Collect. Math. **43**, 1 (1992), 43–53

Unordered Baire-like spaces without local convexity

J. Kakol[1]

Institute of Mathematics, A. Mickiewicz University
Matejki 48/49, 60–769 Poznan, Poland

W. Roelcke

Mathematisches Institut der Universität
Theresienstr. 39, D–8000 München 2, Deutschland

Received December 1, 1991

Abstract

The aim of the present paper is to study the class of tvs which we define by ommiting the word "increasing" in the definition of $*$-suprabarrelled spaces. We prove that the product of Baire tvs is $*$-UBL and hence the class of $*$-UBL spaces is strictly larger than the class of Baire spaces.

1. Introduction

Following Saxon [9] a locally convex space (lcs) E is called Baire-like if E is not the union of an increasing sequence of nowhere dense absolutely convex sets. By dropping here the word "increasing" one obtains the definition of an unordered Baire-like space in the sense of Todd and Saxon [12]. In [12], Theorem 2.2, Todd and Saxon proved that E is unordered Baire-like iff for each sequence $(E_n)_{n\in\mathbb{N}}$ of subspaces of E covering E some E_n is both dense and barrelled. In [7] W. Robertson, Tweddle and Yeomans introduced the class of lcs E with the following property:

(db) if E is the union of an increasing sequence of subspaces, then one of them is both dense and barrelled.

[1] This work was undertaken with the support of the AvH

In [15] the spaces with property (db) were called suprabarrelled. In contrast to Baire spaces Baire-like spaces, unordered Baire-like spaces and suprabarrelled spaces enjoy good permanence properties, i.e., products, quotients, countable-codimensional subspaces of such spaces are spaces of the same type, respectively, [9], [12], [15]. Every barrelled space not containing ϕ, i.e., an $\aleph_0$-dimensional vector space with the strongest locally convex topology, is Baire-like [9], Theorem 2.1; hence all metrizable barrelled spaces are Baire-like. In [10] and [4] examples of normed Baire-like spaces which are not unordered Baire-like and unordered Baire-like spaces which are not Baire are given.

Much of the importance of suprabarrelled spaces comes from their connection with the closed graph theorems. It turns out that for the proof of the closed graph theorem of Todd and Saxon [12], Theorem 2, the (db) property is a sufficient condition on the domain space, where the range space is an (LB)-space. Distinguishing examples among suprabarrelled spaces and (unordered) Baire-like spaces are obtained by Saxon and Narayanaswami in [11]. In particular, they showed that every infinite dimensional Fréchet space contains a dense subspace which is suprabarrelled but not unordered Baire-like. The space m_0 of Saxon [10], Ex. 1.4, provides a concrete example of a normed Baire-like space which is not unordered Baire-like. In [16] Valdivia proved that m_0 is even a suprabarrelled space. A natural extension of the (db) property to the class of arbitrary topological vector spaces (tvs) was introduced by Perez Carreras [6]. He called a tvs E $*$-suprabarrelled if for every increasing sequence of subspaces of E covering E there is one of them which is dense and ultrabarrelled. Every $*$-suprabarrelled space is ultrabarrelled. Metrizable (LF)-spaces (such spaces do exist [3], [11]) provide examples of ultrabarrelled spaces which are not $*$-suprabarrelled by an obvious application of Adasch's closed graph theorem [1], 8(6).

