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Weak sequential completeness of sequence spaces

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ABSTRACT

In [4] Köthe and Toeplitz introduced the theory of sequence spaces and established many of the basic properties of sequence spaces by using methods of classical analysis. Later many of these same properties of sequence spaces were reestablished by using "soft proofs" of functional analysis. In this note we would like to point out that an improved version of a classical lemma of Schur due to Hahn can be used to give very short proofs of two of the weak sequential completeness results of Köthe and Toeplitz. One of our proofs actually gives an improvement of one of the completeness results of Köthe and Toeplitz which was obtained by Benett using functional analysis methods and the method of proof is used in §3 to obtain a completeness result for β-duals of vector-valued sequence spaces. One of our completeness results is employed to obtain a more general form of a Hellinger-Toeplitz type theorem for sequence spaces due to Köthe and the second completeness result is employed to obtain another Hellinger-Toeplitz type theorem for sequence spaces which covers additional cases not covered by Köthe's result.

1. Sequential completeness

Throughout this note λ and μ will denote vector spaces of real-valued sequences each containing the vector space φ consisting of all real-valued sequeces which are eventually 0. If $x \in \lambda$, x_k will denote the k^{th} coordinate of x so $x = \{x_k\}$. The α-dual (β-dual) of λ, λ^α (λ^β), consists of all real-valued sequences y such that the series $\sum_{k=1}^{\infty} |y_k\, x_k|$ $\left(\sum_{k=1}^{\infty} y_k\, x_k\right)$ converges for all $x \in \lambda$. If $y \in \lambda^\alpha$ or $y \in \lambda^\beta$ and $x \in \lambda$, we write $y \cdot x = \sum_{k=1}^{\infty} y_k\, x_k$. Since λ contains φ, the pairs $(\lambda, \lambda^\alpha)$ and (λ, λ^β) are in duality with respect to the bilinear pairing $y \cdot x$, $x \in \lambda$, $y \in \lambda^\alpha$ or λ^β. We denote the weak topologies on λ (λ^β or λ^α) induced by these dualities by $\sigma(\lambda, \lambda^\alpha)$, $\sigma(\lambda, \lambda^\beta)$ $\left(\sigma(\lambda^\beta, \lambda), \sigma(\lambda^\alpha, \lambda)\right)$.

The sequence space λ is said to be monotone if $m_0 \lambda = \lambda$, where m_0 is the space of all sequences with finite range and the product $m_0 \lambda$ is understood to be coordinatewise ([3]). If λ is monotone, $x \in \lambda$ and $y \in \lambda^\beta$, then the series $\sum_{k=1}^{\infty} y_k\, x_k$ is subseries convergent and, hence, absolutely convergent so $\lambda^\alpha = \lambda^\beta$. We prove our first sequential completeness result for monotone sequence spaces. In what follows e_j will denote the sequence which has a 1 in the j^{th} coordinate and 0 in the other coordinates.

Theorem 1

If λ is monotone, then $\sigma(\lambda^\alpha, \lambda)$ is sequentially complete.

Proof. Let $\{y^i\}$ be $\sigma(\lambda^\alpha, \lambda)$ Cauchy. Set $\lim_i y^i \cdot e_j = y_j$ for each j and $y = \{y_j\}$. For $x \in \lambda$ and $\sigma \subseteq \mathbb{N}$, we have from the monotonicity of λ that $\lim_i \sum_{j \in \sigma} y_j^i\, x_j$ exists. By the Schur Lemma ([2] 8.2), $\{y_j\, x_j\} \in \ell^1$ and $\lim_i \sum_{j=1}^{\infty} y_j^i\, x_j = \sum_{j=1}^{\infty} y_j\, x_j$. Hence, $y \in \lambda^\alpha$ and $\lim_i y^i \cdot x = y \cdot x$ for $x \in \lambda$. $\square$

This result was originally established by Köthe and Toeplitz for normal sequence spaces by using a gliding hump argument ([4]) (λ is normal or solid if $y \in \lambda$ and $|y_j| \geq |x_j|$ for all j implies $x = \{x_j\} \in \lambda$; normal spaces are obviously monotone). The result for monotone spaces was established by Bennett using several deep theorems of functional analysis due to Grothendieck ([3] Proposition 3). The simple proof above using the Schur Lemma is an interesting contrast to Bennett's proof.

