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On functions of bounded (p,2)-variation

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Dedicated to Prof. M. Cotlar on his 80th birthday.

Abstract

In this paper we introduce the concept of (p,2)-variation which generalizes the Riesz p-variation. The following result is proved: A function $f: [a,b] \longrightarrow \mathbb{R}$ is of bounded (p,2)-variation ($1 < p < \infty$) if and only if f' is absolutely continuous on $[a,b]$ and $f'' \in L_p[a,b]$. Moreover it is shown that the (p,2)-variation of a function f on $[a,b]$ is given by

$$V_p^2(f;[a,b]) = \|f''\|_{L_p[a,b]}^p .$$

Introduction

In the past century, about 1880, C. Jordan ([2]) introduced the notion of a function of bounded variation and established the relation between these functions and monotonic ones. Later on the concept of bounded variation was generalized in various directions. In 1910, F. Riesz ([3]) defined the concept of bounded p-variation ($1 \leq p < \infty$) and proved that, for $1 < p < \infty$, this class coincides with the class of functions f, absolutely continuous with derivative $f' \in L_p[a,b]$. Moreover, the p-variation of a function f on $[a,b]$ is given by $\|f'\|_{L_p[a,b]}^p$, that is:

$$V_p(f;[a,b]) = \|f'\|_{L_p[a,b]}^p .$$

In the year 1908, de la Vallée Poussin ([5]), obtained a generalization of bounded variation, introducing the concept of bounded second variation. It is known that, if a function f is of bounded second variation on $[a,b]$, then f is absolutely continuous on $[a,b]$ and can be expressed as the difference of two convex functions (see, e.g. [4, Theorem 1.1]).

In the present paper we introduce the concept of bounded $(p,2)$-variation ($1 \leq p < \infty$) and prove a characterization of the class $A_p^2[a,b]$ ($1 < p < \infty$) in terms of this concept. $A_p^2[a,b]$ is the class of functions $f:[a,b] \to \mathbb{R}$ for which f' is absolutely continuous on $[a,b]$ and $f'' \in L_p[a,b]$. Moreover the $(p,2)$-variation of a function f on $[a,b]$ is given by $\|f''\|_{L_p[a,b]}^p$, that is

$$V_p^2(f;[a,b]) = \|f''\|_{L_p[a,b]}^p.$$

The obtained characterization can be considered as a "natural" generalization of that given by F. Riesz for the class $A_p[a,b]$. This result provides an alternative characterization for the Sobolev space $W_p^2[a,b]$.

1. Preliminary results

In this section we introduce some definitions and known results concerning the Riesz p-variation ($1 < p < \infty$) and de la Vallée Poussin second-variation.

Let $f:[a,b] \to \mathbb{R}$. For a given partition of the form:

$$\pi: a = t_0 < t_1 < \ldots < t_m = b$$

of [a,b], let:

$$\sigma_p(f;\pi) := \sum_{j=1}^{m} \frac{|f(t_j) - f(t_{j-1})|^p}{|t_j - t_{j-1}|^{p-1}} \qquad (1 < p < \infty).$$

The number

$$V_p(f;[a,b]) := \sup_{\pi} \sigma_p(f;\pi),$$

where the supremum is taken over all partitions π of $[a,b]$, is called the Riesz p-variation of the function f on $[a,b]$.

If $V_p(f;[a,b]) < \infty$, the function f is said to have bounded (or finite) Riesz p-variation. By $BV_p[a,b]$ we shall denote the Banach space of all functions $f:[a,b] \to \mathbb{R}$ for which $V_p(f;[a,b]) < \infty$ and the norm is given by

$$\|f\|_p := |f(a)| + \left(V_p(f;[a,b])\right)^{1/p}.$$

F. Riesz ([3]) introduced the so-called Riesz class $A_p[a,b]$ ($1 < p < \infty$) in the following way: $f \in A_p[a,b]$ if and only if f is absolutely continuous on $[a,b]$ and $f' \in L_p[a,b]$. In the same paper, the following characterization of the class $A_p[a,b]$ was also proved:

Lemma 1.1 (Riesz ([3]))

A real function f defined on the interval $[a,b]$ belongs to the class $A_p[a,b]$ $(1 < p < \infty)$ if and only if $V_p(f;[a,b]) < \infty$. Moreover:

$$V_p(f;[a,b]) = \|f'\|^p_{L_p[a,b]}.$$

In 1908, de la Vallée Poussin ([5]) introduced the class of functions of bounded second-variation, in the following form: let $f:[a,b] \to \mathbb{R}$, for a given partition π of the form

$$\pi: a = a_1 < c_1 \le d_1 < b_1 = a_2 < \ldots < b_{m-1} = a_m < c_m \le d_m < b_m = b, \tag{1.1}$$

let

$$\sigma^2(f;\pi) := \sum_{j=1}^{m} \left| \frac{f(b_j) - f(d_j)}{b_j - d_j} - \frac{f(c_j) - f(a_j)}{c_j - a_j} \right|,$$

and

$$V^2(f;[a,b]) := \sup_{\pi} \sigma^2(f;\pi),$$

where the supremum is taken over all partitions π of the form (1.1).