The aim of the present paper is to study the class of tvs which we define by ommiting the word "increasing" in the definition of $*$-suprabarrelled spaces: We shall say that a tvs is $*$-unordered Baire-like (shortly $*$-UBL) if every sequence of subspaces of E covering E contains a member which is both dense and ultrabarrelled. It is known that the Baire property is not productive [2], [14]. We prove however that the product of Baire tvs is $*$-UBL. Hence the class of $*$-UBL spaces is strictly larger than the class of Baire spaces. In [13], Theorem 5, Valdivia showed that every infinite dimensional Fréchet space E contains a dense subspace F which is not unordered Baire-like (hence not $*$-UBL) but has the following property: Given a sequence $(F_n)_{n\in\mathbb{N}}$ of subspaces of E covering F, there exists $n \in \mathbb{N}$ such that F_n is ultrabarrelled and its closure in F is of finite codimension in F. In particular F

is $*$-suprabarrelled. Now it is clear that none of the following implications can be reversed:

$$\text{Baire} \implies *\text{-UBL} \implies *\text{-suprabarrelled} \implies \text{ultrabarrelled}.$$

Plainly every lcs which is $*$-UBL is unordered Baire-like. On the other hand the sequence space l^p, $0 < p < 1$, endowed with the topology induced on l^p by the original norm $\|\cdot\|_1$ of l^1 provides a concrete example of a normed unordered Baire-like space which is not ultrabarrelled [4], p. 111, and hence not $*$-UBL.

In Section 2 of the present paper we characterize $*$-UBL spaces in terms of Baire-type properties and we give a characterization of the $*$-UBL spaces by means of a Banach-Steinhaus theorem. This generalizes Saxon's Theorem 2.1 of [10]. Section 3 is concerned with products of $*$-UBL spaces.

All tvs considered in this paper are assumed to be infinite dimensional and Hausdorff over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For a tvs (E, τ), $\mathfrak{U}_0(E)$ or $\mathfrak{U}_0(\tau)$ will denote the filter of all neighbourhoods of zero in (E, τ). By a string in (E, τ) we understand (after Adash [1]) a sequence $(U_j)_{j \in \mathbb{N}}$ of balanced and absorbing sets of E such that $U_{j+1} + U_{j+1} \subset U_j$ for all $j \in \mathbb{N}$. A string $(U_j)_{j \in \mathbb{N}}$ is called
(a) closed, if every U_j is τ-closed;
(b) topological, if every U_j is a τ-neighbourhood of zero.

A tvs (E, τ) is called ultrabarrelled [1] if every closed string in E is topological. The following conditions are equivalent:
(1) (E, τ) is ultrabarrelled.
(2) Every linear map from (E, τ) into a metrizable and complete tvs with closed graph is continuous.
(3) Every Hausdorff vector topology ϑ on E which is τ-polar, i.e., $\mathfrak{U}_0(\vartheta)$ has a basis consisting of τ-closed sets, is coarser than τ, [1], p. 32, p. 44.

In the sequel we shall need several times the following result of [12], Theorem 4.1:
(O) If the union of two countable families of subspaces of a tvs E covers E, then one of those families covers E.

2. Some properties of $*$-UBL spaces

The proofs of the following two Propositions are straightforward.

Proposition 2.1

Every Hausdorff quotient of a $*$-UBL space is $*$-UBL.

Proposition 2.2

Let F be a dense subspace of a tvs E. If F is $*$-UBL, then E is $*$-UBL.

Proposition 2.3

Every countable-codimensional subspace F of a $*$-UBL space E is $*$-UBL.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a co-basis of F in E. For every $n \in \mathbb{N}$ let $F_n = F + \operatorname{lin}\{x_1, \ldots, x_n\}$. Observe that there is $n \in \mathbb{N}$ such that F_n is $*$-UBL. If not, then for every $n \in \mathbb{N}$ there exists a sequence $(F_{n,k})_{k\in\mathbb{N}}$ of subspaces of F_n covering F_n such that none of the $F_{n,k}$ is both dense and ultrabarrelled. But $(F_{n,k})_{n,k\in\mathbb{N}}$ covers E, a contradiction. Hence we may assume that F has codimension 1 in E.

