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Approximation by compact operators over spaces of continuous functions

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Abstract

In this paper the author proves that for any compact metrizable space Q, $K(c, C(Q))$ is proximinal in $L(c, C(Q))$.

1. Introduction

If X and Y are two normed linear spaces, then $L(X,Y)$ denotes the Banach space of all bounded linear operators from X to Y, and $K(X,Y)$ denotes the space of all compact operators in $L(X,Y)$. If A is a closed subset of the normed linear space X, then A is said to be "proximinal" in X if for each $x \in X$ there is $y_0 \in A$ such that

$$\|x - y_0\| = d(x, A) = \inf\Big\{ \|x - y\| \,;\, y \in A \Big\}.$$

In this case y_0 is said to be "a best approximation" for x from A. If Q is a compact Hausdorff space then $C(Q)$ denotes the Banach space of all continuous real valued functions defined on Q.

The proximinality of $K(X,Y)$ in $L(X,Y)$ was studied by several authors, for example Halmos [5], Mach and Ward [9], Mach [8], Lau [7], and Cho [2]. In their paper, Mach and Ward [9] showed that $K(\ell_p, \ell_p)$ is proximinal in $L(\ell_p, \ell_p)$ for each $1 \le p < \infty$. At the end of their paper they asked about the proximinality of $K(C(S), C(S))$ in $L(C(S), C(S))$ when S is a compact Hausdorff space. Lau [7] showed that if X^* is uniformly convex, then for any compact Hausdorff space Q, $K(X,C(Q))$ is proximinal in $L(X,C(Q))$. In this important paper the author asked again about the proximinality of $K(C(S),C(Q))$ in $L(C(S),C(Q))$. In trying to solve this problem, Feder [4] proved that if $X = C[0,1]$, ℓ_∞ or $L_\infty[0,1]$ then $K(X,X)$ is not proximinal in $L(X,X)$. Benyamini [1] showed that if c is the space of all convergent sequences of real numbers, then for any compact Hausdorff space Q, $K(C(Q),c)$ is proximinal in $L(C(Q),c)$. He also showed that if $[1,\omega^2]$ is the set of all ordinal numbers less than or equal to ω^2, and S is a compact Hausdorff space satisfying that $C^*(S)$, "the dual space of $C(S)$", contains a copy of $L_1[0,1]$, then $K(C(S),C[1,\omega^2])$ is not proximinal in $L(C(S),C[1,\omega^2])$. Using a version of Tietze's extension theorem, like the one in Kamal [6], one can generalize the last result to obtain

Theorem 1.1

Let Q and S be two compact Hausdorff spaces. If $C^*(S)$ contains a copy of $L_1[0,1]$, and Q has a subset homeomorphic to $[1,\omega^2]$, then $K(C(S),C(Q))$ is not proximinal in $L(C(S),C(Q))$.

In this paper the author proves that if Q is a compact metrizable space, then $K(c,C(Q))$ is proximinal in $L(c,C(Q))$. This result in addition to the known results, may help in finding the general solution of the problem of the proximinality of $K(C(S),C(Q))$ in $L(C(S),C(Q))$.

The rest of this introduction will cover some definitions and known theorems. If X is a normed linear space and Q is a compact Hausdorff space, then $C(Q,\{X^*,\omega^*\})$ denotes the space of all bounded functions $f: Q \to X^*$ such that f is continuous with respect to the ω^* topology on X^*, and $C(Q,X)$ denotes the space of all functions $f: Q \to X$, continuous with respect to the norm defined on X.

Theorem 1.2 (Dunford and Schwartz [3, page 490])

Let Q be a compact Hausdorff space, and let X be a normed linear space. The mapping

$$\alpha: L(X,C(Q)) \longrightarrow C(Q,\{X^*,\omega^*\})$$

defined by $\alpha(T)(q)(x) = T(x)(q)$, for $T \in L(X,C(Q))$, $q \in Q$ and $x \in X$, is an isometric isomorphism from $L(X,C(Q))$ onto $C(Q,\{X^*,\omega^*\})$. Furthermore $\alpha(K(X,C(Q))) = C(Q,X)$.

From Theorem 1.2 one can obtain the following well known result.

