat{T}, \hat{T}_1$ of a given Finsler connection (∇, N) on M^n in several fragments, as follows $T(X,Y) = \hat{T}(\beta X, \beta Y))$, $C(X,Y) = \hat{T}(\gamma X, \beta Y)$, $R^1(X,Y) = \hat{T}_1(\beta X, \beta Y)$, $P^1(X,Y) = \hat{T}(\gamma X, \beta Y)$ and $S^1(X,Y) = \hat{T}_1(\gamma X, \gamma Y)$, for any $X,Y \in \Gamma^\infty(\pi^{-1}TM^n)$. As to the terminology, T, C are referred to as the horizontal and mixed components of $\hat{T}$. There is no "vertical component" of $\hat{T}$ as $\hat{T}(\gamma X, \gamma Y) = 0$ for any $X,Y \in \Gamma^\infty(\pi^{-1}TM^n)$. Note that

$$(2.3) \qquad R^1(X,Y) = -K[\beta X, \beta Y]$$

Thus R^1 depends on the nonlinear connection N alone and $R^1 = 0$ if and only if N is involutive. In local coordinates, if $R^1(X_i, X_j) = R^k_{ij} X_k$ then:

$$R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k.$$

Let ∇ be a connection in $\pi^{-1}TM^n$. Let N_∇ be the distribution consisting of all $X \in T(V(M^n))$ so that $\nabla_X v = 0$, where v is the Liouville vector. If N_∇ is a nonlinear connection on $V(M^n)$ then ∇ is termed regular. Cf. H. Akbar-Zadeh, [2]. Any regular connection ∇ in $\pi^{-1}TM^n$ gives rise to a Finsler connection (∇, N_∇) on M^n.

Let (M^n, E) be a Finslerian manifold and $(\pi^{-1}TM^n, g)$ its induced Riemannian bundle. A connection ∇ in $\pi^{-1}TM^n$ is metric (respectively v-metric) if $\nabla g = 0$ (respectively if $\nabla_{\gamma X} g = 0$ for any $X \in \Gamma^\infty(\pi^{-1}TM^n)$). A Finsler connection (∇, N) on M^n is h-metric if $\nabla_{\beta X} g = 0$ for any $X \in \Gamma^\infty(\pi^{-1}TM^n)$. By the fundamental theorem of Finsler geometry, there exists a unique regular connection ∇ in $\pi^{-1}TM^n$ so that *i)* ∇ is metric, *ii)* $T = S^1 = 0$. This is the Cartan connection of (M^n, E), cf. E. Cartan, [12], S.S. Chern, [13]. Its nonlinear connection N_∇ is the orthogonal complement of the vertical distribution $ker(d\pi)$ in $T(V(M^n))$ with respect to the Sasaki metric, i.e. the Riemannian metric G on $V(M^n)$ defined by:

$$G(X,Y) = g(FX, FY) + g(KX, KY)$$

for any $X, Y \in T(V(M^n))$.

Let (∇, N) be a Finsler connection on M^n. Denote by $\hat{R}$ the curvature 2-form of ∇. It may be decomposed in several fragments by setting $R(X,Y)Z = \hat{R}(\beta X, \beta Y)Z$, $P(X,Y)Z = \hat{R}(\gamma X, \beta Y)Z$ and $S(X,Y)Z = \hat{R}(\gamma X, \gamma\beta Y)Z$, for any $X, Y, Z \in \Gamma^\infty(\pi^{-1}TM^n)$. Note that $\hat{R}(X,Y)Z = R(FX, FY)Z + P(KX, FY)Z - P(KY, FX)Z + S(KX, KY)Z$, for any $X, Y \in \Gamma^\infty(T(V(M^n)))$, $Z \in \Gamma^\infty(\pi^{-1}TM^n)$.

Lemma 2.1

Let (M^n, E) be a Finslerian manifold and $(\pi^{-1}TM^n, g)$ its induced Riemannian bundle. Let ∇ be a v-metric connection in $\pi^{-1}TM^n$. If $S^1 = 0$ then, for any $X \in \Gamma^\infty(\pi^{-1}TM^n)$:

$$\nabla_{\gamma X} v = X \tag{2.4}$$

Proof. Indeed, if

$$\nabla_{\dot{\partial}_i} X_j = C^k_{ij} X_k$$

then $\nabla_{\dot{\partial}_i} g = 0$, $S^i_{jk} = C^i_{jk} - C^i_{kj} = 0$ and the Christoffel process yield:

$$C^i_{jk} = \tfrac{1}{2} g^{im} \dot{\partial}_j g_{km}$$

where $g^{ij} g_{jk} = \delta^i_k$. As g_{ij} are positive-homogeneous of degree 0 it follows that:

$$C^i_{jk} y^j = C^i_{jk} y^k = 0 \tag{2.5}$$

Therefore (2.4) is completely proved. □

Remark. Assume that ∇ is additionally regular. Then the meaning of (2.4) is that $Z \in \ker(d\pi) \to \nabla_Z v$ gives a bundle isomorphism $\ker(d\pi) \cong \pi^{-1}TM^n$ whose inverse is γ.

Lemma 2.2

Let (∇, N) be a Finsler connection on M^n so that i) ∇ is v-metric, ii) $S^1 = 0$, iii) $\nabla_{\beta X} v = 0$ for any $X \in \Gamma^\infty(\pi^{-1}TM^n)$. Then $N = N_\nabla$ and ∇ is regular.

Proof. Indeed, if $X \in (N_\nabla)_u \subseteq T_u(V(M^n)) = N_u \oplus \ker(d_u\pi)$ then $X = \beta Y + \gamma Z$ for some $Y, Z \in \pi_u^{-1}TM^n$. Next $0 = \nabla_X v = \nabla_{\gamma Z} v = Z$, i.e. $X \in N_u$. □

Let ∇ be a regular connection obeying (2.4). Then:

$$R(X,Y)v = R^1(x,y),$$
$$P(X,Y)v = P^1(X,Y), \qquad S(X,Y)v = S^1(X,Y) \tag{2.6}$$

for any $X, Y \in \Gamma^\infty(\pi^{-1}TM^n)$. Indeed, by (2.3) one may perform the following calculation:

$$R(X,Y)v = -\nabla_{[\beta X, \beta Y]} v = -\nabla_{P_v[\beta X,\beta Y]} v = -\gamma^{-1} P_v[\beta X, \beta Y] = R^1(X,Y)$$

In classical language, the torsions R^i_{jk}, P^i_{jk} and S^i_{jk} may be obtained from the horizontal, mixed and vertical curvature tensors R^i_{jkm}, P^i_{jkm} and S^i_{jkm} by contraction with the "supporting element" y^i (e.g. $R^i_{jk} = R^i_{jkm} y^m$). The properties (2.6) are of course enjoyed by the Cartan connection of (M^n, E). Nevertheless we chose to reformulate (2.6) for v-metric regular connections with $S^1 = 0$ since the main application we have in mind concerns the induced connection of an imbedded Finslerian manifold. This is both metric and regular, has a vanishing S^1 torsion tensor field, yet generally does not coincide with the "intrinsic" Cartan connection of the submanifold, cf. e.g. [15].