Now Proposition 2.3 follows from the following three facts: (1) If E is a tvs such that every sequence of subspaces of E covering E contains a dense member, then a subspace F of E of codimension 1 in E has the same property, cf. [12], the proof of Theorem 4.4. (2) (O) from the Introduction. (3) Every finite-codimensional subspace of an ultrabarrelled space is ultrabarrelled, [1], 6(10). □

The main result of this section gives a characterization of $*$-UBL spaces by means of a Banach-Steinhaus theorem. We shall need the following

DEFINITION 2.4. Let E be a tvs. We call a sequence $(V_j)_{j\in\mathbb{N}}$ of balanced subsets of E semistring if $V_{j+1} + V_{j+1} \subset V_j$, $j \in \mathbb{N}$. If moreover every V_j is absorbing in $\operatorname{lin} V_1$, then $(V_j)_{j\in\mathbb{N}}$ will be called an almost string. By a semibarrel (almost barrel) we will call a balanced subset V of E for which there exists a semistring (almost string) $(V_j)_{j\in\mathbb{N}}$ with $V_1 = V$. We will say that E is semi-Baire (almost Baire) if E has no cover by a sequence of nowhere dense semibarrels (almost barrels).

Obviously, Baire tvs $\Rightarrow$ semi-Baire $\Rightarrow$ almost Baire.

Proposition 2.5

A tvs (E, τ) is $*$-UBL iff it is almost Baire.

Proof. Assume that (E, τ) is almost Baire but not $*$-UBL. Then there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of subspaces of E covering E such that none of the E_n is both dense and ultrabarrelled. By assumption there is $n \in \mathbb{N}$ such that E_n is dense. By (O) we may assume that all E_n are dense. For every $n \in \mathbb{N}$ there exists a closed string $(U_j^n)_{j\in\mathbb{N}}$ in E_n such that $U_j^n \notin \mathfrak{U}_0(E_n)$, $j \in \mathbb{N}$. Since $(U_2^n)_{n\in\mathbb{N}}$ is a sequence of almost barrels in E and $(mU_2^n)_{n,m\in\mathbb{N}}$ covers E, there exists $p \in \mathbb{N}$ such that $\operatorname{Int} \overline{U_2^p} \neq \emptyset$. Therefore $U_1^p \in \mathfrak{U}_0(E_p)$, a contradiction.

Now assume that (E,τ) is $*$-UBL. Let $(V^n)_{n\in\mathbb{N}}$ be a sequence of almost barrels covering E. Since $E = \bigcup_{n=1}^{\infty} \operatorname{lin} V^n$, there exists $n \in \mathbb{N}$ such that $\operatorname{lin} V^n$ is both dense and ultrabarrelled. Choose an almost string $(V_j^n)_{j\in\mathbb{N}}$ associated with V^n in the sense of 2.4. Since each V^n is absorbing in $\operatorname{lin} V^n$, the closures W_j^n of V_j^n in $\operatorname{lin} V^n$ for $j \in \mathbb{N}$ form a closed string in $\operatorname{lin} V^n$, whence W_j^n is a neighbourhood of zero in $\operatorname{lin} V^n$. The space $\operatorname{lin} V^n$ is τ-dense, so $\overline{V}^{\,n} \in \mathfrak{U}_0(E)$. Hence (E,τ) is almost Baire. □

Now we are ready to prove the following

Theorem 2.6

A tvs (E,τ) is $*$-UBL iff for every sequence $(\mathfrak{F}_n)_{n\in\mathbb{N}}$ of families of continuous linear maps from (E,τ) into a tvs F such that for each $x \in E$ the set $\{f(x): f \in \mathfrak{F}_n\}$ is bounded for some $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\mathfrak{F}_m$ is equicontinuous.

Proof. Our proof uses some ideas found in [1], p. 38–39. Assume that (E,τ) is $*$-UBL and let $(\mathfrak{F}_n)_{n\in\mathbb{N}}$ be given as in the theorem. For each $n \in \mathbb{N}$ set

$$M_n = \Big\{x \in E\colon \{f(x): f \in \mathfrak{F}_n\} \text{ is bounded in } F\Big\}.$$

By assumption the sequence of subspaces $(M_n)_{n\in\mathbb{N}}$ covers E; hence there exists $m \in \mathbb{N}$ such that M_m is both dense and ultrabarrelled. By [1], 7(3), the family $\mathfrak{F}_m$ is equicontinuous on M_m and consequently $\mathfrak{F}_m$ is equicontinuous on E.

