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A note on regular elements in Calkin algebras

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Abstract

An element a of the Banach algebra A is said to be regular provided there is an element $b \in A$ such that $a = aba$. In this note we study the set of regular elements in the Calkin algebra $C(X)$ over an infinite-dimensional complex Banach space X.

Let A denote a complex Banach algebra with identity 1. An element a in A is said to be regular provided there is an element b in A such that $a = aba$. We say that a is decomposably regular provided the b in the preceding equation can be chosen to be an invertible element in A. Let A^{-1} denote the set of all invertible elements in A. Set $\widehat{A} = \{a \in A\colon a \in aAa\}$ and $A^\bullet = \{a \in A\colon a^2 = a\}$. It is easy to see that

$$A^{-1}A^\bullet = A^\bullet A^{-1} = \{a \in A\colon a \in aA^{-1}a\}. \tag{0.1}$$

For a subset M of A let δM and $cl\, M$ denote, respectively, the boundary and the closure of M. Harte [8, Theorem 1.1] has proved that

$$A^{-1}A^\bullet = \widehat{A} \cap cl\,(A^{-1}). \tag{0.2}$$

Let X be an infinite-dimensional complex Banach space and denote the set of bounded (compact) linear operators on X by $B(X)$ ($K(X)$). The fact that $K(X)$ is

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a closed two-sided ideal in $B(X)$ enables us to define the Calkin algebra over X as the quotient algebra $C(X) = {}^{B(X)}/_{K(X)}$. $C(X)$ is itself a Banach algebra in the quotient algebra norm

$$\|T + K(X)\| = \inf_{K \in K(X)} \|T + K\|. \tag{0.3}$$

We shall use π to denote the natural homomorphism of $B(X)$ onto $C(X)$; $\pi(T) = T + K(X)$, $T \in B(X)$. Throughout this paper $N(T)$ and $R(T)$ will denote respectively the null space and the range space of T. Set $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim {}^{X}/_{R(T)}$. An operator $T \in B(X)$ is Fredholm if $R(T)$ is closed, and both $\alpha(T)$ and $\beta(T)$ are finite. The Fredholm operators $\Phi(X)$ constitute a multiplicative open semigroup in $B(X)$, and by Atkinson's theorem [7, Theorem 3.2.8] we have

$$\Phi(X) = \pi^{-1}(C(X)^{-1}) \tag{0.4}$$

The index of an operator T in $B(X)$ is defined by $i(T) = \alpha(T) - \beta(T)$, if at least one of $\alpha(T)$ and $\beta(T)$ is finite. It is well known that $B(X)^{-1} + K(X) \subset \Phi(X)$, and that $T \in B(X)^{-1} + K(X)$ if and only if $T \in \Phi(X)$ and $i(T) = 0$ [1, Theorem 0.2.2 and Theorem 0.2.8]. In this note we study the set of regular elements in the Calkin algebra $C(X)$.

Theorem 1

If X is a Banach space then

$$\widehat{B(X)} + K(X) = \pi^{-1}(\widehat{C(X)}). \tag{1.1}$$

Proof. Begin with the corresponding result for idempotents ([2, Lemma 1], [9, Lemma 1]):

$$B(X)^{\bullet} + K(X) = \pi^{-1}(C(X)^{\bullet}). \tag{1.2}$$

If $T^2 - T$ is compact then the only possible points of accumulation of its spectrum are 1 and 0: now if

$$P = \frac{1}{2\pi i} \int_{\gamma} (T - zI)^{-1}\, dz \tag{1.3}$$

with 1 inside and 0 outside γ disjoint from the spectrum of T then $P^2 = P$ and there are T', T'' in $B(X)$ (given by contour integrals) with

$$P = T'T = TT', \qquad I - P = T''(I - T) = (I - T)T''. \tag{1.4}$$

Evidently

$$(1.5) \qquad T - P = T(I - P) + (T - I)P = (T^2 - T)(T' + T'') \in K(X),$$

giving (1.2). If more generally $A - ABA \in K(X)$ then $T = BA$ gives $P = P^2$ for which (1.4) holds: now

$$(1.6) \qquad AP(T'B)AP = AP^3 = AP \in \widehat{B(X)} \quad \text{and} \quad A - AP \in K(X). \square$$