The monotonicity condition in Theorem 1 cannot be completely eliminated. For example, $\sigma(c^\alpha, c) = \sigma(\ell^1, c)$ is not sequentially complete (see the remarks following Lemma 7).

A sequence space λ is perfect if $\lambda^{\alpha\alpha} = \lambda$. For perfect spaces we have the following sequential completeness property.

Theorem 2

If λ is perfect, then $\sigma(\lambda, \lambda^\alpha)$ is sequentially complete.

Proof. Let $\{x^i\}$ be $\sigma(\lambda, \lambda^\alpha)$ Cauchy. Set $x_j = \lim_i e_j \cdot x^i$ and $x = \{x_j\}$. For $y \in \lambda^\alpha$ and $\sigma \subseteq \mathbb{N}$, since λ^α is normal, $\lim_i \sum_{j \in \sigma} y_j\, x_j^i$ exists. By the Schur Lemma ([2] 8.2), $\{y_j\, x_j\} \in \ell^1$ and $\lim_i \sum_{j=1}^{\infty} y_j\, x_j^i = \sum_{j=1}^{\infty} y_j\, x_j$. Since λ is perfect, $x \in \lambda$ and $\lim_i y \cdot x^i = y \cdot x$ for $y \in \lambda^\alpha$. $\square$

This result was established by Köthe and Toeplitz by using a gliding hump argument ([4]). A functional analysis proof was given by Köthe in [5], 30.5(3). The proof above gives an interesting contrast to Köthe's methods. The converse of Theorem 2 holds and gives a characterization of perfect sequence spaces ([5]); see the proof of Theorem 3 below.

The proof of Theorem 2 using Schur's Lemma can also be used to treat a similar result for the normal topology of a sequence space. The normal topology of λ, η, is the locally convex topology on λ generated by the family of semi-norms $p_y(x) = \sum_{i=1}^{\infty} |x_i y_i|$, $y \in \lambda^\alpha$ ([5] 30.2). For the normal topology, we have the following result.

Theorem 3

λ is perfect if and only if η is sequentially complete.

Proof. Let $\{x^i\}$ be η-Cauchy in λ. Set $x_j = \lim_i e_j \cdot x^i$ and $x = \{x_j\}$. If $y \in \lambda^\alpha$, $\lim_{i,j} \sum_{k=1}^{\infty} |y_k(x_k^i - x_k^j)| = 0$ so by Schur's Lemma ([2] 8.2), $\{x_j \cdot y_j\} \in \ell^1$ and $x \in \lambda^{\alpha\alpha} = \lambda$ with $p_y(x^i - x) \to 0$.

Let $x \in \lambda^{\alpha\alpha}$ and let $x^i = (x_1, \ldots, x_i, 0, \ldots)$ be the i^{th} section of x. Since $\sum_{i=1}^{\infty} |x_i y_i| < \infty$ for each $y \in \lambda^\alpha$, $\{x^i\}$ is η-Cauchy in λ and must converge to an element in λ which is just x. Hence, λ is perfect. □

Köthe showed that λ is perfect if and only if η is complete so Theorem 3 gives a nice complement to this result ([5] 30.5.7).

2. Hellinger-Toeplitz results

We use the sequential completeness to obtain a form of the classical Hellinger-Toeplitz Theorem for sequence spaces which is due to Köthe and Toeplitz ([4]). Our result gives a generalization of the Hellinger-Toeplitz type result of Köthe and Toeplitz to monotone sequence spaces; the proof also corrects the proof of Köthe given in [6] 34.7(7).

Let A be an infinite (scalar) matrix such that A maps λ into μ, i.e., for each $x \in \lambda$, the formal matrix product $Ax = \{\sum_{j=1}^{\infty} a_{ij} x_j\} \in \mu$. The classical Hellinger-Toeplitz Theorem asserts that if the matrix A maps ℓ^2 into ℓ^2, then A is (norm) continuous. We are interested in theorems which assert that A is either continuous or bounded with respect to the weak topologies of λ and μ; if this is done for $\lambda = \mu = \ell^2$, it will imply that a matrix A mapping ℓ^2 into ℓ^2 is bounded and, therefore, norm continous thus giving the classical Hellinger-Toeplitz theorem.