Now we prove the reverse implication. Let $(V^t)_{t\in A}$ be a basis of balanced τ-neighbourhoods of zero. For every $t \in A$ let $(V_j^t)_{j\in\mathbb{N}}$ be a topological string in (E,τ) such that $V_1^t + V_1^t \subset V^t$. Set $H = \oplus_{t\in A} E_t$ (algebraically), where $E_t = E$ for all $t \in A$. Let $T_t\colon E_t \to H$ be the canonical inclusion, $t \in A$. Assume that $(V^n)_{n\in\mathbb{N}}$ is a sequence of almost barrels in (E,τ) covering E and $(V_j^n)_{j\in\mathbb{N}}$ $(n \in \mathbb{N})$ the associated almost strings. Set

$$W_j^n = \bigoplus_{t\in A} (V_j^n + V_j^t), \qquad n, j \in \mathbb{N}.$$

Then $(W_j^n)_{j\in\mathbb{N}}$ is a string in H, $n \in \mathbb{N}$. Set $F_n = H/\bigcap_{j=1}^{\infty} W_j^n$ and let $h_n\colon H \to F_n$ be the quotient map, $n \in \mathbb{N}$. Since $\bigcap_{j=1}^{\infty} h_n(W_j^n) = 0$, then the sets $h_n(W_j^n)$, $j \in \mathbb{N}$, form a basis of neighbourhoods of zero for a metrizable vector topology ϑ_n on F_n. By (F,ϑ) we denote the product space of the sequence $(F_n,\vartheta_n)_{n\in\mathbb{N}}$. Let f_n be the canonical injection of F_n into F, $n \in \mathbb{N}$. Since $T_t(V_j^t) \subset \oplus_{t\in A}(V_j^n + V_j^t)$, $(\mathfrak{F}_n)_{n\in\mathbb{N}}$ with

$$(*) \qquad \mathfrak{F}_n = \big(f_n \circ h_n \circ T_t\big)_{t\in A}$$

is a sequence of families of continuous linear maps from (E,τ) into (F,ϑ). Let $x \in E$, then $x \in V^n$ for some $n \in \mathbb{N}$. Now $f_n(h_n(T_t(x)))_{t\in A}$ is bounded in F. In fact, let $U \in \mathfrak{U}_0(\vartheta)$. Then, there exists $j \in \mathbb{N}$ such that

$$U \cap f_n(F_n) \supset f_n\big(h_n\big(\bigoplus_{t\in A}(V_j^n + V_j^t)\big)\big).$$

Since V_j^n is absorbing in lin V^n, there exists $\lambda > 0$ such that $\lambda x \in V_j^n$. Consequently, on account of the definition of T_t, one has $\lambda f_m(h_m(T_t(x))) \subset U$. We have proved that for every $x \in E$ there exists $n \in \mathbb{N}$ such that $\{f(x): f \in \mathfrak{F}_n\}$ is bounded in F. By assumption there is $m \in \mathbb{N}$ such that $\mathfrak{F}_m$ is an equicontinuous family from (E,τ) into (F,ϑ). This implies that the family $\{h_m \circ T_t\}_{t\in A}$ is equicontinuous from (E,τ) into (F_m,ϑ_m). Therefore

$$\bigcap_{t\in A}\big(h_m \circ T_t\big)^{-1}\big(h_m(W_j^m)\big) \in \mathfrak{U}_0(\tau) \qquad \text{for all } j \in \mathbb{N}.$$