The number $V^2(f;[a,b])$ is called de la Vallée Poussin second-variation of the function f on $[a,b]$.

If $V^2(f;[a,b]) < \infty$, the function f is said to have bounded (or finite) second-variation and the set of such functions is denoted by $\mathrm{BV}^2[a,b]$.

The following results are also known (see, e.g. [3, Theorem 1.1] or [5]).

Lemma 1.2

If $V^2(f;[a,b]) < \infty$, then there exists a non-negative constant L such that

$$|f(x) - f(y)| \le L\,|x - y| \qquad (x, y \in [a,b]),$$

and the function f can be expressed as a difference of two convex functions.

Remark 1.1. If $V^2(f;[a,b]) < \infty$, then from the standard properties of convex functions (see e.g. [1, p. 271–300]), we have the existence of the right-hand derivative $f'_+(x_0)$ and left-hand derivative $f'_-(x_0)$ for all $x_0 \in (a,b)$.

2. Main result

In this section we introduce the notion of Riesz $(p,2)$-variation and we give a result similar to Riesz Lemma 1.1, for the class $A_p^2[a,b]$.

Let $f\colon[a,b]\to\mathbb{R}$ and $1<p<\infty$. For a given partition π of the form

$$\pi\colon a=a_1<c_1\le d_1<b_1=a_2<\ldots<b_{m-1}=a_m<c_m\le d_m<b_m=b, \tag{1.1}$$

let

$$\sigma_p^2(f;\pi):=\sum_{j=1}^{m}\left|\frac{f(b_j)-f(d_j)}{b_j-d_j}-\frac{f(c_j)-f(a_j)}{c_j-a_j}\right|^p\frac{1}{(b_j-a_j)^{p-1}}$$

and

$$V_p^2(f;[a,b]):=\sup_{\pi}\sigma_p^2(f;\pi),$$

where the supremum is taken over all partitions π of the form (1.1).

The number $V_p^2(f;[a,b])$ is called Riesz $(p,2)$-variation of the function f on $[a,b]$. If $V_p^2(f;[a,b])<\infty$, the function is said to have bounded (or finite) Riesz $(p,2)$-variation and the set of such functions is denoted by $\mathrm{BV}_p^2[a,b]$.

Lemma 2.1

Let $1<p<\infty$. If $V_p^2(f;[a,b])<\infty$, then f has bounded second-variation and

$$V^2(f;[a,b])\le\left(V_p^2(f;[a,b])\right)^{1/p}|b-a|^{1-1/p}.$$

Proof. Let $\pi\colon a=a_1<c_1\le d_1<b_1=a_2<\ldots<b_{m-1}=a_m<c_m\le d_m<b_m=b$ be a partition of $[a,b]$. Then by Hölder's inequality we obtain

$$\sum_{j=1}^{m}\left|\frac{f(b_j)-f(d_j)}{b_j-d_j}-\frac{f(c_j)-f(a_j)}{c_j-a_j}\right|\frac{|b_j-a_j|^{1-1/p}}{|b_j-a_j|^{1-1/p}}$$

$$\le\left(\sum_{j=1}^{m}\left|\frac{f(b_j)-f(d_j)}{b_j-d_j}-\frac{f(c_j)-f(a_j)}{c_j-a_j}\right|^p\frac{1}{|b_j-a_j|^{p-1}}\right)^{\frac{1}{p}}\cdot\left(\sum_{j=1}^{m}|b_j-a_j|\right)^{1-\frac{1}{p}}.$$

Hence:

$$V^2(f;[a,b])\le\left(V_p^2(f;[a,b])\right)^{1/p}|b-a|^{1-1/p}.\ \square$$

By Lemmas 1.2, 1.3 and 2.1 we obtain:

Corollary 2.1

Let $1 < p < \infty$. If $V_p^2(f;[a,b]) < \infty$, then f is absolutely continuous on $[a,b]$ and f can be expressed as a difference of two convex functions.

Lemma 2.2

Let $1 < p < \infty$. If $V_p^2(f;[a,b]) < \infty$, then we have the existence of a derivative $f'(x_0)$ for all $x_0 \in (a,b)$.

