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Embedding sums into products of Banach spaces

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ABSTRACT

In this paper we study the problem of embedding sums $\oplus_I X$ of Banach spaces into large products X^J of the same or different Banach spaces. The first result in this direction corresponds to Saxon [12], who solved it for X finite-dimensional and I countable. For X a Hilbert space it was solved in [2].

In the first part we give solutions to this problem for general Banach spaces, completing in this way [12], [2] and [3]. Then we apply those results to subfactorizations of "diagonal" operators acting between vector valued sequence spaces. As a by-product, a criteria for a Banach space to contain non-separable l_p-spaces is given. In the second part we introduce tensor products in order to replace subfactorization arguments by tensor product statements and show how the preceding tools can serve to explain some pathologies occurring in tensor products of locally convex spaces.

Finally, we give examples and counterexamples showing that most of the classical Banach spaces satisfy the countable embedding ($I = \mathbb{N}$).

Introduction

We start from the following observation: for certain locally convex spaces (in short lcs) E the sum space $\oplus_{\mathbb{N}} E$ is isomorphic to a subspace of some product E^I (obviously I uncountable), while for others such an embedding is not possible.

Examples of the first kind are $E = s(\mathbb{R})$, the space of all rapidly decreasing infinitely differentiable functions on the real line: E is nuclear, then $\oplus_{\mathbb{N}} E$ is also nuclear and, via Komura-Komura's theorem, it is a subspace of some product of copies of $s(\mathbb{R})$; or, for similar reasons, the universal Schwartz space $[l_\infty, \mu(l_\infty, l_1)]$ (see [6, p. 206]). Examples of the second kind are those lcs carrying the weak topology, since a sum space $\oplus_{\mathbb{N}} E$ never carries the weak topology.

This suggest the following problem:

Problem S1. Characterize those lcs E such that $\oplus_{\mathbb{N}} E$ is (isomorphic to) a subspace of some product E^I.

This problem admits two meaningful extensions:

Problem S2. Let E be an lcs. For which spaces F is $\oplus_{\mathbb{N}} E$ a subspace of F^I?

Problem S3. Handle problems 1 and 2 for uncountable sums.

This paper deals with those problems when $E = X$ is a Banach space and it is therefore a continuation of [2], where problems (S1) and (S3) were treated for $X = H$ a Hilbert space.

The organization of the paper is as follows: Firstly, there is a study of the embedding (S1) for finite-dimensional spaces. This corresponds to what we have called Saxon's theorem. The study of projective representations of the space φ leads us to introduce "new" topologies on the sum space. All of them are equivalent in φ but not in uncountable sum spaces φ_d. This is done in §1.

In §2, the preceding definitions are extended to general Banach spaces and the subfactorization techniques are presented. It is shown (Subfactorization theorem) that an embedding $\oplus_{\mathbb{N}} X \to X^J$ is equivalent to a subfactorization, through finite powers of X, of the diagonal operators acting between certain vector valued sequence spaces (those corresponding to the topology under consideration). In the case of uncountable sums (Subspace theorem) this is equivalent to admit those vector valued sequence spaces as subspaces.

In §3 examples and applications are given.

In Part II (§4) we replace subfactorization arguments by tensor products. We re-define the topologies introduced in Part I via tensor products and show how they provide an explanation to some pathologies. Finally, §5 displays some open problems, examples and counterexamples.

Preliminaries

In this section we recollect the basic definitions we will use throughout the paper. Other pertinent definitions will be stated at the appropriate place in the text.

Let Γ be a nonempty index set, and let $\{X_i\}_{i\in\Gamma}$ be a collection of Banach spaces. Let $0 < p \leq +\infty$. We denote by $\left(\sum_{i\in\Gamma} X_i\right)_p$ the Banach (p-Banach if $0 < p < 1$) space formed by all families (x_i), $x_i \in X_i$, such that $\left(\|x_i\|\right) \in l_p(\Gamma)$ or $c_0(\Gamma)$ ($p = +\infty$). The norm (p-norm, if $0 < p < 1$) in this space is

$$\|(x_i)\|_{l_p^s(X,I)} = \left(\sum_{i\in I}\|x_i\|_{X_i}^p\right)^{1/p}$$

When Γ is countable we simply write $\sum_p X_n$. When $X_n = X$, $\forall n$, variants of those definitions are obtained as follows: Let X be a Banach space. Let I be an index set and let $0 < p \leq +\infty$. We denote $l_p^s(X,I)$ to the Banach space formed by all the absolutely-p-summable families (sequences if I is countable) of X; that is, the families $(x_i) \in X^I$ such that the norm (p-norm if $0 < p < 1$):

$$\|(x_i)\|_{l_p^s(X,I)} = \left(\sum_{i\in I}\|x_i\|_{X_i}^p\right)^{1/p}$$

is finite.

For $p = +\infty$ we shall consider the space $c_0^s(X,I)$ of norm-null families of X instead of $l_\infty(X,I)$, the space of all bounded families of X. This space is endowed with the sup-norm.

We denote $l_p^w(X,I)$ th the Banach space formed by all the weakly-p-summable families (sequences, if I is countable) of X; that is, the families $(x_i) \in X^I$ such that the norm (p-norm if $0 < p < 1$):

$$\|(x_i)\|_{l_p^w(X,I)} = \sup\left\{\left(\sum_{i\in I}|\langle f, x_i\rangle|^p\right)^{1/p} : \|f\|_{X^*} \leq 1\right\}$$

is finite.

For $p = +\infty$ we shall consider the space $c_0^w(X,I)$ of weak-null families of X instead of $l_\infty(X,I)$, the space of all bounded families of X. Again, this space is endowed with the sup-norm.

When X is finite-dimensional one simply obtains (in both cases) the classical $l_p(I)$, I uncountable, and l_p, I countable, spaces.

Let $T: X \to Y$ be an operator acting between Banach spaces. Let Z be a Banach space. By a subfactorization of T through Z we mean two operators $B: X \to Z$ and $A: \overline{\operatorname{Im} Z} \to Y$ such that $T = AB$. Note that A need not be definied on all of Z, but only in the closure of the range of B in Z. When A can be defined on the whole Z then we have a factorization of T through Z.

For a general background on locally convex spaces we suggest [8], and also [6]. We follow the notation of [8] concerning projective descriptions of locally convex spaces. If E is a locally convex space, $\mathcal{U}(E)$ denotes a fundamental system of absolutely convex closed neighbourhoods of zero; if $U \in \mathcal{U}(E)$, the completion of the normed space $E_U = \left(E/\ker p_U, \|\cdot\|_U\right)$, where p_U is the seminorm associated to U and $\|\phi_U(x)\|_U = p_U(x)$ and ϕ_U is the quotient map, shall be termed the Banach space associated to U. If $V \in \mathcal{U}(E)$ and $V \subseteq U$ then the canonical linking map $\hat{T}_{VU}$ is the extension to the completions of the operator $T_{VU}: E_V \to E_U$ defined by $T_{VU}\phi_V(x) = \phi_U(x)$.