On the other hand we have

$$\begin{aligned}
&\bigcap_{t\in A}\big(h_m \circ T_t\big)^{-1}\big(h_m(W_j^m)\big)\\
&\quad\subset \bigcap_{t\in A} T_t^{-1}(W_j^m) = \bigcap_{t\in A}(V_j^m + V_j^\tau)\\
&\quad\subset \bigcap_{t\in A}(V_j^m + V^t) = \overline{V}_j^{\,m} \subset \overline{V}^{\,m}.
\end{aligned}$$

This proves that $\overline{V}^{\,m} \in \mathfrak{U}_0(\tau)$. Therefore (E,τ) is almost Baire. By Proposition 2.5 (E,τ) is $*$-UBL. □

Remark 2.7. (a) By the above proof, the condition in 2.6 goes over into an equivalent one if the spaces F are restricted to be (complete and) metrizable.
(b) The convex case of Theorem 2.6, i.e., for locally convex spaces E and F, is essentially contained in the characterization of the unordered Baire-like property obtained by Saxon [10], Theorem 2.1.

3. Product of $*$-UBL spaces

It is known that the product of two normed Baire spaces may not be Baire [2], [14], but every product of Baire-like, unordered Baire-like, or $*$-suprabarrelled spaces is a space of the same type, respectively, [9], [12], [15]. In this section we prove that every product of semi-Baire spaces is $*$-UBL and finite products of $*$-UBL spaces are $*$-UBL. Hence, in particular, every product of Baire spaces is $*$-UBL. If J is a subset of an index set I, we identify canonically $\prod_{i\in J} E_i$ with a subspace of $\prod_{i\in I} E_i$.

We shall need the following

Lemma 3.1 (Pfister)

Let $E = \prod_{i\in I} E_i$ be the product of nonempty topological spaces E_i. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of E covering E. Then there exists $n \in \mathbb{N}$ and a finite subset J of I such that $K_n \supset \prod B_i$, where $B_i = E_i$ for all $i \in I \setminus J$ and B_i consists exactly one point for $i \in J$.

Proof. It is enough to consider the case when each E_i is endowed with the discrete topology τ_i and apply the Baire category theorem for the Baire space $\prod_{i\in I}(E_i, \tau_i)$. □

Corollary 3.2 ([3], [1], [9])

Every product of barrelled (ultrabarrelled), or Baire-like spaces is a space of the same type, respectively.

Corollary 3.3

Let $E = \prod_{i\in I} E_i$ be the product of tvs E_i and let $(V^n)_{n\in\mathbb{N}}$ be a sequence of almost barrels of E covering E, with associated almost strings $(V_j^n)_{n\in\mathbb{N}}$. Then for every $j \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and a finite subset J of I such that

$$\overline{V}_j^n \supset \prod_{i\in I\setminus J} E_i.$$

Proposition 3.4

Let E_1, E_2 be $*$-UBL spaces. Then the product $E_1 \times E_2$ is $*$-UBL.

Proof. With a view to 2.5 let $(V^n)_{n\in\mathbb{N}}$ be a sequence of almost barrels covering $E_1 \times E_2$ with associated almost strings $(V_j^n)_{j\in\mathbb{N}}$. Assume that $\operatorname{Int}\overline{V}^n = \emptyset$ for all $n \in \mathbb{N}$. Let $\mathcal{F} = \{\operatorname{lin} V_2^n : n \in \mathbb{N}\}$ and put

$$\mathbb{N}_i = \{n \in \mathbb{N}: \operatorname{Int}_i(\overline{V}_2^n \cap E_i) = \emptyset\}, \quad \mathcal{F}_i = \{\operatorname{lin} V_2^n : n \in \mathbb{N}_i\}, \qquad i = 1, 2,$$

where Int_i denotes the interior with respect to E_i. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ by an indirect argument. Using (O) from Introduction we may assume that

$$E_1 = \bigcup_{p=1}^{\infty} \bigcup_{n\in\mathbb{N}_1} pV_2^n \cap E_1.$$

By assumption $\operatorname{Int}_1(\overline{V}_2^n \cap E_1) \neq \emptyset$ for some $n \in \mathbb{N}_1$, a contradiction. By 2.5 this ends the proof. □

Theorem 3.5

Every product $E = \prod_{i\in I} E_i$ of semi-Baire spaces is a $*$-UBL space.