Lemma 1.3

If X is a normed linear space, and Q is a compact Hausdorff space, then $K(X,C(Q))$ is proximinal in $L(X,C(Q))$ if and only if $C(Q,X)$ is proximinal in $C(Q,\{X^*,\omega^*\})$.

In this paper, ℓ_1, c, and c_0 are the classical Banach sequence spaces, and unless it is mentioned otherwise, the ω^*-topology on ℓ_1 is the ω^*-topology induced by c. In this topology each $x = (x_1, x_2, \ldots)$ in ℓ_1 corresponds to the linear functional $\tilde{x}$ in c^* defined by

$$\tilde{x}(y) = \left(x_1 \cdot \lim_{i\to\infty} \alpha_i\right) + \sum_{i=2}^{\infty} x_i \alpha_{i-1}, \qquad \text{for } y = (\alpha_1, \alpha_2, \ldots) \in c.$$

For each $i = 1, 2, \ldots$, if $y_i = (y_1^i, y_2^i, \ldots)$ in ℓ_1, $y_i' = (y_2^i, y_3^i, \ldots)$, and the sequence $\{y_i\}$ converges to y_0 in the ω^*-topology, on ℓ_1, then it is obvious that the sequence $\{y_i'\}$ converges to $y_0' = (y_2^0, y_3^0, \ldots)$ with respect to the ω^*-topology induced on ℓ_1 by c_0.

Proposition 1.4 (Mach [8])

Let $\{y_i\}$ be a bounded sequence in ℓ_1, that converges to zero with respect to the ω^*-topology induced by c_0, and let $x \in \ell_1$, then $\lim_{i\to\infty}\left(\|y_i - x\| - \|y_i\| - \|x\|\right) = 0$.

Let Q be a compact metrizable space, and let $f \in C(Q, \{\ell_1, \omega^*\})$. For each $q \in Q$, $f(q) \in \ell_1$, so one may assume that $f(q) = \big(f_1(q), f_2(q), \ldots\big)$ where $f_i(q)$ is a bounded real valued function on Q. For each $q_0 \in Q$, and $x \in \ell_1$, define,

$$r(f, q_0, x) = \lim_{i\to\infty} \sup\left\{\|f(q) - x\|;\ d(q, q_0) < \tfrac{1}{i}\right\}$$

where $d(q, q_0)$ is the distance between q and q_0. The asymptotic radius of f at q_0 is defined by:

$$\operatorname{ar}(f, q_0) = \inf\{r(f, q_0, x);\ x \in \ell_1\},$$

and $r(f) = \sup\{\operatorname{ar}(f, q);\ q \in Q\}$.

If $\operatorname{ar}(f, q_0)$ is attained then the asymptotic center of f at q_0 is defined by:

$$\operatorname{ac}(f, q_0) = \{x \in \ell_1;\ r(f, q_0, x) = \operatorname{ar}(f, q_0)\},$$

and

$$\Gamma(f,q_0) = \{x \in \ell_1;\ r(f,q_0,x) \le r(f)\}.$$

The proof of the following lemma can be obtained from the basic definition of the asymptotic center, and lemma 1.2 *i* and *ii* of Benyamini [1].

Lemma 1.5

Let Q be a compact metrizable space, and let $f \in C(Q,\{\ell_1,\omega^*\})$.

i) If $q \in Q$ and $\{q_i\}$ is any sequence in Q that converges to q, then $\overline{\lim}\ \mathrm{ar}(f,q_i) \le \mathrm{ar}(f,q)$.

ii) If there is $g \in C(Q,\ell_1)$ such that $g(q) \in \Gamma(f,q)$ for each $q \in Q$, then g is a best approximation for f from $C(Q,\ell_1)$.

2. The proximinality of $K(c,C(Q))$ in $L(c,C(Q))$

In order to show that $K(c,C(Q))$ is proximinal in $L(c,C(Q))$, it is enough, by Lemma 1.3, to show that for each $f \in C(Q,\{\ell_1,\omega^*\})$, there is $g \in C(Q,\ell_1)$ such that $d(f,C(Q,\ell_1)) = \|f-g\|$.

The following Lemma will be used in the proof of the main theorem.

Lemma 2.1

Let Q be a compact metrizable space, $f \in C(Q,\{\ell_1,\omega^*\})$, and let $q \in Q$. Then, there exists a real number β such that $(\beta, f_2(q), f_3(q), \ldots) \in \mathrm{ac}(f,q)$.