The space E admits a representation as (a dense subspace of, when it is not complete) the projective limit of the Banach spaces associated to the neighbourhoods corresponding to some $\mathcal{U}(E)$ and their respective linking maps. This we write as:

$$E = \lim_{\leftarrow} \hat{T}_{VU}(\hat{E}_V) \qquad U, V \in \mathcal{U}(E)$$

Given a family $\{E_i\}_{i \in I}$ of locally convex spaces, the sum space $\oplus_I E_i$ is the subspace of $\prod_{i \in I} E_i$ consisting of only those elements $x = (x_i)$, $x_i \in E_i$, which have finitely many non-zero x_i. We denote by $I_i: E_i \to \oplus_I E_i$ the mapping sending $x_i \in E_i$ to the element of $\oplus_I E_i$ whose i-th coordinate is equal to x_i, and all of whose other coordinates vanish. The inductive p-topology is the finest locally p-convex topology on $\oplus_I E_i$ making all the embeddings I_i continuous. A base of zero-neighbourhoods for the inductive p-topology is formed by the sets $\Gamma_p I_i(\mathcal{U}_{i,j})$, where Γ_p means "absolutely p-convex cover", and $\{\mathcal{U}_{i,j}\}$ is a base of zero-neighbourhoods in E_i.

Given two locally convex spaces E and F we can consider two natural topologies on the tensor product space $E \otimes F$:

The ε-topology, defined by the system of seminorms:

$$\varepsilon_{U,V}\left(z = \sum_i x_i \otimes y_i\right) = \sup\left\{ \left|\langle \varphi \otimes \phi, z\rangle\right| : p_{U^\circ}(\varphi) \leq 1,\, p_{V^\circ}(\phi) \leq 1 \right\}$$

where U and V are 0-neighbourhoods in E and F respectively.

The associated Banach space to $U \otimes V$ in $E \otimes_\varepsilon F$ is $\hat{E}_{(U \otimes V)} = \hat{E}_V \bar{\otimes}_\varepsilon \hat{E}_U$ (see [6]).

The π-topology, defined by the system of seminorms:

$$\pi_{U,V}\Big(z = \sum_i x_i \otimes y_i\Big) = \inf\Big\{\sum_i p_U(x_i)p_V(y_i)\Big\}$$

(the infimum is taken over all representations of z) where U and V are 0-neighbourhoods in E and F respectively.

The associated Banach space to $U \otimes V$ in $E \otimes_\pi F$ is $\hat{E}_{(U\otimes V)} = \hat{E}_V \bar{\otimes}_\pi \hat{E}_U$ (see [6]).

We recall that an lcs E is said to be nuclear if $E \otimes_\varepsilon X = E \otimes_\pi X$ for all Banach spaces X (see [6]).

Part I. Subfactorizations

§1. Finite dimensional Banach spaces

When $X = \mathbb{K}$, the scalar field, the sum space $\oplus_{\mathbb{N}}\mathbb{K}$ is usually noted φ. In Theorem 1.4 of [12], Saxon proves, among other things, that:

If E is any lcs not carrying the weak topology, then φ is a subspace of any product E^I when $\operatorname{card} I \geq 2^{\aleph_0}$

which we shall refer to as Saxon's theorem. With this we can consider solved problems 1 and 2 for X a finite-dimensional locally convex space.

Let us look at problem 3. When $\operatorname{card} I = d$, the sum space $\oplus_I\mathbb{K}$ shall be denoted φ_d. Does there exist an analogue of Saxon's result for φ_d? The answer is no, and in a quite strong sense: for instance, φ_d is not a Schwartz space, and thus cannot be embedded into any product of Schwartz spaces [3]. See Corollary 1 for deeper information.

The proof Saxon gives of his result relies upon the locally convex structure of the space E. We focus here our attention on the locally convex structure of φ. We see that $\varphi = \varprojlim D_\sigma(l_1)$, $\sigma \in l_\infty^+$, that is, it is the (reduced) projective limit of diagonal operators $D_\sigma : l_1 \to l_1$ with σ running through $l_\infty^+ = \{x \in l_\infty : x_n > 0\ \forall n\}$. The natural ordering for sequences is: $(x_n) \leq (y_n)$ if and only if, for some constant $c > 0$, $y_n \leq c x_n$. With this ordering l_p^+ and c_0^+ are cofinal in l_∞^+; therefore, by factorization of suitable D_σ through l_p or c_0 one obtains (see [8, p. 231]):

$$\varphi = \varprojlim D_\sigma(l_p) = \varprojlim D_\sigma(c_0), \qquad \sigma \in l_\infty^+, \quad 0 < p \leq +\infty$$

(which incidentally gives another proof of Saxon's theorem: by using the Dvoretzky-Rogers theorem, diagonal operators $D_\sigma : l_2 \to l_2$, $\sigma \in c_0$, subfactorize through any infinite-dimensional Banach space X (see [1]), which gives the embedding $\varphi \to X^I$ for Banach spaces. When E is an lcs, the result follows by considering the map $\prod \phi_U: E^I \to \prod E_U$ and the fact that if $f: A \to B$ is surjective and B contains φ then also A contains φ).

We now pass to φ_d. It is well-known that there are other topologies which can be considered in a sum space besides the inductive one: the so-called box-topology (see [6]) or topological direct sum, in the terminology of [8], the inductive p-topology, $0 < p < 1$ (the strongest locally-p-convex topology), etc. It is also well-known that all of them agree on φ and are different on φ_d, d uncountable (see [6]).

This, and the above representation formulae for φ suggest that a topology τ_p could be definied on φ_d for each $0 < p \leq +\infty$ as follows (convention: when $p = +\infty$, $l_p(I)$ means $c_0(I)$):

$$[\varphi_d, \tau_p] = \lim_{\leftarrow} D_\sigma(l_p(I)), \qquad \sigma \in l_\infty^+(I),$$

for which an explicit system of seminorms (p-seminorms if $0 < p < 1$) is given by:

$$q_{p,\sigma}(x) = \left\|\sigma^{-1}x\right\|_{l_p(I)}, \qquad \sigma \in l_\infty^+(I).$$

It is easy to see that τ_1 is the inductive topology, τ_∞ is the box-topology, and for $0 < p < 1$, τ_p is the strongest locally-p-convex topology on φ_d. All of them coincide on φ. All of them are different on φ_d: one has that for $q < p$, $\tau_p \leq \tau_q$ as clearly follows from the expressions of their seminorms. To see they are different, assume $q < p$ and notice that τ_p is the kernel topology corresponding to the family of diagonal maps:

$$D_{\sigma^{-1}}: \varphi_d \longrightarrow l_p(I) \qquad \sigma \in l_\infty^+(I).$$

Therefore the equality $\tau_p = \tau_q$ on φ_d would imply by a density argument the continuity of some map:

$$D_{\sigma^{-1}}: l_p(I) \longrightarrow l_q(I) \qquad \sigma \in l_\infty^+(I)$$

which is impossible since I in uncountable (see also [6]).

