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On the nonexistence of CR functions on Levi-flat CR manifolds

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Abstract

We show that no compact Levi-flat CR manifold of CR codimension one admits a continuous CR function which is nonconstant along leaves of the Levi foliation. We also prove the nonexistence of certain CR functions on a neighborhood of a compact leaf of some Levi-flat CR 3-manifolds, and apply it to showing that some foliated 3-manifolds cannot be embedded as smooth Levi-flat real hypersurfaces in complex surfaces.

In CR theory it is natural to ask whether or not a given CR manifold has a rich collection of CR functions. This problem has been extensively investigated in the strictly pseudoconvex case. In this short note we consider it in the Levi-flat case. We show that no compact Levi-flat CR manifold of CR codimension one admits non-trivial continuous CR functions (§1). We also obtain another kind of nonexistence result in dimension three, and use it for showing that some smooth codimension one foliations on 3-manifolds cannot be realized as Levi foliations of smooth Levi-flat real hypersurfaces in complex surfaces (§2).

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1. Nonexistence of continuous CR functions

An odd dimensional real manifold M is called a Levi-flat CR manifold of CR codimension one if M has a codimension one foliation $\mathcal{F}$ whose tangent bundle has an integrable complex structure. $\mathcal{F}$ is called the Levi foliation on M. A complex valued continuous function on M is called a continuous CR function if it satisfies the tangential Cauchy-Riemann equations in the distributional sense. By the well-known Weyl lemma, this simply means that the function is holomorphic along each leaf of $\mathcal{F}$.

The first result of this note is the following.

Theorem 1

Every continuous CR function on a closed Levi-flat CR manifold of CR codimension one is constant along leaves of the Levi foliation.

By a result of Wolak [5], this theorem implies further that such a CR function is constant on each connected component of the complement of the union of all compact leaves.

Proof of Theorem. Let M be a closed Levi-flat CR manifold of CR codimension one with the Levi foliation $\mathcal{F}$. Let h be a continuous CR function on M. Passing to a double cover if necessary, we may suppose that $\mathcal{F}$ is transversely orientable. We denote by C the union of all leaves of $\mathcal{F}$ along which h is constant. Now, suppose by contradiction that $C \neq M$. We first note that C is closed. In fact, let L be any leaf in $\overline{C}$ and take any two points p and q of L. Choose a sequence $\{x_n\}$ of points of C so that $x_n \to p$ as $n \to \infty$. Then, by a standard argument in foliation theory, one can find a sequence of points y_n such that x_n and y_n lie on the same leaf for each n and that $y_n \to q$ as $n \to \infty$. Since $h(x_n) = h(y_n)$ for each n, it follows that $h(p) = h(q)$, implying that $L \subset C$ as desired. Now let U be any connected component of $M - C$. Since the border δU of U (See [3, p. 408] for definition) consists of finitely many leaves contained in C ([3, Proposition 2]), by composing a suitable polynomial with h if necessary, we may assume without loss of generality that h vanishes identically on δU (hence, necessarily, on $\overline{U} - U$). Then, $|h||\overline{U}$ must take a nonzero maximum at some point z of U. But this contradicts the Maximum Principle, because h is holomorphic along the leaf through z. The proof of the theorem is complete. □

Question. Does Theorem 1 hold for compact Levi-flat CR manifolds of higher CR codimensions?

2. Nonrealizability as Levi foliations

In this section, we will first prove a kind of nonexistence result for CR functions with certain properties on some 3-dimensional Levi-flat CR manifold, and then will apply it to showing the nonembeddability of some foliated 3-manifolds as Levi-flat real hypersurfaces in complex surfaces.

For a leaf L of a transversely oriented, codimension one foliation, we denote by $\mathcal{H}_+(L)$ the holonomy group of L on the positive side.

Lemma

Let M be a 3-dimensional Levi-flat CR manifold and $\mathcal{F}$ the Levi foliation on M. Suppose that $\mathcal{F}$ is transversely oriented and has a compact leaf K such that $\mathcal{H}_+(K)$ is generated by a contraction. Then there does not exist a continuous CR function f on a neighborhood of K such that f vanishes on K and that $\operatorname{Re} f > 0$ on the positive side of K.