Proof. Let $x = (x_1,x_2,\ldots) \in \ell_1$, and let $z = (x_1, f_2(q), f_3(q), \ldots)$; it will be shown that $r(f,q,x) \ge r(f,q,z)$.

Let $\{q_i\}$ be a sequence in Q that converges to q such that: $r(f,q,z) = \lim_{i\to\infty}\|f(q_i)-z\|$. Then

$$r(f,q,x) \ge \overline{\lim}\,\|f(q_i)-x\| = \overline{\lim}\left(|f_1(q_i)-x_1| + \sum_{k=2}^{\infty}|f_k(q_i)-x_k|\right).$$

For each $p \in Q$, let $f'(p) = (f_2(p), f_3(p), \ldots)$, $x' = (x_2,x_3,\ldots)$ and let $w = x' - f'(q)$. Then

$$\begin{aligned}\sum_{k=2}^{\infty}|f_k(q_i)-x_k| &= \|f'(q_i)-x'\| = \|f'(q_i)-f'(q)-w\| \\ &= \|f'(q_i)-f'(q)\| + \|w\| + \|f'(q_i)-f'(q)-w\| - \|w\| - \|f'(q_i)-f'(q)\|.\end{aligned}$$

Since $f \in C(Q, \{\ell_1, \omega^*\})$, then $\{f'(q_i)\}$ converges to $f'(q)$ with respect to the ω^*-topology induced on ℓ_1 by c_0, thus by Proposition 1.4,

$$\lim_{i\to\infty}\Big(\|f'(q_i) - f'(q) - w\| - \|w\| - \|f'(q_i) - f'(q)\|\Big) = 0.$$

Therefore

$$\begin{aligned} r(f,q,x) &\geq \overline{\lim}\Big(|f_1(q_i) - x_1| + \|f'(q_i) - f'(q)\| + \|w\|\Big) \\ &= \overline{\lim}\,\|f(q_i) - z\| + \|w\| \geq r(f,q,z). \end{aligned}$$

Thus $\mathrm{ar}(f,q) = \inf\Big\{r(f,q,z);\ z = \big(\alpha, f_2(q), f_3(q), \ldots\big) \text{ and } \alpha \in \mathbb{R}\Big\}$. So by a simple compactness argument one can show that there is a real number β such that $(\beta, f_2(q), f_3(q), \ldots) \in \mathrm{ac}(f,q)$. □

Lemma 2.2

Let Q, f and q be as in Lemma 2.1, a and ε be two positive numbers. If $\mathrm{ar}(f,q) = a$, *and* $r(f,q,0) = a + \varepsilon$, *then there is $x \in \mathrm{ac}(f,q)$ such that $\|x\| \leq 2\varepsilon$.*

Proof. We show first that $\sum_{k=2}^{\infty}|f_k(q)| \leq \varepsilon$. For each number α, let $y(\alpha) = (\alpha, f_2(q), f_3(q), \ldots)$, and let $\{q_i\}$ be a sequence in Q that converges to q, such that $\lim_{i\to\infty}\|f(q_i) - y(0)\| = r(f,q,y(0))$. As in Lemma 2.1, let $f'(p) = \big(f_2(p), f_3(p), \ldots\big)$. Then $\{f'(q_i)\}$ converges to $f'(q)$ with respect to the ω^*-topology induced on ℓ_1 by c_0. Thus by Proposition 1.4,

$$\lim_{i\to\infty}\Big(\|(f'(q_i) - f'(q)) + f'(q)\| - \|f'(q)\| - \|f'(q_i) - f'(q)\|\Big) = 0.$$

Therefore

$$\begin{aligned} \sum_{k=2}^{\infty}|f_k(q)| = \|f'(q)\| &= \lim_{i\to\infty}\Big(\|f'(q_i)\| - \|f'(q_i) - f'(q)\|\Big) \\ &= \lim_{i\to\infty}\Big(\sum_{k=1}^{\infty}|f_k(q_i)| - \Big[|f_1(q_i)| + \sum_{k=2}^{\infty}|f_k(q_i) - f_k(q)|\Big]\Big) \\ &\leq \overline{\lim}\sum_{k=1}^{\infty}|f_k(q_i)| - r(f,q,y(0)). \end{aligned}$$