§2. Infinite-dimensional Banach spaces

If we pass to locally convex sums of general Banach spaces, let us firstly verify that the Banach spaces associated to the natural fundamental system of 0-neighbourhoods of $\oplus_{\mathbb{N}}X$ for the inductive p-topology are isomorphic with $l_p^s(X)$ and that under this isomorphism the linking maps are "diagonal" operators

$$D_{\sigma^{-1}}: l_1^s(X) \longrightarrow l_1^s(X), \quad D_\sigma\big((x_n)\big) = (\sigma_n x_n).$$

To show this, let $\mathcal{U}$ be the basis of zero neighbourhoods in $\oplus_{\mathbb{N}}X$ for the finest locally p-convex topology formed by all the sets $W = \Gamma_p\big(\bigcup_{n\in\mathbb{N}} I_n(\sigma_n B_X)\big)$, where $\sigma = (\sigma_n)$ ranges over l_∞^+ and B_X is the closed unit ball of X. Let us denote by q_W the p-norm gauge of W, and by B_p the closed unit ball of the p-norm $\|\cdot\|_p$ of $l_p(X)$. For the "diagonal" injection $D_{\sigma^{-1}}: [\oplus_{\mathbb{N}}X, q_W] \to l_p^s(X)$ defined by $D_{\sigma^{-1}}(x) = (\sigma_n^{-1}x_n)$ we clearly have $D_{\sigma^{-1}}(W) \subseteq B_p \cap \oplus_{\mathbb{N}}X$. Conversely, if $\eta = (\eta_n) \in B_p \cap \oplus_{\mathbb{N}}X$, then all but finitely many η_n are zero and $\sum_n |\eta_n|^p \leq 1$. It follows that $(\eta_n\sigma_n) \in W$, because W is absolutely p-convex, and, furthermore $D_{\sigma^{-1}}(\eta\sigma) = \eta$. Thus $D_{\sigma^{-1}}(W) = B_p \cap \oplus_{\mathbb{N}}X$ and $D_{\sigma^{-1}}$ is a topological isomorphism onto a dense subspace of $l_p^s(X)$. This gives that the completion of $[\oplus_{\mathbb{N}}X, q_W]$ is l_p^s. If $V = \Gamma_p\big(\bigcup_{n\in\mathbb{N}} I_n(\eta_n B_X)\big)$ is another neighbourhood in $\oplus_{\mathbb{N}}X$ with $\eta_n \leq \sigma_n$ for all $n \in \mathbb{N}$, then the linking map $\hat{T}_{VW}: [\oplus_{\mathbb{N}}X, q_V] \to [\oplus_{\mathbb{N}}X, q_W]$ is equivalent to the "diagonal" $D_{\sigma^{-1}\eta}$ on $l_p^s(X)$.

Therefore, via the factorization argument we used before, the associated Banach spaces can also be chosen isomorphic with $l_p^s(X)$, $0 < p < +\infty$, or $c_0^s(X)$ with diagonal linking maps:

$$\oplus_{\mathbb{N}}X = \varprojlim D_\sigma\big(l_p^s(X)\big), \qquad \sigma \in l_\infty^+, \quad 0 < p \leq +\infty$$

We consider now uncountable sums of Banach spaces. Let X be a Banach space. In the same spirit as above we can consider on $\oplus_I X$ the topologies:

$$[\oplus_I X, \tau_p^s] := \varprojlim D_\sigma\big(l_p^s(X,I)\big) \qquad \sigma \in l_\infty^+(I)$$

for which an explicit system of seminorms is given by the formulae:

$$q_{p,\sigma}^s(x) = \Big(\sum_I \|\sigma_i^{-1}x_i\|^p\Big)^{1/p} = \|\sigma^{-1}x\|_{l_p^s(X,I)}$$

(when $p = +\infty$ it has to be understood $c_0^s(X,I)$ with the sup norm).

But we can moreover consider the topologies obtained replacing l_p^s by l_p^w:

$$[\oplus_I X, \tau_p^w] := \varprojlim D_\sigma\big(l_p^w(X,I)\big) \qquad \sigma \in l_\infty^+(I)$$

for which an explicit system of seminorms is given by the formulae:

$$q_{p,\sigma}^w(x) = \sup_{\|a\|\leq 1}\Big(\sum_I |\langle a, \sigma_i^{-1}x_i\rangle|^p\Big)^{1/p} = \|\sigma^{-1}x\|_{l_p^w(X,I)}$$

(when $p = +\infty$ it has to be understood $c_0^w(X,I)$ with the sup norm).

Relationships among τ_p^w and τ_p^s topologies

When $I = \mathbb{N}$ the factorization argument gives $\tau_p^s = \tau_q^s = \tau_p^w = \tau_q^w$ for all values of p and q. In general $\tau_p^w \leq \tau_q^s$; when $p > q$, one also has the relations $\tau_p^w \leq \tau_q^w$ and $\tau_p^s \leq \tau_q^s$. When I is uncountable and $p \neq q$, it is clear again that $\tau_p^w \neq \tau_q^w$, $\tau_p^s \neq \tau_q^s$ and that $\tau_p^w \neq \tau_q^s$, since they induce different topologies on φ_d. To see that $\tau_p^s \neq \tau_p^w$ we use a

Lemma

Let I be an uncountable set, and $D_\sigma: l_p^w(X,I) \to l_p^s(X,I)$, $\sigma \in l_\infty^+(I)$, a diagonal map. Then $\operatorname{Im} D_\sigma \subseteq l_p^s(X,N)$, where N denotes some countable subset of I.

Proof. Since $\sigma_i > \varepsilon$ when i belongs to an uncountable set I_0, it is possible, by the Dvoretzky-Rogers theorem, to choose a sequence $(x_i)_{i\in I_0}$ in X weakly-p-summable but not strongly-p-summable. In this form, $D_\sigma((x_i))$ cannot be strongly-p-summable.

We finish the proof of $\tau_p^s \neq \tau_p^w$: the equality $\tau_p^s = \tau_p^w$ would imply the continuity of some

$$D_{\sigma^{-1}}: l_p^w(X,I) \to l_p^s(X,I)$$

by a usual density argument. □

Let us denote by τ_0 the topology induced by the topological product X^I on $\oplus_I X$ and by τ the inductive topology. We have:

Proposition

Let $1 \leq p \leq +\infty$. $\tau_0 < \tau_{\text{box}} = \tau_\infty^w = \tau_\infty^s < \tau_p^w < \tau_p^s < \tau_1^s = \tau$.