We first establish a result which asserts that $A: \lambda \to \mu$ is $\sigma(\lambda, \lambda^\beta) - \sigma(\mu, \mu^\beta)$ continuous.

Theorem 4

If $\sigma(\lambda^\beta, \lambda)$ is sequentially complete, then A is $\sigma(\lambda, \lambda^\beta) - \sigma(\mu, \mu^\beta)$ continuous.

Proof. Let a^i be the i^{th} row of A so $a^i \in \lambda^\beta$ by hypothesis. For any matrix $A = [a_{ij}]$ mapping λ into μ, for $y \in \mu^\beta$ write $y A = \{\sum_{i=1}^\infty a_{ij} y_i\}_j$, provided the series converge ($y A = A^{\mathrm{T}} y$ with the obvious definition of the transpose of A, A^{T}). Define, $A_n: \lambda \to \mu$ by $A_n x = \sum_{i=1}^n (a^i \cdot x) e_i$ so if $y \in \mu^\beta$, then $y A_n = \{\sum_{i=1}^n a_{ij} y_i\}_j \in \lambda^\beta$. If $x \in \lambda$, $y \in \mu^\beta$, then $y \cdot A x = \lim_n y \cdot A_n x = \lim_n y A_n \cdot x$. Hence, $\{y A_n\}$ is $\sigma(\lambda^\beta, \lambda)$ Cauchy and by sequential completeness must converge to an element of λ^β say, z, and we have $y \cdot A x = z \cdot x$. Therefore, if $\{x^\rho\}$ is a net in λ which is $\sigma(\lambda, \lambda^\beta)$ convergent to 0, then $\{A x^\rho\}$ is $\sigma(\mu, \mu^\beta)$ convergent to 0, and the weak continuity follows. □

Remark 5. If λ is monotone, then $\lambda^\alpha = \lambda^\beta$ and $\sigma(\lambda^\alpha, \lambda)$ is sequentially complete by Theorem 1 so Theorem 4 is applicable when λ is monotone. In particular, if $\lambda = m_0$, then λ is monotone and not normal so Theorem 4 gives an improvement of 34.7(7) of Köthe ([6]) where it is assumed that λ is normal and μ has the weaker topology $\sigma(\mu, \mu^\alpha)$ instead of $\sigma(\mu, \mu^\beta)$. Moreover, the proof of 34.7(7) in [6] cites the use of Theorem 2 (30.5(3) in [5]) whereas it is the sequential completeness of $\sigma(\lambda^\alpha, \lambda)$ not the sequential completeness of $\sigma(\lambda^\alpha, \lambda^{\alpha\alpha})$ that is required.

It also follows from Theorem 4 that A is continuous with respect to the Mackey (strong) topologies of both λ and μ in the dualities between (λ, λ^β) and (μ, μ^β) ([6] 32.2). When λ and μ are some of the classical sequence spaces, e.g. ℓ^2, this gives the continuity of A with respect to the norm topologies; in particular, this gives the classical Hellinger-Toeplitz Theorem for ℓ^2.

We next establish a result concerning the boundedness of A with respect to weak topologies; our result covers the case when $\lambda = c$ which is not included in Theorem 4. For this we require the general uniform boundedness principle of [7] or [2]. If (E, τ) is a topological vector space, a sequence $\{x_k\}$ in E is τ-$\mathcal{K}$ convergent if every subsequence of $\{x_k\}$ has a further subsequence $\{x_{m_k}\}$ such that the series $\sum x_{m_k}$ is τ-convergent in E. A τ-$\mathcal{K}$ convergent sequence converges to 0, but the converse in general does not hold ([2] §3). A subset $B \subseteq E$ is τ-$\mathcal{K}$ bounded if whenever $\{x_k\} \subseteq B$ and $t_k \to 0$, then the sequence $\{t_k x_k\}$ is τ-$\mathcal{K}$ convergent ([1], [2] §3). A τ-$\mathcal{K}$ bounded set is bounded, but in general a bounded set needn't be τ bounded ([2] §3). A space (E, τ) in which every τ bounded set is τ-$\mathcal{K}$ bounded is called an

$\mathcal{A}$-space ([7]). For $\mathcal{A}$-spaces we have the following form of the uniform boundedness principle. If F is a topological vector space and Γ is a family of continuous linear operators from E into F which is pointwise bounded on E and if E is an $\mathcal{A}$-space, then Γ is uniformly bounded on bounded subsets of E ([7] Corollary 4).