Proof. Suppose there exists such a function f. Let N be a positive-sided neighborhood of K on which f is defined. We may assume that some given leaf L of the restricted foliation $\mathcal{F}|N$ is a Riemann surface with just one end such that ∂L is a circle. Let $L^* = L \cup \Gamma$ be the Royden compactification of L, and let Γ and Δ ($\subset \Gamma$) be the ideal boundary and the harmonic boundary of L respectively (see [4, Chapter III]). We argue two cases separately. Case (I): $\Delta = \emptyset$. Then we can apply Theorem 2H in [4, p. 160] for $G = L$ and $u = \operatorname{Re} f$ to obtain that $\operatorname{Re} f(z) \geq \min_{w \in \partial L} \operatorname{Re} f(w)$ for all $z \in L$. The RHS of this inequality is a positive constant because ∂L is compact, while the LHS can be arbitrarily near 0 as z approaches Γ. This is a contradiction. Case (II): $\Delta \neq \emptyset$. Then by Theorem 4A in [4, p. 172] there exists a harmonic probability measure μ on Γ such that $\operatorname{supp} \mu = \Delta$. In this case, since f vanishes identically on Γ, we can apply Theorem 5H in [4, p. 194] for $E = \Gamma$ and $G = L$ to obtain that f vanishes identically on L. A contradiction. This completes the proof. □

Here is a simple example which shows that the condition $\operatorname{Re} f > 0$ cannot be dropped in the above lemma: Consider a CR diffeomorphism $\varphi(z,t) = (z/2, 2t)$ and a CR function $f(z,t) = zt$ on $(\mathbb{C} - \{0\}) \times \mathbb{R}$ with the standard CR structure. Put $M = (\mathbb{C} - \{0\}) \times \mathbb{R}/\varphi$. Then M is naturally a Levi-flat CR manifold and satisfies all the hypotheses in the above lemma. f induces a nonconstant continuous CR function on M which vanishes on the compact leaf.

Question. Does the above lemma hold for every isolated compact leaf K without any condition on $\mathcal{H}_+(K)$?

Recall that a real codimension one smooth submanifold M of a complex manifold X is a Levi-flat hypersurface of X if $TM \cap J(TM)$ defines a foliation (hence a Levi-flat CR structure) on M, where J is the complex structure of X.

Theorem 2

Let $\mathcal{F}$ be a transversely oriented codimension one C^∞ foliation on an oriented 3-dimensional manifold M. Suppose that $\mathcal{F}$ has a compact leaf K such that $\mathcal{H}_+(K)$ is generated by a single diffeomorphism germ which is infinitely tangent to the identity at 0. Then M cannot be embedded in any complex surface X in such a way that M is a C^∞ Levi-flat hypersurface of X having $\mathcal{F}$ as the Levi foliation.

Remark. In the case when K is a torus leaf, this theorem is proved in [1, Theorem 1]. More generally, in [2, Theorem 3] it is proved that if K is a torus leaf with nontrivial holonomy of the Levi foliation of a C^∞ Levi-flat hypersurface in a complex surface, then every element of the holonomy group of K except the identity must be finitely tangent to the identity at 0.

Proof. Contrary to the conclusion of the theorem, suppose that $(M, \mathcal{F})$ can be realized as a C^∞ Levi-flat hypersurface. Then by [1, Proof of Theorem 1], there is a holomorphic function f defined on a neighborhood of K in X with divisor K such that $d(\operatorname{Re} f)|M$ does not vanish on K. By the hypothesis on $\mathcal{H}_+(K)$, one can find a compact leaf K' (maybe K itself) arbitrarily near K such that $\mathcal{H}_+(K')$ is generated by a contraction. Since K' is compact and f is holomorphic on K', f must be constant on K'. Put $f' = f - f(K')$. Then, f' vanishes on K' and, since K' is near K, $d(\operatorname{Re} f')|M$ does not vanish on K'. Hence, replacing f' with $-f'$ if necessary, we may assume that $\operatorname{Re} f' > 0$ on the positive-side of K' in M. But then the existence of the CR function $f'|M$ contradicts the above lemma. $\square$

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