Proof. It is clear that τ_∞^s is the box-topology and also that τ_1^s is the inductive topology (for $0 < p < 1$, τ_p^s is the inductive-p-topology). The relation $\tau_p^w \leq \tau_p^s$ is inmediate and we have just seen they are different. That $\tau_0 < \tau_{\text{box}} = \tau_\infty^w = \tau_\infty^s$ is inmediate as well (the equality $\tau_\infty^w = \tau_\infty^s$ also follows, as we shall see in Part II of the equality $c_0^s(X) = c_0 \bar{\oplus}_\varepsilon X$. □

It is worth to mention that since all those topologies are finer than τ_0 and coarser than τ they have a basis of neighbourhoods of 0 consisting of τ_0-closed sets ([6, p. 81]). They are therefore complete ([6, p. 59]), and this gives us the correctness of their definitions:

Lemma

For $1 \leq p \leq +\infty$, the topologies τ_p^w and τ_p^s are complete.

We return to our original problems.

Subfactorization Theorem

Let X be a Banach space. Then $\oplus_{\mathbb{N}} X$ is a subspace of Y^I if and only if for some (all), $0 < p \leq +\infty$, and all σ belonging to some cofinal subset of l_∞^+ the diagonal morphisms $D_\sigma : l_p^a(X) \to l_p^a(X)$, $a = s$ or w, can be subfactorized through some finite product Y^n.

Proof. The if part is clear. Note that it already implies $\operatorname{card} I \geq 2^{\aleph_0}$.

On the other hand, an embedding $\oplus_{\mathbb{N}} X \to Y^I$ implies, choosing suitable 0-neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ in $\oplus_{\mathbb{N}} X$, and $\mathcal{W}$ in X^I, a diagram:

$$\begin{array}{ccccc} (\oplus_{\mathbb{N}} X)_{\mathcal{U}} = l_p^s(X) & \longrightarrow & \left(\oplus_{\mathbb{N}} X\right)_{\mathcal{W} \cap \oplus_{\mathbb{N}} X} & \longrightarrow & l_p^s(X) = (\oplus_{\mathbb{N}} X)_{\mathcal{V}} \\ & & \downarrow & & \\ & & Y^n = (Y^I)_{\mathcal{W}} & & \end{array}$$

where the upper arrow is a diagonal operator D_σ with $\sigma \in l_\infty^+$. This gives a subfactorization of D_σ through X^n. □

The subfactorization theorem admits an equivalent formulation for uncountable sums.

Subfactorization Theorem for uncountable sums, case s

Let X be a Banach space and let $0 < p \leq +\infty$. Then $\left[\oplus_I X, \tau_p^s\right]$ is a subspace of X^J if and only if for all σ belonging to some cofinal subset of $l_\infty^+(I)$ the diagonal morphisms $D_\sigma : l_p^s(X, I) \to l_p^s(X, I)$ can be subfactorized through some finite product X^n.

Subfactorization Theorem for uncountable sums, case w

Let X be a Banach space and let $0 < p \leq +\infty$. Then $\left[\oplus_I X, \tau_p^w\right]$ is a subspace of X^J if and only if for all σ belonging to some cofinal subset of $l_\infty^+(I)$ the diagonal morphisms $D_\sigma : l_p^w(X, I) \to l_p^w(X, I)$ can be subfactorized through some finite product X^n.

However, in the uncountable case something more can be said:

Subspace theorem

Let X, Y be Banach spaces and I an uncountable set with cofinality $\operatorname{card} I > \aleph_0$. Then:

(1) $\left[\oplus_I X, \tau_p^s\right]$ is a subspace of Y^J if and only if $l_p^s(X, I)$ is a subspace of some finite product Y^n and $\operatorname{card} J \geq 2^{\operatorname{card} I}$.

(2) $\left[\oplus_I X, \tau_p^w\right]$ is a subspace of Y^J if and only if $l_p^w(X, I)$ is a subspace of some finite product Y^n and $\operatorname{card} J \geq 2^{\operatorname{card} I}$.

Proof. We shall write $l_p^a(X,I)$, $a = w$ or s, to denote $l_p^w(X,I)$ and $l_p^s(X,I)$. Notice an obvious fact: if $\operatorname{card} I = \operatorname{card} I_0$ then, $l_p^a(X,I) = l_p^a(X,I_0)$. Next observe that if we have a diagonal operator

$$D_\sigma\colon l_p^a(X,I) \longrightarrow l_p^a(X,I) \qquad \sigma \in l_\infty^+(I)$$

then some $\varepsilon > 0$ exists such that $\sigma_i > \varepsilon$ for all i belonging to some set I_0 with $\operatorname{card} I_0 = \operatorname{card} I$ (by the cofinality condition imposed on $\operatorname{card} I$). If $j\colon l_p^a(X,I_0) \to l_p^a(X,I)$ denotes the canonical inclusion, the composition

$$l_p^a(X,I_0) \xrightarrow{\ j\ } l_p^a(X,I) \xrightarrow{\ D_\sigma\ } l_p^a(X,I)$$

gives an isomorphism: $\operatorname{Im} D_\sigma \circ j = l_p^a(X,I_0)$ and

$$\varepsilon\|x\| \leq \big\|D_\sigma \circ j\big\| \leq \|\sigma\|_\infty \, \|x\|$$

Thus a subfactorization of D_σ through Y^n implies a factorization of the identity of $l_p^a(X,I_0)$ through a certain subspace Z of Y^n. In this way, $l_p^a(X,I_0)$ should be isomorphic to some complemented subspace of Z, which is, in turn, a subspace of Y^n.

The other implication follows easily from the definition:

$$\big[\oplus_I X, \tau_p^a\big] = \varprojlim D_\sigma\big(l_p^a(X,I)\big),$$

implies that $\big[\oplus_I X, \tau_p^a\big]$ embedds into some product $\big(l_p^a(X,I)\big)^J$, which in turn, embedds into Y^J. □

§3. Applications

It is an open problem to characterize those Banach spaces containing l_p as a subspace. The following corollary gives, for many I uncountable, an equivalence for this problem:

Corollary 1

Let $1 \leq p \leq +\infty$. Let X be a stable ($X \times X \cong X$) Banach space and I an uncountable set with cofinality $\operatorname{card} I > \aleph_0$. $l_p(I)$ is a subspace of X if and only if $\big[\varphi_d, \tau_p\big]$ embedds into a large product X^I with $\operatorname{card} I \geq 2^d$.

From the proof of the subspace theorem it follows:

Corollary 2

If a diagonal operator $D_\sigma: l_p(I) \to l_p(I)$, I uncountable and with cofinality $\operatorname{card} I > \aleph_0$, *is subfactorized through X, then X contains a copy of $l_p(I)$.*

Remarks 1. The case $0 < p < 1$ of corollary 1 had to be ruled out for obvious reasons. The above characterization has no counterpart for l_p since φ embedds in X^I for any infinite-dimensional Banach space X (Saxon's theorem) as long as $\operatorname{card} J \geq 2^{\aleph_0}$.