3. Imbedding equations

A Minkowski space is a real vector space V, $\dim_{\mathbb{R}} V = n$, carrying a Minkowskian norm $\|\xi\|$, $\xi \in V$, i.e. *i)* $\|\xi\| \geq 0$ and $\|\xi\| = 0 \iff \xi = 0$, *ii)* $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$, *iii)* $\|\lambda\xi\| = \lambda\|\xi\|$, $\lambda > 0$, $\xi \in V$ and *iv)* there is a basis $\{e_1, \ldots, e_n\}$ in V such that the function $f\colon \mathbb{R}^n \to [0, +\infty)$ defined by $f(y^1, \ldots, y^n) = \|y^i e_i\|$, for any $(y^1, \ldots, y^n) \in \mathbb{R}^n$, is smooth along $y \neq 0$, that is $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

Let $(V, \|\cdot\|)$ be a Minkowski space. Then the statement *iv)* holds for any other choice of linear basis in V. For practical purposes, several violations of the axioms *i)*–*iv)* are tacitly admited. For instance, let $V = \mathbb{R}^n$ and $\|\xi\| = \left(\prod_{i=1}^n y^i\right)^{1/n}$, where $\xi = (y^1, \ldots, y^n)$. This is the Berwald-Moór metric, cf. [5]. Note that *i)* is not satisfied. Also, if n is even and $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathbb{R}^n$, then f in *iv)* is not defined on the whole of $\mathbb{R}^n$.

Let (M^n, E) be a Finslerian manifold. Each tangent space $T_x(M^n)$, $x \in M^n$, has a natural structure of Minkowski space induced by E. Indeed, if $u \in T_x(M^n)$ we may set $\|u\|_x = E(u)^{1/2}$ and $\|\cdot\|_x$ is a Minkowski norm on $T_x(M^n)$.

Let $(V_1, \|\cdot\|)$, $(V_2, \|\cdot\|)$ be two Minkowski spaces. Then V_1, V_2 are congruent if there is a $\mathbb{R}$-linear isomorphism $f: V_1 \to V_2$ such that $\|f(\xi)\|_2 = \|\xi\|_1$, for any $\xi \in V_1$. Note that, given a Finslerian manifold (M^n, E), the tangent spaces at various points of M^n (regarded as Minkowski spaces) are generally not congruent. If this occurs (i.e. there is a Minkowski space $(V, \|\cdot\|)$, $\dim_{\mathbb{R}} V = n$, so that $(T_x(M^n), \|\cdot\|_x) \simeq (V, \|\cdot\|)$ for any $x \in M^n$) then (M^n, E) is termed a Finsler space modeled on a Minkowski space, cf. Y. Ichijyo, [23]. An example of Finsler space modeled on a Minkowski space is furnished by the concept of (V, H)-manifold. Let $(V, \|\cdot\|)$ be a n-dimensional Minkowski space and $G = \{T \in \mathrm{GL}(n, \mathbb{R}) : \|T\xi\| = \|\xi\|, \xi \in V\}$. Then G is a Lie group, cf. [23]. Let $H \subseteq G$ be a Lie subgroup. Let M^n be a real n-dimensional manifold carrying a H-structure $B \to M^n$. Then M^n is termed a (V, H)-manifold. One endows (M^n, B) with a Finsler energy as follows. Let $u \in T_x(M^n)$, $x \in M^n$. Let (U, x^i) be a local coordinate neighborhood of x and let $\{X_1, \ldots, X_n\}$ be a cross-section of B defined on U (i.e. a local frame adapted to the H-structure). Then $u = \xi^i X_i$ and we define $L: T(M^n) \to \mathbb{R}$ by $L(u) = \|\xi^i e_i\|$, where $\{e_1, \ldots, e_n\}$ is a fixed basis in V. The definition of L does not depend upon the choice of adapted frame. If $u = y^i(u)\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial x^i} = A_i^j X_j$, the Lagrangian L of a (V, H)-manifold may be also written:

$$L(x, y) = \left\|y^i A_i^j(x) e_j\right\| \tag{3.1}$$

Let (M^n, E) be a Finslerian manifold. Then M^n is a locally Minkowski manifold if there is a C^∞ atlas on M^n with respect to which E depends only on directional arguments. Any Minkowski space is a locally Minkowski manifold, in a natural way. A (V, H)-manifold (M^n, B) is locally Minkowski if and only if the H-structure is integrable, cf. [24, p. 14].

Let $(M^{n+p}, \overline{E})$ be a real $(n+p)$-dimensional Finslerian manifold, $p \geq 1$. Let $\psi: M^n \to M^{n+p}$ be an immersion of a Finslerian manifold (M^n, E) in M^{n+p}. Assume ψ to be isometric, i.e.

$$E(u) = \overline{E}((d\psi)u) \tag{3.2}$$

for any $u \in T(M^n)$. Clearly $d\psi: V(M^n) \to V(M^{n+p})$ is an immersion, as well. The identity (3.2) may be locally written:

$$E(x, y) = \overline{E}\left(\psi^\alpha(x), \frac{\partial \psi^\alpha}{\partial x^i}(x) y^i\right) \tag{3.3}$$

where $u^\alpha = \psi^\alpha(x^1, \dots, x^n)$, $1 \le \alpha \le n+p$, are the local equations of M^n in M^{n+p}. One denotes by (u^α) a local coordinate system on M^{n+p}, while (u^α, v^α) will be the induced local coordinates on $V(M^{n+p})$.

Let $(\pi^{-1}TM^{n+p}, \overline{g})$ be the induced Riemannian bundle of $(M^{n+p}, \overline{E})$. The natural projection $V(M^{n+p}) \to M^{n+p}$ and $V(M^n) \to M^n$ are denoted by the same symbol π. Set:

$$g_{\alpha\beta} = \tfrac{1}{2}\frac{\partial^2 \overline{E}}{\partial v^\alpha \partial v^\beta}$$

Taking derivates of (3.3) with respect to the directional arguments y^i one obtains:

$$g_{ij} = g_{\alpha\beta}\frac{\partial \psi^\alpha}{\partial x^i}\frac{\partial \psi^\beta}{\partial x^j}$$

and consequently $(D\psi)_u : \pi_u^{-1}TM^n \to \pi_{(d\psi)u}^{-1}TM^{n+p}$ defined by

$$(D\psi)_u X = \big((d\psi)u, (d\psi)\tilde{\pi}X\big)$$

for $X \in \pi^{-1}TM^n$, is an isometry of $(\pi^{-1}TM^n, g_u)$ into $(\pi_{(d\psi)u}^{-1}TM^{n+p}, \overline{g}_{(d\psi)u})$.

Let $E(\psi)_u$ be the orthogonal complement of $(D\psi)_u\pi^{-1}TM^n$ in $\pi_{(d\psi)u}^{-1}TM^{n+p}$ with respect to $\overline{g}_{(d\psi)u}$, for any $u \in V(M^n)$. The resulting rank p vector bundle $E(\psi) \to V(M^n)$ is the normal bundle of the given immersion. Then:

$$\pi^{-1}TM^{n+p} = (D\psi)\pi^{-1}TM^n \oplus E(\psi) \tag{3.4}$$

As customary, from now on we shall not distinguish notationally between x and $\psi(x)$, u and $(d\psi)u$, X and $(D\psi)X$, Z and $\big(d(d\psi)\big)Z$, etc. Here $x \in M^n$, $u \in T(M^n)$, $X \in \pi^{-1}TM^n$, $Z \in T(V(M^n))$.