Theorem 6

If $(\lambda, \sigma(\lambda, \lambda^\beta))$ is an $\mathcal{A}$-space, then A is $\sigma(\lambda, \lambda^\beta) - \sigma(\mu, \mu^\beta)$ bounded.

Proof. Define $A_n : \lambda \to \mu$ by $A_n x = \sum_{k=1}^{n} (a^k \cdot x) e_k$ as in Theorem 4. Each row, a^k, belongs to λ^β by hypothesis and since the coordinate functionals are weakly continuous, each A_n is weakly continuous. For each $x \in \lambda$, $\{A_n x\}$ is $\sigma(\mu, \mu^\beta)$ convergent to Ax so $\{A_n\}$ is pointwise bounded on λ. Since $(\lambda, \sigma(\lambda, \lambda^\beta))$ is an $\mathcal{A}$-space, by the uniform boundedness theorem for $\mathcal{A}$-spaces described above, $\{Ax : x \in B\}$ is bounded when $B \subseteq \lambda$ is $\sigma(\lambda, \lambda^\beta)$ bounded. □

Any sequentially complete locally convex space is an $\mathcal{A}$-space ([7] Prop. 5) so if λ is sequentially $\sigma(\lambda, \lambda^\beta)$ complete, then Theorem 6 is applicable; this occurs exactly when λ is perfect ([5]).

We show that Theorem 6 is applicable to the case when $\lambda = c$. In this case $\lambda^\alpha = \lambda^\beta = \ell^1$ and $(c, \sigma(c, \ell^1))$ is an $\mathcal{A}$-space (under the pairing between c and $c^\alpha = \ell^1$). This follows from the following observation.

Lemma 7

Let σ and τ be two vector topologies on the vector space X with $\tau \supseteq \sigma$. Suppose (X, τ) is an $\mathcal{A}$-space and σ and τ have the same bounded sets. Then (X, σ) is an $\mathcal{A}$-space.

Proof. If $B \subseteq X$ is σ-bounded, then B is τ-bounded and, hence, τ-$\mathcal{K}$ bounded and σ-$\mathcal{K}$ bounded since $\sigma \supseteq \tau$.

If (λ, τ) is sequentially complete (e.g., an F-space), then (λ, σ) is an $\mathcal{A}$-space for any vector topology $\sigma \subseteq \tau$ such that σ and τ have the same bounded sets (Lemma 7 and [7], Proposition 5). In particular, since c is a B-space under the sup-norm, $(c, \sigma(c, \ell^1))$ is an $\mathcal{A}$-space so Theorem 6 is applicable when $\lambda = c$. Note that in this case c is not monotone and, moreover, $(c, \sigma(c, \ell^1))$ is not sequentially complete so Theorem 4 is also not applicable (if $y_k = \sum_{j=1}^{k} e_j / k \in c^\beta$ then $\lim_k y_k \cdot x = \lim_k x_k$ for each $x \in c$ so $\{y_k\}$ is $\sigma(c^\beta, c)$ Cauchy but $\lim x_k \notin c^\beta$ ([9] Example 7.2.11)). In the case of $\lambda = c$, the matrix map A is norm-$\sigma(\mu, \mu^\beta)$ continuous, and if there is a vector topology τ on μ such that $\mu^\beta = (\mu, \tau)'$ (e.g., if (μ, τ) is a barrelled K-space ([9] 10.5.1)), then A is norm-τ continuous. □

3. Vector sequence spaces

In this section we show that the method of proof of Theorem 1 using the Schur Lemma can be employed to treat vector sequence spaces by employing a vector form of the Schur Lemma. Let X and Y be Hausdorff topological vector spaces, and let $\mathcal{L}(X,Y)$ be the space of all continuous linear operators from X into Y. Let E be a vector space consisting of X-valued sequences, where the operations of addition and scalar multiplication are coordinatewise. If $x \in E$, we write, as before, x_k for the k^{th} coordinate of x. If $x \in X$, $e_j \otimes x$ will denote the sequence with x in the j^{th} coordinate and 0 in the other coordinates. We assume that E contains the span of all such vectors, i.e., E contains the vector space of all X-valued sequences which are eventually 0. Following Maddox we define the β-dual of E (with respect to Y) to be $E^{\beta Y} = \{\{T_j\} \subseteq \mathcal{L}(X,Y) : \sum_{j=1}^{\infty} T_j x_j \text{ converges for every } x \in E\}$ ([8]) (here we require $T_j \in \mathcal{L}(X,Y)$ whereas Maddox only requires linearity). If $T \in E^{\beta Y}$ and $x \in E$, we write $T \cdot x = \sum_{k=1}^{\infty} T_k x_k$.