2. The hypothesis on the cofinality of I is necessary in the subspace theorem as well as in corollaries 1 and 2; the following counter example shown to me by P. Domanski shows that:

Counter example. Let $m_0 < m_1 < m_2 < \ldots$ be an increasing sequence of cardinal numbers and let $m = \sup\{m_n : n \in \mathbb{N}\}$. Assume that I_n are pairwisely disjoint sets with $\operatorname{card} I_n = m_n$. Let finally $I = \bigcup_{n \in \mathbb{N}} I_n$.

Consider $p \neq q$, $1 \leq p, q \leq +\infty$, and let X be the Banach space

$$X = \left(\sum \oplus l_p(I_n)\right)_{l_q}.$$

Then

(a) $[\varphi_m, \tau_p]$ is a subspace of X^J for some J

(b) X is isomorphic to $X \times X$

(c) X does not contain a copy of $l_p(I)$

Proof. (a) Let

$$S = \Big\{\sigma = (\sigma_i)_{i \in I} : \forall i \in I \ \sigma_i \in \{2^{-n} : n \in \mathbb{N}\} \text{ and } \forall n \in \mathbb{N} \ \operatorname{card}\{i : \sigma_i = 2^{-n}\} < m\Big\}$$

Let $\sigma \in S$; then $D_\sigma: l_p(I) \to l_p(I)$ factorizes through X, since, without loss of generality (taking subsequences if necessary), we can assume that $\{i : \sigma_i = 2^{-n}\} = I_n$.

Indeed, if $q < p$, $X \subseteq l_p(I)$, and D_σ acts from $l_p(I)$ into X: if $p < q$, $l_p(I) \subseteq X$ and D_σ acts from X into $l_p(I)$.

By the subfactorization theorem, it is enough to show that S is cofinal in $l_\infty^+(I)$, something which is clear.

(b) is obvious.

(c) Assume that $l_p(I) \cong Z \subseteq X$, and let $P_n: X \to \left(\sum_{k\leq n} \oplus l_p(I_n)\right)_{l_q}$ be the standard projection. By a density argument, $P_n|_Z$ cannot be an isomorphism. Construct then a normalized sequence $(x_n)_n \subseteq Z$ and an increasing sequence $(r_n) \subseteq \mathbb{N}$ such that $r_0 = 0$ and $\|(P_{r_{n+1}} - P_{r_n})(x_n) - x_n\| \leq 2^{-n}$, which means that (x_n) is a basic sequence equivalent to $((P_{r_{n+1}} - P_{r_n})(x_n))_n$, in turn equivalent to the canonical basis of l_q. Contradiction. □

Corollary 3

A diagonal operator $D_\sigma: l_p(I) \to l_p(I)$, I uncountable, cannot be subfactorized through $l_q(I)$ whenever $1 \leq p \neq q \leq +\infty$.

To see this, recall that l_p and l_q spaces are totally incomparable (see [10]); or else (see [9, Th. 13, p. 129]) that any continuous operator from a subspace of $l_r(I)$ to $l_s(J)$, $r > s$, is compact, while D_σ cannot be compact. Note, however, that our method also works for: $p > 1 \geq q$; $p < 1 \leq q$ and $p < q \leq 1$. Probably Corollary 3 is true for all values of p and q greater than zero (see also the example in [2]).

Remarks. If $T: l_p(I) \to l_q(J)$ is a continuous operator, I and J uncountable sets, and $p > q$, $p > 1$, then $\operatorname{Im} T \subset l_q(N)$, where N is a countable subset of J. There are counterexamples for $p = 1 > q$ (see [2]). When T is a diagonal operator then the hypothesis $p > q$ suffices.

Clearly $[\varphi_d, \tau_p]$ is a subspace of a large product of copies of $l_p(J)$, $\operatorname{card} J \geq d$. Corollaries 1 and 2 assert that this is esentially the only possible case.

When $X = H$ is a Hilbert space we can complete [2]:

Corollary 4

Let H be a Hilbert space. Then:

1. $[\oplus_I H, \tau_p^s]$ *is a subspace of* H^J *if and only if we have one of the following alternatives*
   (a) $I = \mathbb{N}$, $\dim H = +\infty$ *and* $\operatorname{card} J \geq 2^{\aleph_0}$
   (b) $p = 2$, $\dim H > d$ *and* $\operatorname{card} J \geq 2^d$
2. $[\oplus_I H, \tau_p^w]$ *is a subspace of* H^J *if and only if*
   (a) $I = \mathbb{N}$, $\dim H = +\infty$ *and* $\operatorname{card} J \geq 2^{\aleph_0}$

Proof. The reason for 1.(a) follows from the Subfactorization theorem and corollary 1 or 2 since τ_p^s induces τ_p in φ_d.

1.(b) is true since $[\oplus_I H, \tau_p^s]$ has associated Banach spaces which are Hilbert spaces.

For τ_p^w we see that 2.(a) remains true since $[\oplus_I H, \tau_p^w]$ is not a subspace of H^J when I is uncountable. This is so due to the fact that $l_2^w(H, I)$ is not a Hilbert space (see [6]). □

Remark. A similar result holds for $l_r(\Lambda)$ instead of $l_2(\Lambda)$ replacing $p = 2$ by $p = r$ in 1.(b). For L_r spaces there is not such a pure result.

§4. Part II. Tensor products

Our purpose in this section is two-fold: on one side, to replace factorization arguments by tensor product statements in the results of Part I; on the other hand, to show that the topologies introduced in Part I apperar as suitable topologies on tensor products. It is in this way that τ^w and τ^s topologies could provide an explanation to some pathologies in the theory of locally convex tensor products.

We first recall some different topologies which can be considered in tensor products with an $l_p(I)$ space:

The ε-topology in $l_p(I) \otimes X$ is that induced by $l_p^w(X, I)$. Its adherence (completion) will be noted $l_p(I)\bar{\otimes}_\varepsilon X$. When $p = 1$ this space is the space of summable sequences of X (see [5]).

The p-topology in $l_p(I) \otimes X$ is that induced by $l_p^s(X, I)$. Its adherence (completion) will be noted $l_p(I)\bar{\otimes}_p X$. Since $c_0^s(X, I)$ is a subspace of $c_0^w(X, I)$, both induce the same topology on $c_0(I) \otimes X$: the ε-topology because of the formula $c_0^s(X, I) \cong c_o(I)\bar{\otimes}_\varepsilon X$ (see [5]).

In the tensor product spaces $X \otimes Y$ many crossnorms (i.e., norms satisfying $\|x \otimes y\| = \|x\| \, \|y\|$) can be defined. The strongest of such norms is denoted π (projective topology). We note $X\bar{\otimes}_\pi Y$ to its completion. The coarsest of such norms is denoted ε (inductive topology). We note $X\bar{\otimes}_\varepsilon Y$ to its completion.