Note that the corresponding result for "decomposable regularity" fails: if $T \in B(X)$ is Fredholm with non zero index then

$$(1.7) \qquad \pi(T) \in C(X)^{-1} \subseteq C(X)^{\bullet}C(X)^{-1} \quad \text{but} \quad T \notin B(X)^{\bullet}B(X)^{-1} + K(X);$$

however

Theorem 2

If X is a Banach space then

$$(2.1) \qquad B(X)^{\bullet}\Phi(X) + K(X) = \pi^{-1}\big(C(X)^{\bullet}C(X)^{-1}\big)$$

and

$$(2.2) \qquad \widehat{B(X)} \cap cl\,\Phi(X) + K(X) = \pi^{-1}\big(\widehat{C(X)} \cap cl\,C(X)^{-1}\big).$$

Proof. By (0.4) it follows that $B(X)^{\bullet}\Phi(X) + K(X) \subset \pi^{-1}\big(C(X)^{\bullet}C(X)^{-1}\big)$. To prove the second inclusion of (2.1), suppose that $T \in \pi^{-1}\big(C(X)^{\bullet}C(X)^{-1}\big)$. From (1.2) and (0.4), it follows that there are $E \in B(X)^{\bullet}$, $S \in \Phi(X)$ and $K \in K(X)$ with $T = ES + K$. This completes the proof of (2.1).

The inclusion '$\subset$' of (2.2) follows from (0.4). To prove the second inclusion of (2.2), suppose that $A \in \pi^{-1}\big(\widehat{C(X)} \cap cl\,C(X)^{-1}\big)$. From (1.1) it follows that there are $B \in \widehat{B(X)}$ and $K_0 \in K(X)$ with $A = B + K_0$. Since $\pi(A) = \pi(B) \in cl\,C(X)^{-1}$ then there is (A_n) in $\Phi(X)$ for which

$$(2.3) \qquad \big\|B - A_n + K(X)\big\| \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Given $\varepsilon > 0$, choose n such that $1/n < \varepsilon/2$ and $\|B - A_n + K(X)\| < \varepsilon/2$. It follows that there is $K \in K(X)$ such that $\|B - A_n + K\| < \|B - A_n + K(X)\| + 1/n$. Set $B_\varepsilon = A_n - K$. It is clear that $B_\varepsilon \in \Phi(X)$ and $\|B - B_\varepsilon\| < \varepsilon$. $\square$

From (2.1) and (2.2), together with (0.2), it follows that

$$\widehat{B(X)} \cap cl\,\Phi(X) + K(X) = B(X)^{\bullet}\Phi(X) + K(X) \tag{2.4}$$

We can be more precise:

Theorem 3

If X is a Banach space then

$$\widehat{B(X)} \cap cl\,\Phi(X) = B(X)^{\bullet}\Phi(X). \tag{3.1}$$

Proof. Suppose that $A \in \widehat{B(X)} \cap cl\,\Phi(X)$. Now there are A' in $B(X)$ and B in $\Phi(X)$ such that $A = AA'A$, $A' = A'AA'$ and $I + (B - A)A' \in B(X)^{-1}$. From (0.4) it follows that there are $\overline{B}$ in $B(X)$ and K in $K(X)$ such that $B\overline{B} = I + K$. Set $A'' = A' + (I - A'A)\overline{B}(I - AA')$. Now $A = AA''A$, and from (0.4) and the proof of (0.2), we have that $A'' \in \Phi(X)$. Thus

$$\widehat{B(X)} \cap cl\,\Phi(X) \subset \{A \in B(X) \colon A \in A\Phi(X)A\}. \tag{3.2}$$