Proof. First we consider the case when $I = \mathbb{N}$. Let $(V^n)_{n\in\mathbb{N}}$ be a sequence of almost barrels covering E with associated almost strings $(V_j^n)_{j\in\mathbb{N}}$ and assume that $\operatorname{Int}\overline{V}^n = \emptyset$ for all $n \in \mathbb{N}$. Set

$$\mathbb{N}_1 = \Big\{n \in \mathbb{N}: \overline{V}_2^n \supset \prod_{m\geq p(n)} E_m \text{ for some } p(n) \in \mathbb{N}\Big\}, \qquad \mathbb{N}_2 = \mathbb{N} \setminus \mathbb{N}_1.$$

Let $\mathcal{F}_i = \{\operatorname{lin} V_2^n : n \in \mathbb{N}_i\}$, $i = 1, 2$, and $\mathcal{F} = \{\operatorname{lin} V_2^n : n \in \mathbb{N}\}$. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Assume that $\mathcal{F}_1$ does not cover E. Since $\mathcal{F}$ covers E, the family $\mathcal{F}_2$ covers E by (O). Hence $E = \bigcup_{n\in\mathbb{N}_2} \bigcup_{p=1}^{\infty} p\overline{V}_2^n$. Using Corollary 3.3 we get a contradiction to the definition of $\mathbb{N}_2$. Hence we may assume that for every $n \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that

$$(X) \qquad \overline{V}_2^n \supset \prod_{m\geq p} E_m.$$

Let

$$\mathbb{N}_m = \big\{n \in \mathbb{N}: \operatorname{Int}_m\big(\overline{V}_{j_n}^n \cap E_m\big) = \emptyset \text{ for some } j_n \in \mathbb{N}\big\}, \quad m \in \mathbb{N}.$$

Set

$$\mathcal{B}_m = \big\{\overline{V}_2^n : n \in \mathbb{N}_m\big\}, \qquad \mathcal{K} = \big\{\overline{V}_2^n : n \in \mathbb{N}\big\}.$$

Using (X) one shows indirectly that $\mathcal{K} = \bigcup_{m=1}^{\infty} \mathcal{B}_m$. Fix an arbitrary $x \in E$. Then there is $n \in N$ such that $x \in V^n$. Since $\overline{V}_2^n \in \mathcal{B}_m$ for some $m \in \mathbb{N}$, we find $j \in \mathbb{N}$ such that $\operatorname{Int}_m(\overline{V}_j^n \cap E_m) = \emptyset$. Since $\operatorname{lin} V_{j+1}^n = \operatorname{lin} V^n$, we obtain that $x \in tV_{j+1}^n$ for some $t \in \mathbb{N}$.

Hence

$$E = \bigcup_{t,m=1}^{\infty} \bigcup_{(n,j)\in S_m} tV_{j+1}^n, \quad \text{where} \quad S_m := \big\{(n,j) \in \mathbb{N} \times \mathbb{N}: \operatorname{Int}_m(\overline{V}_j^n \cap E_m) = \emptyset\big\}.$$

We prove that there exists $m \in \mathbb{N}$ such that

$$(XX) \qquad E_m = \bigcup_{t=1}^{\infty} \bigcup_{(n,j)\in S_m} t\overline{V}_j^n \cap E_m.$$

Suppose that for every $m \in \mathbb{N}$ there exists $x_m \in E$ such that

$$x_m \notin \bigcup_{t=1}^{\infty} \bigcup_{(n,j)\in S_m} t\overline{V}_j^n \cap E_m.$$