A crossnorm τ is called a tensornorm if $\|T \otimes S \colon X \otimes_\tau Y \to X \otimes_\tau Y\| \leq \|T\| \, \|S\|$.

In the case of a tensor product with an $l_p(I)$ space, the ε norm induces the ε-topology. When $p = 1$ the π norm induces the 1-topology. The p-topologies are crossnorms, and therefore intermediate between ε and π; for $p \neq 1$ the p-norms are not tensornorms.

We give a new proof for the embedding of $\oplus_{\mathbb{N}} H$ into H^J (see [2]) replacing subfactorizations by tensors from where it follows a partial solution to problem 1.

We shall say that a tensornorm τ suitable to be defined in tensor products $E \otimes F$ of locally convex spaces is called "reasonable" when $E \bar{\otimes}_\tau F = \varprojlim \hat{E}_U \bar{\otimes}_\tau \hat{E}_V$, whenever τ can be defined in the spaces involved.

Proposition

Let X be a Banach space such that for some reasonable tensornorm τ, $X \bar{\otimes}_\tau X = X$. Then $\oplus_{\mathbb{N}} X$ is a subspace of X^I, $\operatorname{card} I \geq 2^{\aleph_0}$.

Proof. Since ε respects subspaces, $\varphi \otimes_\varepsilon X$ embedds into $X^I \otimes_\varepsilon X$ by Saxon's theorem. Since τ is a tensornorm, the canonical inclusion

$$\varphi \otimes X = \varphi \otimes_\tau X \longrightarrow X^I \otimes_\tau X$$

is continuous. Since $\varepsilon \leq \tau$ on $X^I \otimes X$, the same happens with the induced topologies on $\varphi \otimes X$. The former is ε, and the latter is coarser than $\tau = \varepsilon$. Therefore both are equal. This proves that $\oplus_{\mathbb{N}} X = \varphi \bar{\otimes} X$ embedds into $X^I \bar{\otimes}_\tau X$.

Since τ is a tensornorm, the topology τ respects complemented subspaces. Therefore $X^n = X \bar{\otimes}_\tau \mathbb{K}^n$ is a complemented subspace of $X \bar{\otimes}_\tau X = X$. From this, it follows that $X^2 \bar{\otimes}_\tau X$ is a (complemented) subspace of $X \bar{\otimes}_\tau X$. Therefore, since τ is reasonable, $X^I \bar{\otimes}_\tau X = \varprojlim X^n \bar{\otimes}_\tau X$ embedds into $\left[X \bar{\otimes}_\tau X\right]^I = X^I$. □

When X is an L_p space then $\tau = p$ is allowed in that proof: $\varphi \otimes X$ is still a subspace of $X^I \bar{\otimes}_\tau X$ essentially by the same argument ($\tau = p$ is intermediate between ε and π). It can be directly checked that $X^I \bar{\otimes}_\tau X$ embedds into $\left[X \bar{\otimes}_\tau X\right]^I$. This covers again the situation for $X = l_p(\Lambda)$ since then $X \bar{\otimes}_p X = X$.

It is obvious that if $l_p^s(X)$ or $l_p^w(X)$ embed into X^n, then $\oplus_{\mathbb{N}} X$ embedds into X^I. We can combine global and local approach to obtain:

Proposition

Let X be a Banach space such that for some tensornorm τ and for some $0 < p < +\infty$ $l_p \otimes_\tau X$ embedds into X^n for some n. Then $\oplus_{\mathbb{N}} X$ embedds into X^I, $\operatorname{card} I > 2^{\aleph_0}$.

Proof. We simply need to look at the diagram:

$$\begin{array}{ccccc}
l_p \otimes_\pi X & \longrightarrow & l_p \otimes_\tau X & \longrightarrow & l_p \otimes_\varepsilon X \\
\uparrow & & \uparrow & & \uparrow \\
l_p \otimes_\pi X & \longrightarrow & l_p \otimes_\tau X & \longrightarrow & l_p \otimes_\varepsilon X \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

$$\varprojlim D_\sigma \otimes I(l_p \otimes_\pi X) \longrightarrow \varprojlim D_\sigma \otimes I(l_p \otimes_\tau X) \longrightarrow \varprojlim D_\sigma \otimes I(l_p \otimes_\varepsilon X), \qquad \sigma \in l_\infty^+,$$
$$\varphi \bar{\otimes}_\pi X \longrightarrow \varprojlim D_\sigma \otimes I(l_p \otimes_\tau X) \longrightarrow \varphi \bar{\otimes}_\varepsilon X$$

which proves the equality $\oplus_{\mathbb{N}} X = \varphi \otimes X = \varprojlim D_\sigma \otimes I(l_p \otimes_\tau X)$. The subfactorization argument finishes the proof. □

We turn now to see what happens with uncountable sums to show how the topologies appearing in Part I can be introduced via tensor products:

$$\begin{aligned} [\oplus_I X, \tau_1^s] &= \varprojlim D_\sigma (l_1^s(X, I)) && \sigma \in l_\infty^+(I) \\ &= \varprojlim D_\sigma \otimes I(l_1(I) \bar{\otimes}_\pi X) && \sigma \in l_\infty^+(I) \\ &= [\varphi_d, \tau_1] \bar{\otimes}_\pi X \end{aligned}$$

and

$$\begin{aligned} [\oplus_I X, \tau_1^w] &= \varprojlim D_\sigma (l_1^w(X, I)) && \sigma \in l_\infty^+(I) \\ &= \varprojlim D_\sigma \otimes I(l_1(I) \bar{\otimes}_\varepsilon X) && \sigma \in l_\infty^+(I) \\ &= [\varphi_d, \tau_1] \bar{\otimes}_\varepsilon X \end{aligned}$$

To extend the above lines to other τ_p-topologies we need to use the p-topologies on the tensor product

$$\begin{aligned} [\oplus_I X, \tau_p^s] &= \varprojlim D_\sigma (l_p^s(X, I)) && \sigma \in l_\infty^+(I) \\ &= \varprojlim D_\sigma \otimes I(l_p(I) \bar{\otimes}_p X) && \sigma \in l_\infty^+(I) \\ &= [\varphi_d, \tau_p] \bar{\otimes}_{\tau(p)} X \end{aligned}$$

where the $\tau(p)$ topology is given by the seminorms:

$$\begin{aligned} \tau_{p,\sigma}\left(\sum_{k=1}^N w_k \otimes x_k\right) &= \left\| \sum_{k=1}^N D_{\sigma^{-1}} w_k \otimes x_k \right\|_p \\ &= \left\| \sum_{k=1}^N (\sigma_n^{-1} w_{k,n})_n \otimes x_k \right\|_p \\ &= \left\| \left(\sum_{k=1}^N \sigma_k^{-1} w_{k,n} x_k \right)_n \right\|_{l_p^s(X)} \end{aligned}$$

and therefore intermediate between the ε and the π topologies on $\varphi_d \otimes X$.