But $r(f,q,y(0)) \geq \mathrm{ar}(f,q) = a$, and $\overline{\lim}\,\sum_{k=1}^{\infty}|f_k(q_i)| \leq r(f,q,0) = a + \varepsilon$, so

$$\sum_{k=2}^{\infty}|f_k(q)| \leq (a + \varepsilon) - a = \varepsilon.$$

On the other hand since $\lim_{i\to\infty}\|f(q_i)-y(0)\| = r(f,q,y(0))$, then

$$r(f,q,y(0)) = \lim_{i\to\infty}\Big(|f_1(q_i)| + \sum_{k=2}^{\infty}|f_k(q_i)-f_k(q)|\Big)$$
$$= \lim_{i\to\infty}\Big[\Big(|f_1(q_i)| + \sum_{k=2}^{\infty}|f_k(q_i)| - \sum_{k=2}^{\infty}|f_k(q)|\Big) - \Big(\sum_{k=2}^{\infty}|f_k(q_i)| - \sum_{k=2}^{\infty}|f_k(q)| - \sum_{k=2}^{\infty}|f_k(q_i)-f_k(q)|\Big)\Big]$$

But

$$\lim_{i\to\infty}\Big(\sum_{k=2}^{\infty}|f_k(q_i)| - \sum_{k=2}^{\infty}|f_k(q)| - \sum_{k=2}^{\infty}|f_k(q_i)-f_k(q)|\Big)$$
$$= \lim_{i\to\infty}\Big(\|f'(q_i)-f'(q)+f'(q)\| - \|f'(q)\| - \|f'(q_i)-f'(q)\|\Big) = 0.$$

So

$$r(f,q,y(0)) = \lim_{i\to\infty}\Big[|f_1(q_i)| + \sum_{k=2}^{\infty}|f_k(q_i)|\Big] - \sum_{k=2}^{\infty}|f_k(q)|$$
$$\le r(f,q,0) - \sum_{k=2}^{\infty}|f_k(q)| = a+\varepsilon - \sum_{k=2}^{\infty}|f_k(q)| = a+\varepsilon'$$

where $0 \le \varepsilon' \le \varepsilon$. If $\varepsilon' = 0$ then one can choose $x = y(0)$, otherwise one may assume that $\varepsilon' > 0$.

Secondly we show that there exists a real number β such that $|\beta| \le \varepsilon$ and $y(\beta) \in \mathrm{ac}(f,q)$. If this is true, then $x = y(\beta)$ is the required element. If there are two real numbers β_1, β_2 such that $\beta_1\beta_2 < 0$ and $y(\beta_1) \in \mathrm{ac}(f,q)$ and $y(\beta_2) \in \mathrm{ac}(f,q)$, then for $\alpha = |\beta_2|/(|\beta_1|+|\beta_2|)$ one has $\alpha\beta_1 + (1-\alpha)\beta_2 = 0$.

So if $\{q_i\}$ is any sequence in Q that converges to q then

$$\overline{\lim}\Big(|f_1(q_i)| + \sum_{k=2}^{\infty}|f_k(q_i)-f_k(q)|\Big)$$
$$= \overline{\lim}\Big(\big|f_1(q_i) - (\alpha\beta_1 + (1-\alpha)\beta_2)\big| + \sum_{k=2}^{\infty}|f_k(q_i)-f_k(q)|\Big)$$
$$\le \alpha\,\overline{\lim}\Big[|f_1(q_i)-\beta_1| + \sum_{k=2}^{\infty}|f_k(q_i)-f_k(q)|\Big]$$
$$+ (1-\alpha)\,\overline{\lim}\Big[|f_1(q_i)-\beta_2| + \sum_{k=2}^{\infty}|f_k(q_i)-f_k(q)|\Big] \le a.$$