Set

$$A = \{x \in E: x(m) = \lambda_m x_m,\ |\lambda_m| \le 1,\ \lambda_m \in \mathbb{K},\ m \in \mathbb{N}\}.$$

Then A is absolutely convex and compact in E. Endow $\operatorname{lin} A$ with the Banach topology generated by the Minkowski functional of the set A. Since $\operatorname{lin} A = \bigcup_{t,m=1}^{\infty} \bigcup_{(n,j)\in S_m} tV_{j+1}^n \cap \operatorname{lin} A$, then by the Baire category theorem we find $t, m \in \mathbb{N}$, and $(n, j) \in S_m$ such that $A \subset t\overline{V}_j^n$, a contradiction (since $x_m \in A$ for all $m \in \mathbb{N}$). Therefore (XX) holds indeed. Since E_m is semi-Baire, $\operatorname{Int}_m(\overline{V}_j^n \cap E_m) \neq \emptyset$ for some $(n, j) \in S_m$, a contradiction to the definition of S_m. We have proved that E is almost Baire for $I = \mathbb{N}$, hence $*$-UBL by Proposition 2.5.

Now we consider the case when I is an arbitrary index set. Let E_0 be the subspace of E formed by vectors which have at most countably many non-zero components. Clearly E_0 is dense in E. In order to prove that E is $*$-UBL it is enough to show that E_0 is $*$-UBL by Proposition 2.2. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of subspaces of E_0 covering E_0. Let H_n be the closure of F_n in E, $n \in \mathbb{N}$. By Lemma 4.9 of [12] we have $E = \bigcup_{n=1}^{\infty} H_n$. By Lemma 3.1 there exists $n \in \mathbb{N}$ and a finite set $J(n)$ of I such that $H_n \supset \prod_{i\in I\setminus J(n)} E_i$. Using (O) one obtains that the family $\{H_n: H_n \supset \prod_{i\in I\setminus J(n)} E_i\}$ covers E. So we may assume that $H_n \supset \prod_{i\in I\setminus J(n)} E_i$ for all $n \in \mathbb{N}$. Now an indirect argument using Lemma 9.2.9 of [3] and the fact that the spaces E_i are $*$-UBL shows that $H_n = E$ for some $n \in \mathbb{N}$. Hence F_n is dense in E_0. Using (O) from Introduction we may assume that every F_n is dense in E_0. Suppose that none of the F_n is ultrabarrelled. Then for every $n \in \mathbb{N}$ there exists a closed string $(U_j^n)_{j\in\mathbb{N}}$ in F_n which is not topological. On the other hand we prove that there exists $m \in \mathbb{N}$ such that $(\overline{U}_j^m \cap E_0)_{j\in\mathbb{N}}$ is a closed string in E_0. Since E_0 is ultrabarrelled [5], this is a contradiction.

We have only to show that, for some $m \in \mathbb{N}$, $\overline{U}_j^m \cap E_0$ is absorbing in E_0 for all $j \in \mathbb{N}$. Suppose that for every $n \in \mathbb{N}$ there exists $j_n \in \mathbb{N}$ and $x_n \in E_0$ which is not absorbed by $\overline{U}_{j_n}^n$. Let $J_n = \{i \in I: x_n(i) \neq 0\}$ and put $M = \prod_{i\in J} E_i$, where $J = \bigcup_{n=1}^{\infty} J_n$. Since $(F_n \cap M)_{n\in\mathbb{N}}$ covers M and M is $*$-UBL (by the countable case), there exists $m \in \mathbb{N}$ such that $F_m \cap M$ is both dense in M and ultrabarrelled. Hence $(U_j^m \cap M)_{j\in\mathbb{N}}$ is a topological string in $F_m \cap M$, consequently $(\overline{U_j^m \cap M})_{j\in\mathbb{N}}$ is topological in M. Therefore $\overline{U}_{j_m}^m$ absorbs x_m, a contradiction. This completes the proof. □

Corollary 3.6

Every product of Baire spaces is $*$-UBL.