On the other hand:

$$\begin{aligned}[\oplus_I X, \tau_p^w] &= \lim_{\leftarrow} D_\sigma\big(l_p^w(X,I)\big) && \sigma \in l_\infty^+(I)\\ &= \lim_{\leftarrow} D_\sigma \otimes I\big(l_p(I)\bar{\otimes}_\varepsilon X\big) && \sigma \in l_\infty^+(I)\\ &= [\varphi_d, \tau_p]\bar{\otimes}_\varepsilon X\end{aligned}$$

Remarks. When $I = \mathbb{N}$ we obtain again the equality of the τ_p-topologies, since φ is a nuclear space and therefore ε and π coincide on $\varphi \otimes X$. When I is uncountable the subspace theorem is a crude manifestation of the nonnuclearity of φ_d.

In [6, p. 334] the equality $(\oplus_I E_i) \otimes_\pi X = \oplus_I(E_i \otimes_\pi X)$ is considered. But in pag. 352–353 things appear to be not so clear for the ε-topology.

The reason for such difficulties is that the inductive topology is τ_1^s, not τ_1^w, and therefore it has a good behaviour against the π-topology but not against the ε-topology. If we replace τ_1^s by τ_1^w we obtain the equivalent formula

$$\left(\oplus_I E_i, \tau_1^w\right) \otimes_\varepsilon X = \left[\oplus_I\big(E_i \otimes_\varepsilon X\big), \tau_1^w\right]$$

Since for uncountable embeddings (S3) can be considered solved with the subspace theorem, in the next section we shall concentrate in the seemingly most difficult case: the embedding S1.

§5. Examples, counterexamples and open problems

1. Characterize those infinite dimensional Banach spaces X such that $\oplus_{\mathbb{N}} X$ cannot be embedded into X^I ?

From part II we know that if for some tensornorm τ, $X\bar{\otimes}_\tau X = X$ then $\oplus_{\mathbb{N}} X$ embedds into X^I. There are however Banach spaces such that $X\bar{\otimes}_\tau X \neq X$ for all crossnorms: Let P be Pisier's space [11] whose main feature is that $P \otimes_\varepsilon P = P \otimes_\pi P$. Should we have $P = P\bar{\otimes}_\tau P$ then $P\bar{\otimes}_\varepsilon P\bar{\otimes}_\varepsilon P = P\bar{\otimes}_\pi P\bar{\otimes}_\pi P$, and following [7], P should be nuclear.

On the other hand, an embedding of $\oplus_{\mathbb{N}} X$ into X^I would imply that for some $k \in \mathbb{N}$, X^n is a subspace of X^k for all n. This can be shown as follows: from the proof of the subfactorization theorem, we have a diagram:

$$\begin{array}{ccccc}(\oplus_{\mathbb{N}} X)_{\mathcal{U}} = l_p^s(X) & \longrightarrow & \left(\oplus_{\mathbb{N}} X\right)_{\mathcal{W}\cap\oplus_{\mathbb{N}} X} & \longrightarrow & l_p^s(X) = (\oplus_{\mathbb{N}} X)_{\mathcal{V}}\\ & & \downarrow & & \\ & & X^k = (X^I)_{\mathcal{W}} & & \end{array}$$

If $i_n\colon X^n \to l_p(X)$ denotes the canonical inclusion of X^n into the first n positions, and $p_n\colon l_p(X) \to X^n$ the projection onto the first n coordinates, we see that, for all n, we have a diagram

$$\begin{array}{ccccccccc} X^n & \longrightarrow & l_p^s(X) & \longrightarrow & \left(\oplus_{\mathbb{N}} X\right)_{W \cap \oplus_{\mathbb{N}} X} & \longrightarrow & l_p^s(X) & \longrightarrow & X^n \\ & & & & \downarrow & & & & \\ & & & & X^k & & & & \end{array}$$

where the horizontal arrow is an isomorphism. Therefore one has that, for all n, X^n is isomorphic to a subspace of X^k.

Now it is possible to show that James space J does not satisfy the countable embedding (S1). A description of James space J can be seen in [10]. Here it is enough to know that this space has the property that $\dim J^{**}/J = 1$. The following proof is due to P. Domanski, who goes on to show that $J^k \subseteq J^n$ if and only if $k \le n$.

Proof. Let $k > n$ and assume that $J^k \subseteq J^n$; then $\left(J^k\right)^{**} \subseteq \left(J^n\right)^{**}$ and

$$\dim \left(J^k\right)^{**}/J^k = k, \text{ and } \dim \left(J^n\right)^{**}/J^n = n.$$

An utilization of the Hahn-Banach theorem implies that $\left(J^k\right)^{**}/J^k \subseteq \left(J^n\right)^{**}/J^n$, which is a contradiction. □

Remark. Another observation is that if X is tensorstable (i.e., if $X\bar{\otimes}_\tau X = X$ for some reasonable crossnorm) then X^n is isomorphic to some complemented subspace of X for all $n \in \mathbb{N}$. All this suggests that X should be stable. This seems to imply that spaces satisfying (S1) are intermediate between the tensortable and the stable Banach spaces.

Concerning stability properties, it is clear that finite products of Banach spaces satisfying (S1) also satisfy (S1). This is equally true for the ε tensor product (choosing τ_∞^s) or the π tensor product (choosing τ_1^s). Since finite-dimensional spaces are always complemented, the property (S1) does not passes to complemented subspaces.

It is not difficult to see that p-sums of Banach spaces satisfying (S1) also satisfy (S1):

Proposition

Let $\{X_n\}$ be a sequence of Banach spaces satisfying the embedding (S1). Let $0 < p < +\infty$. The quasi-Banach space $\sum_p X_n$ satisfies the embedding (S1).

Proof. The sum space $\oplus_{\mathbb{N}} \sum_p X_n$ has associated Banach spaces isometric with $l_p^s(\sum_p X_n) = \sum_p l_p^s(X_n)$. Under this new isomorphism the diagonal map D_σ is transformed into the diagonal operator $D_{(\sigma,\sigma,\dots)} = \sum D_\sigma^n$, where D_σ^n is D_σ acting on $l_p^s(X_n)$. Using the subfactorization theorem we see that since D_σ^n subfactorizes through X_n, $\sum D_\sigma^n$ subfactorizes through $\sum_p X_n$ and this last space satisfies (S1). □

Remark. With some extra work for the notations but no change in the proof, this proposition is valid for the p-sum $\sum_p X_i$ of an uncountable quantity of Banach spaces.