Proof. By Corollary 2.1 and Remark 1.1 we have the existence of a right-hand derivative $f'_+(x_0)$ for all $x_0 \in [a,b)$ and the left-hand derivative $f'_-(x_0)$ for all $x_0 \in (a,b]$.

Suppose that there exists $x_0 \in (a,b)$ such that

$$\alpha_{x_0} := \left|f'_+(x_0) - f'_-(x_0)\right| > 0.$$

By the definition of $(p,2)$-variation we have:

$$\begin{aligned} V_p^2(f;[a,b]) &\geq \lim_{h\to 0}\left|\frac{f(x_0+h)-f(x_0)}{h} - \frac{f(x_0)-f(x_o-h)}{h}\right|^p \frac{1}{2^{p-1}|h|^{p-1}} \\ &= \frac{|\alpha_{x_0}|^p}{2^{p-1}} \lim_{h\to 0} \frac{1}{|h|^{p-1}} = +\infty. \end{aligned}$$

Consequently, the function f has a derivative $f'(x_0)$ for all $x_0 \in (a,b)$. □

Lemma 2.3

Let $1 < p < \infty$. If $V_p^2(f;[a,b]) < \infty$, then $f' \in \mathrm{BV}_p[a,b]$. Moreover

$$V_p(f';[a,b]) \leq V_p^2(f;[a,b]).$$

Thus f' is absolutely continuous on $[a,b]$ and $f'' \in L_p[a,b]$, that is $f \in A_p^2[a,b]$.

Proof. Let $\pi: a = a_1 < c_1 \leq d_1 < b_1 = a_2 < \ldots < b_{m-1} = a_m < c_m \leq d_m < b_m = b$ be a partition of $[a,b]$. Let $h > 0$ be such that

$$0 < h \leq \min\left\{\frac{b_j - a_j}{2}\right\}_{j=1}^m.$$

We have

$$\sum_{j=1}^{m}\left|\frac{f(b_j)-f(b_j-h)}{h}-\frac{f(a_j+h)-f(a_j)}{h}\right|^p\frac{1}{|b_j-a_j|^{p-1}}\leq V_p^2(f;[a,b]).$$

Hence, letting $h\to 0$, and by Lemma 2.2 we obtain:

$$\sum_{j=1}^{m}\frac{|f'(b_j)-f'(a_j)|^p}{|b_j-a_j|^{p-1}}\leq V_p^2(f;[a,b]).$$

Now, by Lemma 1.1 we have $f'\in \mathrm{BV}_p[a,b]$ and thus,

$$V_p(f';[a,b])=\|f''\|^p_{L_p[a,b]}\leq V_p^2(f;[a,b]).\ \square$$

Now we prove that if $f\in A_p^2[a,b]$ then $f\in \mathrm{BV}_p^2[a,b]$. Moreover,

$$V_p^2(f;[a,b])\leq \|f''\|^p_{L_p[a,b]}.$$

Lemma 2.4

Let $1<p<\infty$. If $f\in A_p^2[a,b]$, then $f\in \mathrm{BV}_p^2[a,b]$. Moreover,

$$V_p^2(f;[a,b])\leq \|f''\|^p_{L_p[a,b]}.$$

Proof. Let $\pi\colon a=a_1<c_1\leq d_1<b_1=a_2<\ldots<b_{m-1}=a_m<c_m\leq d_m<b_m=b$ be a partition of $[a,b]$. Since we may assume that f' is continuous on $[a,b]$ we have that

$$\left|\frac{f(b_j)-f(d_j)}{b_j-d_j}-\frac{f(c_j)-f(a_j)}{c_j-a_j}\right|^p=|f'(\tau_j^+)-f'(\tau_j^-)|^p=\left|\int_{\tau_j^-}^{\tau_j^+}f''(\sigma)\,d(\sigma)\right|^p$$
$$\leq\int_{a_j}^{b^j}|f''(\sigma)|^p\,d\sigma\cdot(b_j-a_j)^{p-1},$$

where τ_j^+ and τ_j^- are points in the intervals (d_j,b_j) and (a_j,c_j).

Thus

$$V_p^2(f;[a,b]):=\sup_\pi \sigma_p^2(f;[a,b])\leq \|f''\|^p_{L_p[a,b]}.\ \square$$

By Lemmas 2.3 and 2.4 we obtain the main result:

Theorem

Let $1<p<\infty$. A real function f defined on the interval $[a,b]$ belongs to the class $A_p^2[a,b]$ if and only if $f\in \mathrm{BV}_p^2[a,b]$. Moreover,

$$V_p^2(f;[a,b])=\|f''\|^p_{L_p[a,b]}.$$

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