Further, if $A \in A\Phi(X)A$ then there exists an operator S in $\Phi(X)$ such that $A = ASA$. Again, from (0.4) it follows that there are S_1 in $B(X)$, K_1 and K_2 in $K(X)$, such that $SS_1 = I + K_1$ and $S_1S = I + K_2$. Thus $ASS_1 = A + AK_1$, which implies that $AS(S_1 - AK_1) = ASS_1 - ASAK_1 = ASS_1 - AK_1 = A$. Since $AS \in B(X)^{\bullet}$ and $S_1 - AK_1 \in \Phi(X)$, it follows that

$$\{A \in B(X) \colon A \in A\Phi(X)A\} \subset B(X)^{\bullet}\Phi(X). \tag{3.3}$$

By [6, Theorem 5.2] we have that $B(X)^{\bullet}\Phi(X) \subset \widehat{B(X)}$. Further, if $A \in B(X)^{\bullet}\Phi(X)$ there are P in $B(X)^{\bullet}$ and C in $\Phi(X)$ such that $A = PC$. Set $A_n = (P + (I - P)/n)C$, $n = 1, 2, \ldots$. Now $A_n \to A$ as $n \to \infty$, and $(P + (I - P)/n) \in B(X)^{-1}$, $n = 1, 2, \ldots$. Thus $A_n \in \Phi(X)$, which implies that

$$B(X)^{\bullet}\Phi(X) \subset \widehat{B(X)} \cap cl\,\Phi(X). \tag{3.4}$$

Thus (3.1) follows at once from (3.2), (3.3) and (3.4). □

Let us remark that from the proof of Theorem 3 it is easy to see that

$$\{A \in B(X) \colon A \in A\Phi(X)A\} = B(X)^{\bullet}\Phi(X) = \Phi(X)B(X)^{\bullet}. \tag{3.5}$$

Corollary 4

Let X be a Banach space and $A \in B(X)$. Then the following conditions are equivalent:

(4.1) $A \in \delta\Phi(X)$,

(4.2) $A = PB, \quad P \in B(X)^\bullet \setminus \Phi(X) \quad \text{and} \quad B \in \Phi(X)$,

(4.3) $A = CQ, \quad Q \in B(X)^\bullet \setminus \Phi(X) \quad \text{and} \quad C \in \Phi(X)$.

Proof. By Theorem 3 and (3.5). □

For any Hilbert space X, let $\dim_H X$ denote the Hilbert dimension of X, that is the cardinality of an orthonormal basis of X. We set $\mathrm{nul}_H(T) = \dim_H N(T)$ and $\mathrm{def}_H(T) = \dim_H R(T)^\perp$ for $T \in B(X)$. If X is a separable Hilbert space, then with connection according to Theorem 3 we have

Theorem 5

Let X be a separable Hilbert space. Then

$$\widehat{B(X)} \cap cl\,\Phi(X) = \Phi(X) \cup \{T \in B(X)\colon \mathit{nul}_H(T) = \mathit{def}_H(T) \text{ and } R(T) \text{ closed}\}. \tag{5.1}$$

Proof. By [3, Theorem 4 and Remark 5] we have that $cl\,\Phi(X) = \Phi(X) \cup cl\,B(X)^{-1}$. Further, by [5, Theorem 2.9] or [4, Proposition 1] if operator $T \in B(X)$ has closed range, then $T \in cl\,B(X)^{-1}$ if and only if $\mathrm{nul}_H(T) = \mathrm{def}_H(T)$. Hence, it follows that

$$\begin{aligned} &\widehat{B(X)} \cap cl\,\Phi(X) \\ &= \widehat{B(X)} \cap \left(\Phi(X) \cup cl\,B(X)^{-1}\right) \\ &= \Phi(X) \cup \left(\widehat{B(X)} \cap cl\,B(X)^{-1}\right) \\ &= \Phi(X) \cup \{T \in B(X)\colon \mathrm{nul}_H(T) = \mathrm{def}_H(T) \text{ and } R(T) \text{ closed}\}. \end{aligned} \tag{5.2}$$ □

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