Thus $r(f,q,y(0)) = a$ which contradicts the fact that $r(f,q,y(0)) = a+\varepsilon'$, and that $\varepsilon' > 0$. So without loss of generality one may assume that if $y(\beta) \in \mathrm{ac}(f,q)$, then $\beta > 0$. It will be shown that there exists $\beta \leq \varepsilon$ such that $y(\beta) \in \mathrm{ac}(f,q)$. Assume not, then for each β such that $y(\beta) \in \mathrm{ac}(f,q)$, one has $\beta = \varepsilon' + \varepsilon'' > \varepsilon$. Thus $r(f,q,y(\varepsilon')) > a$. Let $\{q_i\}$ be a sequence in Q converging to q and satisfying that $\lim_{i\to\infty}\|f(q_i) - y(\varepsilon')\| > a$. If $\underline{\lim}\ f(q_i) \geq \varepsilon'$ then:

$$\begin{aligned}
a &< \lim_{i\to\infty}\left(\left|f_1(q_i) - \varepsilon'\right| + \sum_{k=2}^{\infty}\left|f_k(q_i) - f_k(q)\right|\right)\\
&= \overline{\lim}\left(\left|f_1(q_i)\right| + \sum_{k=2}^{\infty}\left|f_k(q_i) - f_k(q)\right|\right) - \varepsilon'\\
&\leq r(f,q,y(0)) - \varepsilon' = a + \varepsilon' - \varepsilon' = a,
\end{aligned}$$

and if $\{q_i\}$ has a subsequence $\{s_i\}$ for which $f(s_i) - \varepsilon' < 0$ for each i, then for any $\beta = \varepsilon' + \varepsilon''$, if $\varepsilon'' > 0$ then

$$\begin{aligned}
r(f,q,y(\beta)) &\geq \overline{\lim}\left(\left|f_1(s_i) - (\varepsilon' + \varepsilon'')\right| + \sum_{k=2}^{\infty}\left|f_k(s_i) - f_k(q)\right|\right)\\
&= \overline{\lim}\left(\left|f_1(s_i) - \varepsilon'\right| + \sum_{k=2}^{\infty}\left|f_k(s_i) - f_k(q)\right|\right) + \varepsilon''\\
&> a + \varepsilon''. \ \square
\end{aligned}$$

Lemma 2.3

Let Q be a compact metrizable space, and let $f \in C(Q,\{\ell_1,\omega^*\})$. The set valued function $\Gamma: Q \to 2^{\ell_1}$ defined by $\Gamma(q) = \Gamma(f,q)$ is lower semicontinuous.

Proof. Let F be a closed subset of ℓ_1, and let $G = \{q \in Q; \Gamma(q) \subseteq F\}$. It will be shown that G is closed. Let $q_0 \in \overline{G}$ and let us show that $q_0 \in G$; that is, if $x_0 \in \Gamma(q_0)$ then $x_0 \in F$. Let $\{q_i\}$ be a sequence in G converging to q_0. If $\{q_i\}$ has a subsequence $\{t_i\}$ satisfying that $\lim_{i\to\infty} \mathrm{ar}(f,t_i) < r(f)$, then without loss of generality one may assume that there is a positive number $\varepsilon_0 > 0$, such that $\mathrm{ar}(f,t_i) + \varepsilon_0 \leq r(f)$ for each i. Let $\varepsilon_i = 1/i$, then for each i there is a neighbourhood U_i of q_0 in Q, such that for each $q \in U_i$, $\|f(q) - x_0\| < r(f) + 1/i$. Choose a subsequence $\{s_i\}$ of $\{t_i\}$ satisfying that U_i is a neighbourhood of s_i. For each fixed $i \geq 1$, let $y_i \in \mathrm{ac}(f,s_i)$, and let $x_i = (\frac{\varepsilon_0}{\varepsilon_0+1/i})x_0 + (\frac{1/i}{\varepsilon_0+1/i})y_i$. Let us see that $x_i \in \Gamma(s_i)$. Let $\varepsilon > 0$ be given. Then there is a neighbourhood U of s_i in U_i, such that for each $q \in U$, $\|f(q) - y_i\| \leq \operatorname{ar}(f, s_i) + \varepsilon(\frac{\varepsilon_0 + 1/i}{1/i})$. But then for each $q \in U$;