Remark 3.7. (a) We do not know whether countable products of $*$-UBL spaces are $*$-UBL. From the proof of Theorem 3.5 it follows that a product $\prod_{i \in I} E_i$ of $*$-UBL spaces is $*$-UBL iff for every countable subset J of I the product $\prod_{i \in J} E_i$ is $*$-UBL.

(b) As an immediate consequence of the definition of $*$-UBL spaces, the proof of Adasch's closed graph theorem [1], 10(11), p. 57–58, a) yields the following result:

(+) Let E be a $*$-UBL space and F a vector space. Assume that for every $n \in \mathbb{N}$ $f_n: F_n \to F$ is an injective linear map from an infra-s-space F_n into F. If $F = \bigcup_{n=1}^{\infty} f_n(F_n)$ and F is endowed with the finest vector topology such that all f_n are continuous, then every linear map $f: E \to F$ with closed graph is continuous and f induces a continuous map of E into $f_n(F_n)$ for some $n \in \mathbb{N}$, where $f_n(F_n)$ is provided with the final topology with respect to f_n.

Our Theorem 3.5 combined with (+) leads to a generalization of Adasch's closed graph theorem [1], 10(11).

ADDED IN PROOF. $*$-UBL spaces were also considered by J. Izquierdo, *Sobre ciertas clases de espacios hipertonelados*, Tesis Doctoral, Valencia 1982.

References

1. N. Adasch, B. Ernst, D. Keim, Topological vector spaces, *Lecture Notes in Math.* **639** (1978).
2. J. Arias de Reyna, Normed barely Baire spaces, *Israel J. Math.* **42** (1982), 33–36. Correction, *Israel J. Math.* **50** (1985), 264.
3. J. Bonet, P. Perez Carreras, *Barrelled locally convex spaces*, North-Holland Math. Studies, Amsterdam 1987.
4. P. Dierolf, S. Dierolf, L. Drewnowski, Remarks and examples concerning unordered Baire-like spaces and ultrabarrelled spaces, *Colloq. Math.* **39** (1978), 110–117.
5. S. Dierolf, The barrelled spaces associated with a bornological space need not be bornological, *Bull. London Math. Soc.* **12** (1980), 60–62.
6. P. Perez Carreras, Sobre ciertas clases de espacios vectoriales topologicos, *Rev. Real Acad. Ci.* **76** (1982), 585–594.
7. W.J. Robertson, I. Tweddle, F.E. Yeomans, On the stability of barrelled topologies III, *Bull. Austr. Math. Soc.* **22** (1980), 99–112.
8. W. Roelcke, S. Dierolf, On the three-space problem for topological vector spaces, *Collect. Math.* **32** (1981), 3–25.
9. S.A. Saxon, Nuclear and product spaces, Baire-like spaces and the strongest locally convex topology, *Math. Ann.* **197** (1972), 87–106.
10. S.A. Saxon, Some normed barrelled spaces which are not Baire, *Math. Ann.* **209** (1974), 153–160.

11. S.A. Saxon, P.P. Narayanaswami, Metrizable (LF)-spaces, (db)-spaces, and the separable quotient problem, *Bull. Austr. Math. Soc.* **23** (1981), 65–80.
12. A. Todd, S.A. Saxon, A property of locally convex Baire spaces, *Math. Ann.* **206** (1973), 23–34.
13. M. Valdivia, Classes of barrelled spaces related with the closed graph theorem, *Portug. Math.* **40** (1981), 345–366.
14. M. Valdivia, Products of Baire topological vector spaces, *Fundamenta Math.* **75** (1985), 71–80.
15. M. Valdivia, On suprabarrelled spaces, *Proc. Funct. Anal. holomorphy and approx. theory, Lecture Notes in Math.* **843** (Rio de Janeiro 1978).
16. M. Valdivia, On certain normed barrelled spaces, *Ann. Inst. Fourier* **29** (1979), 39–56.