Examples of Banach (or quasi-Banach) spaces satisfying the embedding (S1) are what we could call "extended sequence spaces"; that is: $l_p(I)$, $l_p^s(X,I)$, $l_p^w(X,I)$ and $l_p(I)\bar{\otimes}_p X$, for X any Banach space, I any index set, as well as $l_p(I)\bar{\otimes}_\varepsilon X$ and $l_p(I)\bar{\otimes}_\pi X$. It is also clear that we could replace l_p spaces for suitable general (even nonlocally convex) sequence spaces λ (provided some non-very-restrictive conditions: λ^+ cofinal in l_∞^+, etc. (see [3]). Finally, and this is what suggests the name, tensortable Banach spaces, that is, Banach spaces X such that $X = X\bar{\otimes}_\tau X$ for some tensornorm τ.

Due to the well-known isomorphism $l_p^w(X) = L(l_q, X)$ $(p^{-1} + q^{-1} = 1)$ and $l_p^w(X^*) = L(X^*, l_p)$ we have that spaces such as $L(H)$, the space of all bounded operators on a Hilbert space satisfies the embedding (S1).

We can add to our list some usual function spaces such as $L_p(a,b)$, $H^1(U)$ or $\mathcal{C}[0,1]$. In the first two cases the proof can be performed writing those spaces as $l_p^s(X)$ for some Banach space X:

Let m be a finite or infinite cardinal. It is not hard to check that $L_p([0,1]^m, \mathbb{C}) = l_p^s(L_p([0,1]^m, \mathbb{C}))$ or that $L_p(\mathbb{R}) = l_p^s(L_p[0,1])$, (see [9, 14, Th. 9, Corollary, Th. 10 and Th. 14]).

The space $H^1(U)$ is isomorphic to $l_1^s(H^1(U))$.

If we use general structure theorems we see that most of the classical Banach spaces satisfy the embedding (S1). Recall that a Banach lattice is said to be an abstract L_p space if whenever $x \wedge y = 0$ $\|x + y\|^p = \|x\|^p + \|y\|^p$. We quote [9, p. 136]:

Any abstract L_p space X is linearly isometric to

$$\left[l_p(\Gamma, \mathbb{C}) \oplus \left(\bigoplus \sum_{\alpha \in A} L_p([0,1]^{m_\alpha}, \mathbb{C}) \right)_p \right]$$

for some index set Γ and some set of cardinal $m_\alpha \geq \aleph_0$.

From this and the remark after the latter proposition we get that:

Any abstract L_p space satisfies the embedding (S1).

Therefore, spaces such as $L_p(0,1)\otimes_a X$, $a=\varepsilon,\pi,p$, satisfy the embedding (S1). This includes spaces of Bochner integrable functions $L_p(\mu,X)=L_p(\mu)\bar{\otimes}_p X$.

For $\mathcal{C}(K)$-spaces we prove slightly less: choosing τ^s_∞, $\oplus_{\mathbb{N}}\mathcal{C}(K)$ has associated Banach spaces isomorphic to $c^s_0(\mathcal{C}(K))=c^s_0\bar{\otimes}_\varepsilon\mathcal{C}(K)$, which are therefore subspaces of $\mathcal{C}(K)\bar{\otimes}_\varepsilon\mathcal{C}(K)=\mathcal{C}(K\times K)$. This space, in turn, embedds into $\mathcal{C}(K)$ if K is uncountable and metrizable (Milutin's theorem, see [9, p. 85]). In this way:

$\oplus_{\mathbb{N}}\mathcal{C}(K)$ embedds into $\mathcal{C}(K\times K)^I$.

For K uncountable and metrizable $\mathcal{C}(K)$ satisfies (S1).

Therefore this covers the situation for vector-valued continuous function spaces such as $\mathcal{C}([0,1],X)=\mathcal{C}([0,1])\bar{\otimes}_\varepsilon X$.

2. An extension of the above theory for Fréchet locally convex spaces

Notice that the techniques developped in this paper do not usually work in arbitrary locally convex spaces: the subfactorization technique encounters the following problem: if E is a locally convex space admitting a projective description $E=\varprojlim E_U$, then $\oplus_{\mathbb{N}}E$ admits a projective description as:

$$\oplus_{\mathbb{N}}E\colon\ \cdots\longrightarrow l^s_p(E_V)\longrightarrow l^s_p(E_U)$$

where the linking maps have the form $D_\sigma\otimes T_{VU}$.

To see where the difficulty lies, one could observe that even in a simple and seemingly harmless case as $E=\Lambda(P)$, a Köthe sequence space, where the spaces E_V can be chosen isomorphic to l_1 and the linking mappings a tensor product $D_\sigma\otimes D_\eta$ of diagonal operators, things can be complicated: if $E=(s)$, the space of rapidly decreasing sequences, then E is isomorphic to $s(\mathbb{R})$, and, as already mentioned in the introduction, E satisfies the embedding (S1). We remark, however, that in [13] a $\Lambda_1(\alpha)$-space is constructed for which the embedding $\otimes_{\mathbb{N}}\Lambda_1(\alpha)\to\Lambda_1(\alpha)^J$ is not possible. Following [13], it is possible to choose a certain $\Lambda_1(\alpha)$ space in such a form that, for a certain sequence space λ, the space $\Lambda_1(\alpha)$ is λ-nuclear but the sum space $\oplus_{\mathbb{N}}\Lambda_1(\alpha)$ is not λ-nuclear. The space λ can be choosen regular enough so that the class of λ-nuclear spaces is a variety, i.e., is closed under the formation of isomorphic images, quotients, subspaces and arbitrary products. Especially, it follows that $\oplus_{\mathbb{N}}\Lambda_1(\alpha)$ cannot be embedded as a subspace of a product $\Lambda_1(\alpha)^I$.

Concerning the tensor product approach, notice that the equality $\varphi\otimes E=\oplus_{\mathbb{N}}E$ does not necessarily hold for non gDF spaces (in the terminology of [6]).

Final remarks. It could be of some interest to say a word about the problem of τ_p-topologies and duality. If $\oplus_I E_i$ is endowed with the box topology then the dual space can be shown to be isomorphic to the subspace of $\prod_{i\in I} E_i^*$ having a countable number of non-vanishing coordinates. We quote from [6, p. 173]: "No systematic discussion of this duality seems to exist in the literature". It is not difficult to see that if we denote by $l_q(\sigma I) = \{\xi : \sigma\xi \in l_q(I)\}$ then $[\oplus_I X, \tau_p^s]^* = \bigcup_{\sigma\in l_\infty^+(I)} l_{p^*}^s(X^*, \sigma I)$, where $1/p + 1/p^* = 1$. When $p = +\infty$, $[\oplus_I X, \tau_{\text{box}}]^* = \bigcup_{\sigma\in l_\infty^+(I)} l_1^s(X^*, \sigma I)$, and when $p = 1$ then $[\oplus_I X, \tau_1^s]^* = \bigcup_{\sigma\in l_\infty^+(I)} l_\infty(X^*, \sigma I)$. Moreover, it seems that the natural topologies to be considered in those dual spaces are those induced by the seminorms $d_{\sigma,p^*}(\xi) = \left(\sum_I \|\sigma_i\xi_i\|^{p^*}\right)^{1/p^*}$.

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