Let $\overline{\nabla}$ be the Cartan connection of $(M^{n+p}, \overline{E})$. We recall (cf. e.g. (1.1) in [15, p. 3] or (3.1) in [1, p. 276]) the Gauss and Weingarten formulae:

$$\overline{\nabla}_X Y = \nabla_X Y + \hat{H}(X,Y) \tag{3.5}$$

$$\overline{\nabla}_X \xi = -\hat{A}_\xi X + \nabla_X^\perp \xi \tag{3.6}$$

for any $X \in \Gamma^\infty(T(V(M^n)))$, $Y \in \Gamma^\infty(\pi^{-1}TM^n)$, $\xi \in \Gamma^\infty(E(\psi))$. Here ∇, $\hat{H}$, $\hat{A}_\xi$ and $\nabla^\perp$ are respectively the induced connection, the second fundamental form (of ψ), the Weingarten operator (associated with the normal section ξ) and the normal connection (in $E(\psi)$). Note that, for any $\xi \in \Gamma^\infty(E(\psi))$, $\hat{A}_\xi$ is a cross-section in $T^*(V(M^n)) \otimes \pi^{-1}TM^n$.

Let $v, \overline{v}$ be the Liouville vectors of M^n, M^{n+p} respectively. Then $(D\psi)_u v(u) = \overline{v}((d_x\psi)u)$, for any $u \in V(M^n)$, $x = \pi(u)$. Therefore, the customary assumption that M^n is tangent to the "supporting element" of M^{n+p} (cf. e.g. [27, p. 108]) is superfluous. In the sequel, we do not distinguish notationally between v and $\overline{v}$.

Let $\overline{F}: T(V(M^{n+p})) \to \pi^{-1}TM^{n+p}$ be the bundle morphism induced by $d\pi: T(V(M^{n+p})) \to T(M^{n+p})$. The following diagram has commutative squares:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^{-1}TM^n & \xrightarrow{\gamma} & T(V(M^n)) & \xrightarrow{F} & \pi^{-1}TM^n & \longrightarrow & 0 \\ & & \downarrow D\psi & & \downarrow d(d\psi) & & \downarrow D\psi & & \\ 0 & \longrightarrow & \pi^{-1}TM^{n+p} & \xrightarrow{\overline{\gamma}} & T(V(M^{n+p})) & \xrightarrow{\overline{F}} & \pi^{-1}TM^{n+p} & \longrightarrow & 0 \end{array}$$

We shall not distinguish notationally between γ and $\overline{\gamma}$, respectively F and $\overline{F}$.

For any $X, Y \in \Gamma^\infty(\pi^{-1}TM^n)$ set:

$$Q(X,Y) = \hat{H}(\gamma X, Y) \tag{3.7}$$

Then Q is the vertical second fundamental form (of ψ). Set $Y = v$ in (3.5). Then:

$$\overline{\nabla}_{\gamma X} v = \nabla_{\gamma X} v + Q(x, v)$$

for any $X \in \Gamma^\infty(\pi^{-1}TM^n)$. As $\overline{\nabla}$ is v-metric and $\overline{S}^1 = 0$, it follows that $\overline{\nabla}$ enjoys the property (2.4). Thus:

$$\nabla_{\gamma X} v = X \tag{3.8}$$

$$Q(X, v) = 0 \tag{3.9}$$

Set:

$$W_\xi X = \hat{A}_\xi \gamma X$$

Then W_ξ is the vertical Weingarten operator. It is related to Q by the identity:

$$\overline{g}(Q(X,Y), \xi) = g(W_\xi X, Y)$$

for any $X, Y \in \Gamma^\infty(\pi^{-1}TM^n)$, $\xi \in \Gamma^\infty(E(\psi))$. As $\overline{S}^1 = 0$ it follows (by (3.5)) that $S^1 = 0$ and Q is symmetric (so that $(W_\xi)_u: \pi_u^{-1}TM^n \to \pi_u^{-1}TM^n$ is self-adjoint with respect to g_u, $u \in V(M^n)$).

Lemma 3.1

Let $(M^{n+p}, \overline{E})$ and (M^n, E) be two Finslerian manifolds and $\psi: M^n \to M^{n+p}$ an isometric immersion. Let ∇ be the induced connection in $(\pi^{-1}TM^n, g)$ and $N_\nabla = \{X : \nabla_X v = 0\}$. Then N_∇ is a nonlinear connection on $V(M^n)$ (and therefore ∇ is regular).

Proof. Indeed, if $Y \in N_u \cap V_u$, $u \in V(M^n)$ then $Y = \gamma X$ for some $X \in \pi_u^{-1}TM^n$ and $0 = \nabla_Y v = \nabla_{\gamma X} v = X$, so that the sum $N_u + V_u$ is direct. As $V_u = \ker(d_u\pi) \subseteq T_u(V(M^n))$ and $\dim_{\mathbb{R}} V_u = n$, it is sufficient to check that $\dim_{\mathbb{R}} N_u = n$. Let ∇ be an arbitrary connection in $\pi^{-1}TM^n$. Let $X \in T(V(M^n))$. In local coordinates $X = A^i\partial_i + B^i\dot{\partial}_i$. Then $\nabla_X v = 0$ is equivalent to:

$$A^i\Gamma^k_{i0} + B^i(\delta^k_i + y^jC^k_{ij}) = 0 \tag{3.10}$$

where $\nabla_{\partial_i}X_j = \Gamma^k_{ij}X_k$, $\Gamma^k_{i0} = \Gamma^k_{ij}y^j$. If ∇ obeys to (2.5) then (3.10) yields:

$$B^k = -A^i\Gamma^k_{i0}$$

and therefore $(N_\nabla)_u$ is spanned by the tangent vectors $(\partial_i - \Gamma^j_{i0}\dot{\partial}_j)(u)$. Thus N_∇ is a C^∞-differentiable n-distribution. Clearly, these considerations apply to the case of the induced connection. Thus, the induced connection is regular. □

Let $\beta: \pi^{-1}TM^n \to N$ be the corresponding horizontal lift. Set:

$$H(X,Y) = \hat{H}(\beta X, Y) \tag{3.11}$$

for any $X, Y \in \Gamma^\infty(\pi^{-1}TM^n)$. Then H is the horizontal second fundamental form. It is related to the horizontal Weingarten operator:

$$A_\xi X = \hat{A}_\xi \beta X$$

by the identity:

$$\overline{g}(H(X,Y),\xi) = g(A_\xi X, Y)$$

Set $N(X) = H(X,v)$, $N_0 = N(v)$. Cf. [27], N and N_0 are termed the normal curvature vector and the normal curvature, respectively. Let $\overline{\beta}: \pi^{-1}TM^{n+p} \to \overline{N}$ be the horizontal lift corresponding to $\overline{N} = N_{\overline{\nabla}}$, i.e. to the nonlinear connection of the Cartan connection of $(M^{n+p}, \overline{E})$. Then:

$$\beta X = \overline{\beta} X + \gamma N(X) \tag{3.12}$$

for any $X \in \Gamma^\infty(\pi^{-1}TM^n)$. Indeed, let $X \in \pi^{-1}TM^n$. Then $\beta X \in N_u \subseteq T_u(V(M^n)) \subseteq T_u(V(M^{n+p})) = \overline{N}_u \oplus \overline{V}_u$ so that $\beta X \doteq \overline{\beta}\overline{Y} + \gamma\overline{Z}$ for some $\overline{Y}, \overline{Z} \in \pi_u^{-1}TM^{n+p}$. Here N_u is short for $(N_\nabla)_u$. Applying F leads to $\overline{Y} = X$, (as $F \circ \beta =$ identity, $F \circ \gamma = 0$). Next:

$$0 = \nabla_{\beta X} v = \overline{\nabla}_{\beta X} v - H(X,v) = \overline{\nabla}_{\gamma\overline{Z}} v - N(X) = \overline{Z} - N(X)$$

so that $\overline{Z} = N(X)$, and (3.12) is completely proved. From (3.12) and $\overline{T} = 0$ (by the Gauss formula (3.5)) one may derive:

$$T(X,Y) + H(X,Y) - H(Y,X) = \overline{C}(N(X),Y) - \overline{C}(N(Y),X) \tag{3.13}$$

for any $X, Y \in \Gamma^\infty(\pi^{-1}TM^n)$. Thus, the induced connection ∇ satisfies all axioms determining the Cartan connection of (M^n, E) except for $T = 0$. From (3.13) it follows that generally $T \neq 0$. Also H fails to be symmetric, in general; accordingly A_ξ is not self-adjoint.

Let $\overline{B}$, B be the curvature 2-forms of the Cartan connection $\overline{\nabla}$ of $(M^{n+p}, \overline{E})$, respectively of the induced connection ∇. Consider the horizontal, mixed and vertical components $\overline{R}$, $\overline{P}$ and $\overline{S}$ of $\overline{B}$ (built with the use of $\overline{\beta}$, γ), respectively the fragments R, P and S of B (built with the use of β in (3.12) and γ). As a consequence of (3.5)–(3.6) one obtains:

$$\begin{aligned}(3.14)\quad & \overline{R}(X,Y)Z + \overline{P}(N(X),Y)Z - \overline{P}(N(Y),X)Z + \overline{S}(N(X),N(Y))Z \\ &= R(X,Y)Z + A_{H(X,Z)}Y - A_{H(Y,Z)}X + (\nabla_{\beta X}H)(Y,Z) \\ &\quad - (\nabla_{\beta Y}H)(X,Z) + H(T(X,Y),Z) + Q(R^1(X,Y),Z) \\ (3.15)\quad & \overline{P}(X,Y)Z + \overline{S}(X,N(Y))Z \\ &= P(X,Y)Z + A_{Q(X,Z)}Y - W_{H(Y,Z)}X + (\nabla_{\gamma X}H)(Y,Z) \\ &\quad - (\nabla_{\beta Y}Q)(X,Z) + H(C(X,Y),Z) + Q(P^1(X,Y),Z) \\ (3.16)\quad & \overline{S}(X,Y)Z = S(X,Y)Z + W_{Q(X,Z)}Y - W_{Q(Y,Z)}X \\ &\quad + (\nabla_{\gamma X}Q)(Y,Z) - (\nabla_{\gamma Y}Q)(X,Z)\end{aligned}$$

for any $X, Y, Z \in \Gamma^\infty(\pi^{-1}TM^n)$. Cf. also [16, p. 90] or [1, p. 277–288].

4. Umbilical submanifolds

Let $\psi: M^n \to M^{n+p}$ be an isometric immersion of a Finslerian manifold (M^n, E) into another $(M^{n+p}, \overline{E})$. Then ψ is totally-umbilical if $H = g \otimes \mu$ where $\mu = \text{trace}(H)/n$ is the mean curvature vector of ψ.

If $L = E^{1/2}$ is the Lagrangian of M^n, define the Finsler 1-form $\iota = (dL) \circ \gamma$, $\iota \in \Gamma^\infty(\pi^{-1}T^*M^n)$. Note that $(\pi_u^{-1}TM^n)^* \cong \pi_u^{-1}T^*M^n$ (a $\mathbb{R}$-linear isomorphism), for any $u \in V(M^n)$. Here $\pi^{-1}T^*M^n \to V(M^n)$ is the pullback of the cotangent bundle $T^*(M^n)$ by π. If $\iota_i = \iota X_i$ then $\iota_i = \partial L/\partial y^i$. Finally, note that:

$$g(X, v) = L\,\iota(X) \tag{4.1}$$

for any $X \in \Gamma^\infty(\pi^{-1}TM^n)$. Repeated contraction with the supporting element in $H = g \otimes \mu$ gives $N = L\iota \otimes \mu$ and $N_0 = L^2\mu$ (as $\iota(v) = L$). The condition that ψ is totally-umbilical is customary written:

(4.2) $$H = L^{-2} g \otimes N_0$$

If this is the case then, for any $X \in \Gamma^\infty(\pi^{-1}TM^n)$, $\xi \in \Gamma^\infty((E(\psi))$, one has:

$$A_\xi X = L^{-2}\,\overline{g}(N_0, \xi)\, X$$

If ψ is totally-umbilical then (3.13) furnishes:

$$T(X,Y) = \overline{C}(N(X),Y) - \overline{C}(N(Y),X)$$

or:

(4.3) $$LT = 2\iota \wedge \overline{C}(N_0, \cdot)$$

One problem in Finsler geometry (cf. e.g. B.T.M. Hassan, [22]) is to classify the submanifolds of $(M^{n+p}, \overline{E})$ for which the induced connection and the intrinsic Cartan connection of (M^n, E) coincide (i.e. $T = 0$). If this is the case then (4.3) yields:

(4.4) $$\overline{C}(N_0, \cdot) = \lambda\,\iota$$

for some $\lambda \in C^\infty(V(M^{n+p}))$. Yet $0 = \overline{C}(N_0, v) = \lambda\,\iota(v) = \lambda L$ so that $\lambda = 0$ and (4.4) reduces to:

(4.5) $$H^\alpha_{00} C^\rho_{\alpha\beta} = 0$$

where $H(X_i, X_j) = H^\alpha_{ij} X_\alpha$ and $H^\alpha_{00} = H^\alpha_{ij} y^i y^j$. It is an open problem whether totally-umbilical submanifolds (with $T = 0$) may be classified via (4.5). We proceed with several simplifying assumptions. Recall (cf. [28, p. 159]) that a Finslerian manifold (M^n, E) is locally Minkowski if and only if $R_{ijkm} = 0$ and $C_{ijk|m} = 0$. Short bars indicate h-covariant derivates. Also, by (17.22)-(17.23) in [28, p. 144] it follows that $P_{ijk} = 0$, $P_{ijkm} = 0$. Let $(M^{n+p}, \overline{E})$ be a locally Minkowski manifold. Then $\overline{R} = 0$, $\overline{P} = 0$, $\overline{P}^1 = 0$ and $\overline{\nabla}_X \overline{C} = 0$, for any $X \in \Gamma^\infty(\overline{N})$. Let $\psi: M^n \to M^{n+p}$ be a totally-umbilical isometric immersion. The Gauss-Codazzi equations (3.14) of M^n in M^{n+p} become:

(4.6) $$R(X,Y)Z = A_{H(Y,Z)}X - A_{H(X,Z)}Y$$

(4.7) $$(\nabla_{\beta Y} H)(X,Z) - (\nabla_{\beta X} H)(Y,Z) = H(T(X,Y),Z) + Q(R^1(X,Y),Z)$$

for any $X, Y, Z \in \Gamma^{\infty}(\pi^{-1}TM^n)$. Taking into account (4.2) and the identity:

$$A_{N_0}X = L^{-2}\|N_0\|^2 X \tag{4.8}$$

the Gauss equation (4.6) may be written:

$$R(X,Y)Z = L^{-4}\|N_0\|^2\left\{g(Y,Z)X - g(X,Z)Y\right\} \tag{4.9}$$

Set $Z = v$ in (4.9) and use (2.6) so that obtain:

$$R^1(X,Y) = L^{-3}\|N_0\|^2\left\{\iota(Y)X - \iota(X)Y\right\} \tag{4.10}$$

In [1, p. 279] the authors introduced the horizontal scalar curvature r of (M^n, E), $r = g^{ij}R_{ij}$, $R_{jk} = R^i_{ijk}$. From (4.9) it follows that:

Proposition 4.1

Any totally-umbilical submanifold M^n of a locally Minkowski manifold M^{n+p} has nonnegative horizontal scalar curvature:

$$r = n(n-1)L^{-4}\|N_0\|^2 \tag{4.11}$$

Let (∇, N) be a Finsler connection on (M^n, E). Let $K \in C^{\infty}(V(M^n))$ be positive-homogeneous of degree 0 in the y^i's. Then (∇, N) is said to be a Finsler connection of scalar curvature K if its R^1 torsion tensor field is given by:

$$R^i_{jk} = K_j h^i_k - K_k h^i_j \tag{4.12}$$

where $h^i_j = \delta^i_j - \iota^i \iota_j$, $\iota^i = y^i/L$, and K_i is given by:

$$3K_i = L\left(L\frac{\partial K}{\partial y^i} + 3K\iota_i\right) \tag{4.13}$$

This slightly generalizes the situation in [29, p. 552]. Indeed, if the Cartan connection of (M^n, E) is of scalar curvature then (M^n, E) is a Finsler space of scalar curvature.

Proposition 4.2

Let $\psi: M^n \to M^{n+p}$ be an isometric immersion between two Finslerian manifolds (M^n, E) and $(M^{n+p}, \overline{E})$. If ψ is totally umbilical and the induced connection of M^n in M^{n+p} is of scalar curvature K then:

$$K = -L^{-4}\|N_0\|^2 \tag{4.14}$$

Proof. Indeed, comparing (4.10), (4.12) we obtain:

$$L^{-3}f^2\left(\iota_k\delta^i_j - \iota_j\delta^i_k\right) = K_j h^i_k - K_k h^i_j$$

where $f = \|N_0\|$. Contraction with y^k gives:

$$L^{-3}f^2\left(L\delta^i_j - \iota_j y^i\right) = -K_k y^k h^i_j \tag{4.15}$$

as $h^i_k y^k = 0$. Note that (4.13) gives $K_i y^i = L^2 K$. This and suitable contraction of indices in (4.15) lead to (4.14). □

By the Euler theorem on positive-homogeneous functions $\|v\| = L$. The induced connection is metric so that:

$$0 = \left(\nabla_{\beta X} g\right)(v, v) = (\beta X)\left(\|v\|^2\right) - 2g\left(\nabla_{\beta X} v, v\right) = 2\iota(\beta X)(L)$$

It follows that:

$$(dL) \circ \beta = 0 \tag{4.16}$$

From (4.2) and (4.16) we obtain:

$$\nabla_{\beta X} II = L^{-2} g \otimes \nabla^{\perp}_{\beta X} N_0 \tag{4.17}$$

for any $X \in \Gamma^\infty\left(\pi^{-1}TM^n\right)$. Of course (4.17) holds for a totally-umbilical submanifold in an arbitrary ambient Finslerian manifold. As to the Codazzi equation (4.7), by (4.17) it may be written:

$$\begin{aligned} g(X,Z)\nabla^{\perp}_{\beta Y} N_0 &- g(Y,Z)\nabla^{\perp}_{\beta X} N_0 \\ &= L^2\left\{II\left(T(X,Y),Z\right) + Q\left(R^1(X,Y),Z\right)\right\} \end{aligned} \tag{4.18}$$

for any $X, Y \in \Gamma^\infty\left(\pi^{-1}TM^n\right)$.

The vertical distribution $V = \gamma(\pi^{-1}TM^n)$ is involutive. Thus:

$$F[\gamma X, \gamma H(Y,v)] = 0$$

Using this and the Gauss formula (3.5) one may derive:

$$\overline{C}(X,Y) = C(X,Y) + Q(X,Y) \tag{4.19}$$

for any $X,Y \in \Gamma^\infty(\pi^{-1}TM^n)$. Taking into account (4.3), the symmetries of the Cartan tensor $\overline{C}$, and (4.19), it follows that:

$$\begin{aligned} &II(T(X,Y),Z) \\ &= L^{-3}\Big\{\iota(X)\overline{g}(Q(Y,Z),N_0) - \iota(Y)\overline{g}(Q(X,Z),N_0)\Big\}N_0 \end{aligned} \tag{4.20}$$

As a consequence of (4.10), (4.20), the Codazzi equation (4.18) of a totally-umbilical submanifold M^n in a locally Minkowski manifold M^{n+p} becomes:

$$\begin{aligned} &L\Big\{g(X,Z)\nabla^\perp_{\beta Y}N_0 - g(Y,Z)\nabla^\perp_{\beta X}N_0\Big\} \\ &= \iota(Y)\Big\{\|N_0\|^2 Q(X,Z) - \overline{g}(Q(X,Z),N_0)N_0\Big\} \\ &\quad - \iota(X)\Big\{\|N_0\|^2 Q(Y,Z) - \overline{g}(Q(Y,Z),N_0)N_0\Big\} \end{aligned} \tag{4.21}$$

for any $X,Y,Z \in \Gamma^\infty(\pi^{-1}TM^n)$.

Let ∇ be a connection in $\pi^{-1}TM^n$. A regular curve $\alpha: (-\varepsilon,\varepsilon) \to M^n$, $\varepsilon > 0$, is an autoparallel curve of ∇ if:

$$\nabla_{\frac{dV(\alpha)}{dt}} v = 0$$

along α. Here $V(\alpha)$ is the natural lift of α, i.e. the curve $V(\alpha): (-\varepsilon,\varepsilon) \to V(M^n)$, defined by $V(\alpha)(t) = \frac{d\alpha}{dt}(t)$, $|t| < \varepsilon$, while v is the Liouville vector. In other words, α is autoparallel if its natural lift $V(\alpha)$ is a horizontal curve, i.e.

$$\frac{dV(\alpha)}{dt}(t) \in (N_\nabla)_{V(\alpha)(t)}, \qquad |t| < \varepsilon$$

Let N be a nonlinear connection on $V(M^n)$. A curve $C: (-\varepsilon,\varepsilon) \to V(M^n)$, $\varepsilon > 0$, is a N-path if:

$$\frac{dC}{dt}(t) \in N_{C(t)}, \qquad |t| < \varepsilon$$

Thus an autoparallel curve of a regular connection ∇ is a regular curve in M^n whose natural lift is a N_∇-path. A geodesic of a Finslerian manifold (M^n, E) is by definition an autoparallel curve geodesic of its Cartan connection. Any geodesic of a Minkowski space V is a straight line in V.