$$\begin{aligned}
\|f(q) - x_i\| &\leq \left(\frac{\varepsilon_0}{\varepsilon_0 + \frac{1}{i}}\right)\|f(q) - x_o\| + \left(\frac{\frac{1}{i}}{\varepsilon_0 + \frac{1}{i}}\right)\|f(q) - y_i\| \\
&\leq \left(\frac{\varepsilon_0}{\varepsilon_0 + \frac{1}{i}}\right) \cdot \left(r(f) + \tfrac{1}{i}\right) + \left(\frac{\frac{1}{i}}{\varepsilon_0 + \frac{1}{i}}\right) \cdot \left(\operatorname{ar}(f, s_i) + \left(\frac{\varepsilon_0 + \frac{1}{i}}{\frac{1}{i}}\right) \cdot \varepsilon\right) \\
&\leq \left(\frac{\varepsilon_0}{\varepsilon_0 + \frac{1}{i}}\right) \cdot \left(r(f) + \tfrac{1}{i}\right) + \left(\frac{\frac{1}{i}}{\varepsilon_0 + \frac{1}{i}}\right) \cdot \left(r(f) - \varepsilon_0\right) + \varepsilon \\
&= r(f) + \varepsilon.
\end{aligned}$$

Thus $x_i \in \Gamma(s_i) \subseteq F$. But now it is obvious that $\{x_i\}$ converges to x_0, and since F is closed, it follows that $x_0 \in F$.

If $\underline{\lim}\, \operatorname{ar}(f, q_i) = r(f)$, then $\lim_{i\to\infty} \operatorname{ar}(f, q_i) = r(f)$, thus by Lemma 1.5i, $\operatorname{ar}(f, q_0) = r(f)$. Let $\varepsilon_i = 1/i$ and let U_i be a neighbourhood of q_0 in Q satisfying that for each $q \in U_i$ one has, $\|f(q) - x_0\| \leq r(f) + 1/i$. Choose a subsequence $\{t_i\}$ of $\{q_i\}$ such that for each $i = 1, 2, \ldots$, U_i is a neighbourhood for t_i. By the fact that $\lim_{i\to\infty} \operatorname{ar}(f, t_i) = r(f)$ and Lemma 1.5i there is a sequence $\{\delta_i\}$ of non-negative numbers for which $\lim_{i\to\infty} \delta_i = 0$, and for each $i = 1, 2, \ldots$, one has $\operatorname{ar}(f, t_i) + \delta_i = r(f)$. Let g be a function defined on Q by $g(q) = f(q) - x_0$. Then $g \in C(Q, \{\ell_1, \omega^*\})$ and for each $i = 1, 2, \ldots$,

$$r(g, t_i, 0) = r(f, t_i, x_0) \leq r(f) + \tfrac{1}{i} = \operatorname{ar}(f, t_i) + \left(\delta_i + \tfrac{1}{i}\right).$$

Also $\operatorname{ar}(g, t_i) = \operatorname{ar}(f, t_i)$, thus by Lemma 2.2 taking $q = t_i$, $a = r(f, t_i)$, $\varepsilon = \delta_i + 1/i$, and $f = g$ there is $y_i \in \operatorname{ac}(g, t_i)$ such that $\|y_i\| \leq 2(\delta_i + 1/i)$. For each $i = 1, 2, \ldots$, let $x_i = x_0 + y_i$, then it is obvious that $x_i \in \operatorname{ac}(f, t_i)$; that is, $x_i \in \Gamma(t_i) \subseteq F$, and since $\{x_i\}$ converges to x_0, it follows that $x_0 \in F$. Thus Γ is lower semicontinuous. □

Theorem 2.4

If Q is a compact metrizable space, then $K(c, C(Q))$ is proximinal in $L(c, C(Q))$.

Proof. Let $f \in C(Q, \{\ell_1, \omega^*\})$. By Lemma 1.3, it is enough to show that there is $g \in C(Q, \ell_1)$ such that, $\|f - g\| = d(f, C(Q, \ell_1))$. The set valued function $\Gamma(q)$ defined in Lemma 2.3 is lower semicontinuous, thus as in Michael [10] one can show that there is $g \in C(Q, \ell_1)$ such that $g(q) \in \Gamma(q)$ for each $q \in Q$. But then by Lemma 1.5ii, $\|f - g\| = d(f, C(Q, \ell_1))$. Thus $K(c, C(Q))$ is proximinal in $L(c, C(Q))$. □

Corollary 2.5

Let Q be a compact metrizable space and S be a compact Hausdorff space. If S is the union of finitely many convergent sequences, then $K(C(S), C(Q))$ is proximinal in $L(C(S), C(Q))$.

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