The space E is said to be monotone if $m_0 E = E$, where the product $m_0 E$ is coordinatewise. We say the pair (X,Y) has the Banach-Steinhaus property if $\{T_j\} \subseteq \mathcal{L}(X,Y)$ and $\lim T_j x = T x$ exists for each $x \in X$ implies that $T \in \mathcal{L}(X,Y)$, i.e., if the conclusion of the classical Banach-Steinhaus Theorem holds. For example, if X is an F-space or if X is barrelled and Y is locally convex, (X,Y) has the Banach-Steinhaus property.

For monotone spaces we have the following generalization of Theorem 1.

Theorem 8

Let E be monotone and let (X,Y) have the Banach-Steinhaus property. If $\{T^k\} \subseteq E^{\beta Y}$ is such that $\lim_k T^k \cdot x$ exists for each $x \in E$ and if

$$T_j x = \lim_k T^k \cdot (e_j \otimes x) = \lim_k T_j^k x \qquad \text{for } x \in X,$$

then $T = \{T_j\} \in E^{\beta Y}$ and $\lim_k T^k \cdot x = T \cdot x$ (note $T_j \in \mathcal{L}(X,Y)$ by the Banach-Steinhaus property).

Proof. Let $x \in E$. By the monotonicity of E, $\lim_k \sum_{j \in \sigma} T_j^k x_j$ exists for each $\sigma \subseteq \mathbb{N}$. Since $T_j x_j = \lim_k T_j^k x_j$, the vector form of the Schur Lemma given in Theorem 8.1 of [2] implies that $\sum_{j=1}^{\infty} T_j x_j$ converges and $\lim_k \sum_{j=1}^{\infty} T_j^k x_j = \sum_{j=1}^{\infty} T_j x_j$. Thus, $T = \{T_j\} \in E^{\beta Y}$ and $\lim_k T^k \cdot x = T \cdot x$ for every $x \in E$. □

If $\sigma(E^{\beta Y}, E)$ denotes the weakest topology on $E^{\beta Y}$ such that the map $x \to T \cdot x$ is continuous for each $x \in E$, then Theorem 8 asserts that $\sigma(E^{\beta Y}, E)$ is sequentially complete. Theorem 1 shows that this occurs in the scalar case when E is monotone.

Note that the duality methods of Bennett ([3]) cannot be used to obtain Theorem 8 whereas the general vector form of the Schur Lemma yields the result very easily.

References

1. P. Antosik, On uniform boundedness of families of mappings, *Proceedings of the Conference on Convergence Structures*, Szczyrk (1979), 2–16.
2. P. Antosik and C. Swartz, Matrix Methods in Analysis, *Lecture Notes in Mathematics* **1113**, Springer-Verlag, Heidelberg, 1985.
3. G. Bennett, A new class of sequence spaces with applications in summability theory, *J. reine angew. Math.* **266** (1974), 49–75.
4. G. Köthe and O. Toeplitz, Lineare Räume mit unendlich vielen Koordinaten und Ringe unendlichen Matrizen, *J. reine angew. Math.* **171** (1934), 193–226.
5. G. Köthe, *Topological Vector Spaces* I, Springer-Verlag, New York, 1969.
6. G. Köthe, *Topological Vector Spaces* II, Springer-Verlag, New York, 1979.
7. R. Li and C. Swartz, Spaces for which the Uniform Boundedness Principle holds, *Studia Scien. Math. Hung.*, (to appear).
8. I. Maddox, Infinite Matrices of Operators, *Lecture Notes In Mathematics* **786**, Springer-Verlag, Berlin, 1980.
9. A. Wilansky, *Summability through Functional Analysis*, North Holland, Amsterdam, 1984.