Let $\psi: M^n \to M^{n+p}$ be an isometric immersion of (M^n, E) in $(M^{n+p}, \overline{E})$. Let ∇, ∇' be the induced and the intrinsic (Cartan) connections of (M^n, E) respectively. Then ∇, ∇' have the same autoparallel curves (cf. [22]). The isometric immersion ψ is totally-geodesic if any geodesic of (M^n, E) is also a geodesic of $(M^{n+p}, \overline{E})$.

EXAMPLES.

1) Any hyperplane in a Minkowski space V is a totally-geodesic submanifold of V. It is an open problem whether the converse holds.
2) Let $F: M^n \to M^n$ be an isometry of a Finslerian manifold (M^n, E), i.e. F is a C^∞-diffeomorphism and:
$$E((d_xF)u) = E(u)$$
for any $u \in V(M^n)$, $x \in \pi(u)$. Let $K = \{x \in M^n : F(x) = x\}$ be the fixed point set of F. Any connected component L of K is totally-geodesic in (M^n, E), cf. [17]. The geometry of the second fundamental form of L in (M^n, E) has not, as yet, been studied.
3) By a result of M.G. Brown, [10], if M^n is a real hypersurface (i.e. $p = 1$) then ψ is totally-geodesic if and only if $N_0 = 0$. By a result of O. Varga, [37], $N_0 = 0$ yields $N = 0$, as well (provided that $p = 1$). Yet $\Pi \neq 0$ in general.

Proposition 4.3

Let (M^n, E) be a totally-umbilical real hypersurface ($p = 1$) of the locally Minkowski manifold $(M^{n+1}, \overline{E})$. Then the horizontal scalar curvature of M^n vanishes if and only if M^n is totally-geodesic in M^{n+1}.

Proof. We distinguish two cases. Either $N_0 = 0$, and then M^n is totally-geodesic in M^{n+1}, or $N_0(u_0) \neq 0$ for some $u_0 \in V(M^n)$. If this is the case, there is an open neighborhood U of $x_0 = \pi(u_0)$ in M^n so that $N_0(u) \neq 0$ for any $u \in \pi^{-1}(U)$. In dealing with this case, as all our considerations are local in character, we may assume that $N_0 \neq 0$ everywhere on $V(M^n)$. Let then $f = \|N_0\|$ and choose $f^{-1}N_0 \in \Gamma^\infty(E(\psi))$ as unit normal on M^n. Then $\nabla^\perp(f^{-1}N_0) = 0$ and the Codazzi equation (4.21) turns into:
$$g(X,Z)(\beta Y)(f) - g(Y,Z)(\beta X)(f) = 0$$
For arbitrary X take $Y = Z$, $\|Y\| = 1$, Y orthogonal on X. It follows that:
$$(df) \circ \beta = 0 \tag{4.22}$$
Therefore, locally, f must be a solution of the following system of PDE's:
$$\frac{\partial f}{\partial x^i} - N_i^j(x,y)\frac{\partial f}{\partial y^j} = 0 \tag{4.23}$$
See [29, p. 553], where all solutions of (4.23) are determined, provided that N_j^i are the coefficients of the nonlinear connection of the Cartan connection of a Finsler space

of nonvanishing scalar curvature. As for totally umbilical hypersurfaces of locally Minkowski manifolds, we may solve (4.23) by using the Gauss-Codazzi equations (3.15), as follows. Let M^n be a submanifold of a locally Minkowski manifold M^{n+p}. No assumption on the codimension is necessary as yet. Then (3.15) turns into:

$$\overline{S}(X, N(Y))Z = P(X,Y)Z + A_{Q(X,Z)}Y - W_{H(Y,Z)}X + (\nabla_{\gamma X} H)(Y,Z) - (\nabla_{\beta Y} Q)(X,Z) + H(C(X,Y),Z) + Q(P^1(X,Y),Z) \tag{4.24}$$

As a consequence of (3.9) we have the identity:

$$(\nabla_{\beta X} Q)(Y,v) = 0 \tag{4.25}$$

Set $Z = v$ in (4.24) and use (3.9), (4.25) and $\overline{S}^1 = 0$ such as to yield:

$$P^1(X,Y) = W_{N(Y)}X \tag{4.26}$$

$$(\nabla_{\gamma X} H)(Y,v) + H(C(X,Y),v) = 0 \tag{4.27}$$

for any $X, Y \in \Gamma^\infty(\pi^{-1}TM^n)$. Note that (3.8) gives:

$$(\nabla_{\gamma X} H)(Y,v) = (\nabla_{\gamma X} N)Y - H(Y,X) \tag{4.28}$$

Substitution from (4.28) into (4.27) furnishes:

$$(\nabla_{\gamma X} N)Y = H(Y,X) - N(C(X,Y)) \tag{4.29}$$

Assume now that M^n is totally umbilical. Then $N = L^{-1}\iota \otimes N_0$ and $\iota = (dL) \circ \gamma$ give the identity:

$$(\nabla_{\gamma X} N)Y = L^{-1}((\nabla_{\gamma X}\iota)Y)N_0 + L^{-1}\iota(Y)\nabla^{\perp}_{\gamma X} N_0 - L^{-2}\iota(X)\iota(Y)N_0$$

Therefore one needs the v-covariant derivative of the Finsler 1-form ι. This is given by:

$$L(\nabla_{\gamma X}\iota)Y = h(X,Y) \tag{4.30}$$

as a consequence of $(\nabla_{\gamma X} g)(Y,v) = 0$.

Here $h = g - \iota \otimes \iota$ is the angular metric tensor of (M^n, E), (cf. the terminology in [28, p. 101]). Thus, the v-covariant derivative of the normal curvature vector N is given by:

$$(\nabla_{\gamma X} N)Y = L^{-1}\iota(Y)\nabla^{\perp}_{\gamma X} N_0 + L^{-2}\left\{h(X,Y) - \iota(X)\iota(Y)\right\}N_0$$

Set $Y = v$ and use $C(X, v) = 0$ so that to obtain:

(4.31) $$\nabla^{\perp}_{\gamma X} N_0 = 2L^{-1}\, \iota(X)\, N_0$$

Assume now that $p = 1$. The Codazzi equation (4.31) turns into:

$$f^{-1}(df) \circ \gamma = 2L^{-1}\iota$$

or, equivalently:

$$d\big(\log(fL^{-2})\big) \circ \gamma = 0$$

Consequently, there is $a \in C^{\infty}(M^n)$, $a > 0$, i.e. a scalar field depending on positional arguments alone, so that:

(4.32) $$f = aL^2$$

Apply $\delta_i = \partial_i - N_i^j \dot{\partial}_j$ to (4.32) and use (4.16) and (4.23) such as to yield $\partial_i a = 0$. Thus $a = a_0 = \text{const}$. The (horizontal) Ricci curvature of a Finslerian manifold (M^n, E) is given by $R_{jk} = R^i_{ijk}$. Let M^n be a totally-umbilical hypersurface of the locally Minkowski manifold M^{n+1}. By (4.9), (4.32), it follows that:

(4.33) $$R_{jk} = (n-1)\, a_0^2\, g_{jk}$$

Finally, Proposition 4.3 follows from (4.32)–(4.33). □

Next, we say that (M^n, E) has Ricci curvature $\geq e^2$ if $R_{jk}\xi^j\xi^k \geq e^2$ for any $X = \xi^i X_i \in \Gamma^{\infty}(\pi^{-1}TM^n)$, $\|X\| = 1$. By a result of F. Moalla, [31], if (M^n, E) is a complete (with respect to the distance d in [32, p. 323]) Finslerian manifold of Ricci curvature $\geq e^2 > 0$ then M^n has diameter $\leq \pi(n-1)^{1/2}e^{-1}$. The result in [31] cannot be applied directly to (4.33) which is a contraction of the horizontal curvature tensor of the induced connection (rather than the Cartan connection) of M^n. Yet, we have:

Theorem 4.4

Any complete totally-umbilical hypersurface M^n of a locally Minkowski manifold has diameter $\leq \pi/a_0$, provided the induced and intrinsic connections of M^n coincide.

Consequently M^n is compact and (by applying the same result at the level of the universal covering manifold) has a finite $\pi_1(M^n)$. Also the first Betti number of M^n vanishes, cf. also Corollary 2 in [31, p. 2737].

By (4.14), (4.32), if the induced connection is of scalar curvature K then $K = -a_0^2$. Any 2-dimensional Finslerian manifold is a Finsler space of scalar curvature, cf. [28, p. 183]. We obtain:

Theorem 4.5

Let M^2 be a totally-umbilical surface in a locally Minkowski manifold M^3. If the induced and intrinsic connections of M^2 coincide then either M^2 is totally geodesic (and then M^2 is locally Minkowski, cf. [16, p. 85]) or M^2 is a Finsler space of negative scalar curvature $-a_0^2$.

5. Umbilical CR submanifolds

Let $(M^{2m}, \overline{E})$ be a real $2m$-dimensional Finslerian manifold. A Finslerian almost complex structure J on M^{2m} is an endomorphism $J: \pi^{-1}TM^{2m} \to \pi^{-1}TM^{2m}$ so that $J^2 = -I$. Let $(\pi^{-1}TM^{2m}, \overline{g})$ be the induced Riemannian bundle of $(M^{2m}, \overline{E})$. Then $(M^{2m}, \overline{E}, J)$ is a Kählerian-Finsler space if $\overline{g}(JX, JY) = \overline{g}(X, Y)$, for any $X, Y \in \Gamma^\infty(\pi^{-1}TM^{2m})$, and $\overline{\nabla}J = 0$, where $\overline{\nabla}$ is the Cartan connection of $(M^{2m}, \overline{E})$, cf. [20].

Let $\iota: M^n \subseteq M^{2m}$ be a submanifold of M^{2m} ($p = 2m - n$). Then M^n is a CR submanifold of $(M^{2m}, \overline{E}, J)$ if it carries a pair $(\mathcal{D}, \mathcal{D}^\perp)$ of Finslerian distributions so that *i)* $\mathcal{D}_u^\perp$ is the g_u-orthogonal complement of $\mathcal{D}_u$ in $\pi^{-1}TM^n$, *ii)* $J_u\mathcal{D}_u = \mathcal{D}_u$ and *iii)* $J_u\mathcal{D}_u^\perp \subseteq E(\iota)_u$, for any $u \in V(M^n)$, cf. [6], [18].

Let $(M^n, \mathcal{D}, \mathcal{D}^\perp)$ be a totally-umbilical CR submanifold of the Kählerian-Finsler space $(M^{2m}, \overline{E}, J)$. We assume as usual that the induced and intrinsic connections of M^n coincide. If $(M^{2m}, \overline{E})$ is a Riemannian manifold then, by a result of A. Bejancu, [7], M^n is totally-geodesic in $(M^{2m}, \overline{E})$, provided that M^n is proper (i.e. $\mathcal{D}_u \neq 0$, $\mathcal{D}_u \neq \pi_u^{-1}TM^n$, $u \in V(M^n)$). This, in turn, relies on a result of D.E. Blair & B.Y. Chen, [9], asserting that $\mathcal{D}^\perp$ is involutive. The main difficulty in bringing A. Bejancu's result to Finslerian geometry lies in the fact that $\mathcal{D}^\perp$ is not any longer a distribution on M^n, but rather a Finslerian distribution:

$$\mathcal{D}^\perp: u \in V(M^n) \longrightarrow \mathcal{D}_u^\perp \subseteq \pi_u^{-1}TM^n$$

Cf. also [19]. We shall need the following:

Lemma 5.1

Let v be the Liouville vector of M^n. If M^n is totally umbilical in M^{2m} and $v \in \mathcal{D}$, then $F[\beta X, \beta Y] \in \mathcal{D}^\perp$, for any $X, Y \in \mathcal{D}^\perp$.

We establish the following:

Theorem 5.2

Let $(M^n, \mathcal{D}, \mathcal{D}^\perp)$ be a totally umbilical CR submanifold of the Kählerian-Finsler space $(M^{2m}, \overline{E}, J)$. Assume that $(M^{2m}, \overline{E})$ is locally Minkowski and $\dim_{\mathbb{R}} \mathcal{D}^\perp_u > 1$. Then (M^n, E) is a locally Minkowski manifold immersed in $(M^{2m}, \overline{E})$ as a totally-geodesic submanifold.

Proof. Let tan, nor be the canonical projections associated with (3.4) (where $p = 2m - n$, $\psi = \iota$) and set $aX = \tan(JX)$, $bX = \mathrm{nor}(JX)$, $t\xi = \tan(J\xi)$, $f\xi = \mathrm{nor}(JX)$, for any $X \in \Gamma^\infty(\pi^{-1}TM^n)$, $\xi \in \Gamma^\infty(E(\iota))$. As a consequence of $\overline{\nabla} J = 0$ and of (3.5)–(3.6) we obtain:

$$(5.1) \qquad (\nabla_{\beta X} a)Y = A_{bY}X + tH(X,Y)$$

$$(5.2) \qquad (\nabla_{\gamma X} a)Y = W_{bY}X + tQ(X,Y)$$

$$(5.3) \qquad (\nabla_{\beta X} b)Y = fH(X,Y) - H(X,aY)$$

$$(5.4) \qquad (\nabla_{\gamma X} b)Y = fQ(X,Y) - Q(X,aY)$$

for any $X, Y \in \Gamma^\infty(\pi^{-1}TM^n)$. Then, as M^n is totally-umbilical, (5.1) leads to:

$$A_{bY}X - A_{bX}Y = -aT(X,Y) - aF[\beta X, \beta Y]$$

for any $X, Y \in \mathcal{D}^\perp$. Using Lemma 5.1, the fact that $\mathcal{D}^\perp_u \subseteq \ker(a_u)$, $u \in V(M^n)$, and $T = 0$ (as the induced and intrinsic connections of M^n coincide) we obtain:

$$(5.5) \qquad A_{bY}X = A_{bX}Y$$

for any $X, Y \in \mathcal{D}^\perp$. Note that $t\mu \in \mathcal{D}^\perp$ (where $n\mu = \mathrm{trace}(H)$). Set then $Y = t\mu$ in (5.5) and use the umbilicity of M^n so that to yield:

$$(5.6) \qquad \overline{g}(\mu, bX)t\mu = \overline{g}(\mu, bt\mu)X$$

As $\dim_{\mathbb{R}} \mathcal{D}^\perp_u > 1$, $u \in V(M^n)$, we may consider $X \in \mathcal{D}^\perp$ orthogonal on $t\mu$ and such that $X_u \neq 0$, $u \in V(M^n)$. Then (5.6) reduces to $0 = \overline{g}(\mu, bt\mu)X = -\|t\mu\|^2 X$ so that:

$$(5.7) \qquad t\mu = 0$$

Again as a consequence of $\overline{\nabla}J = 0$ and (3.5)–(3.6) we have:

$$(\nabla_{\beta X}t)\xi = A_{f\xi}X - aA_{\xi}X \tag{5.8}$$

$$(\nabla_{\gamma X}t)\xi = W_{f\xi}X - aW_{\xi}X \tag{5.9}$$

$$(\nabla_{\beta X}f)\xi = -H(X,t\xi) - bA_{\xi}X \tag{5.10}$$

$$(\nabla_{\gamma X}f)\xi = -Q(X,t\xi) - bW_{\xi}X \tag{5.11}$$

for any $X,Y \in \Gamma^{\infty}(\pi^{-1}TM^n)$, $\xi \in \Gamma^{\infty}(E(\iota))$. Set $\xi = \mu$ in (5.8) and take the inner product of the resulting identity with $Y \in \Gamma^{\infty}(\pi^{-1}TM^n)$. We obtain:

$$g((\nabla_{\beta X}t)\mu, Y) = -\|\mu\|^2 g(aX,Y) \tag{5.12}$$

Let $X \in \Gamma^{\infty}(\pi^{-1}TM^n)$ so that $aX \neq 0$ (everywhere on $V(M^n)$). Set $Y = aX$ in (5.12). At this point we may use (5.7) and the fact that t is $\mathcal{D}^{\perp}$-valued while a is $\mathcal{D}$-valued, so that to yield:

$$0 = g(t\nabla_{\beta X}\mu, aX) = \|aX\|^2 \|\mu\|^2$$

and Theorem 5.2 is completely proved. □

Proof of Lemma 5.1. With any Finslerian manifold (M^n, E) we may associate a h-differentiation operator d^h, cf. [36]. That is, if Φ is a Finslerian r-form on M^n then $(d^h\Phi)(X_0, X_1, \dots, X_n) = (d\Phi^H)(\beta X_0, \beta X_1, \dots, \beta X_n)$, for any $X_j \in \Gamma^{\infty}(\pi^{-1}TM^n)$, $0 \le j \le n$. Here $\Phi^H(Z_1,\dots,Z_n) = \Phi(FZ_1,\dots,FZ_n)$, $Z_\alpha \in \Gamma^{\infty}(T(V(M^n)))$, $1 \le \alpha \le n$. Also β is the horizontal lift with respect to the non-linear connection of (M^n, E), cf. [4], the operator d^h satisfies the complex condition $(d^h)^2 = 0$ if and only if $R^1 = 0$. Given a Kählerian-Finsler space $(M^{2m}, \overline{E}, J)$, by a result of [21], $\overline{\nabla}J = 0$ yields $d^h\overline{\Omega} = 0$, where $\overline{\Omega}(X,Y) = \overline{g}(X,JY)$, $X,Y \in \Gamma^{\infty}(\pi^{-1}TM^{2m})$. Consider now $X,Y \in \mathcal{D}^{\perp}$, $Z \in \mathcal{D}$. Then:

$$\begin{aligned} 0 &= 3(d^h\overline{\Omega})(X,Y,Z) \\ &= -\overline{\Omega}(F[\overline{\beta}X,\overline{\beta}Y],Z) - \overline{\Omega}(F[\overline{\beta}Z,\overline{\beta}X],Y) - \overline{\Omega}(F[\overline{\beta}Y,\overline{\beta}Z],X) \end{aligned}$$

Note that $N(X) = 0$ (as $N = L\iota \otimes \mu$ and $v \in \mathcal{D}$) for any $X \in \mathcal{D}^{\perp}$. Thus, by (3.12), $\overline{\beta} = \beta$ on $\mathcal{D}^{\perp}$. We obtain

$$\overline{\Omega}(F[\beta X,\beta Y],Z) = \overline{\Omega}(F[L\iota(Z)\gamma\mu,\beta X],Y) - \overline{\Omega}(F[L\iota(Z)\gamma\mu,\beta Y],X) \tag{5.13}$$

Let $\langle v\rangle_u^\perp$ be the orthogonal complement of $\langle v\rangle_u = \mathbb{R}\cdot(u,u)$ in $\mathcal{D}_u$, $u \in V(M^n)$. Proving Lemma 5.1 amounts to checking that:

$$\overline{\Omega}(F[\beta X,\beta Y],Z) = 0$$

for any $Z \in \mathcal{D}$. This follows easily from (5.13) when $Z \in \langle v\rangle^\perp$. The remaining case is $Z = v$. Set $Z = v$ in (5.13) such as to yield:

$$\overline{\Omega}(F[\beta X,\beta Y],v) = L^2\Big\{\overline{\Omega}(F[\gamma\mu,\beta X],Y) - \overline{\Omega}(F[\gamma\mu,\beta Y],X)\Big\} \tag{5.14}$$

(cf. also (4.16)). Using (3.13) (with M^n totally-umbilical and $T = 0$) gives:

$$\overline{C}(N(X),Y) = \overline{C}(N(Y),X) \tag{5.15}$$

for any $X,Y \in \Gamma^\infty(\pi^{-1}TM^n)$. Set $Y = v$ in (5.15) and use (2.5). As $N(v) = L^2\mu$ this procedure leads to:

$$\overline{C}(\mu,X) = 0$$

or:

$$F[\gamma\mu,\overline{\beta}X] = \overline{\nabla}_{\gamma\mu}X \tag{5.16}$$

for any $X \in \Gamma^\infty(\pi^{-1}TM^n)$. Note that, as a consequence of (3.12) and the fact that $\ker(d\pi)$ is involutive, we may replace $\overline{\beta}X$ in (5.16) by βX. Indeed $F = 0$ on $\ker(d\pi)$. As this point we may substitute from (5.16) into (5.14) such that:

$$\begin{aligned}\overline{\Omega}(F[\beta X,\beta Y],v) &= L^2\Big\{\overline{\Omega}(\overline{\nabla}_{\gamma\mu}X,Y) - \overline{\Omega}(\overline{\nabla}_{\gamma\mu}Y,X)\Big\}\\ &= L^2\Big\{(\gamma\mu)(\overline{g}(X,JY)) - \overline{g}(X,\overline{\nabla}_{\gamma\mu}JY) - \overline{g}(\overline{\nabla}_{\gamma\mu}Y,JX)\Big\} = 0\end{aligned}$$

for any $X,Y \in \mathcal{D}^\perp$. The proof of Lemma 5.1 is complete. □

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