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Some properties of the vector-valued Banach ideal space $E(X)$ derived from those of E and X

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Abstract

Let (Ω, Σ, μ) be an arbitrary measure space, E a Banach ideal space (Köthe function space) on Ω, X a Banach space, and $E(X)$ the "vector-valued Banach ideal space" composed of E and X. By a general method based on semi-embeddings, it is proved for certain properties P of Banach spaces that if E and X have P then $E(X)$ has P as well $(*)$. Examples for P are the analytic Radon-Nikodým property and the property not to contain c_0, thus simplifying results of Bukhvalov. $(*)$ is also true for the type II-Λ-Radon-Nikodým property and the separable complementation property, and somewhat weaker versions of $(*)$ hold for the type I-Λ-Radon-Nikodým property, the property (P) of Costé and Lust-Piquard, and for the near Radon-Nikodým property.

1. Introduction and Preliminaries

1.1 Introduction

The purpose of this note is not so much to prove a definite theorem but to explain a general procedure which sometimes enables one to obtain easy proofs of theorems of the type

$$(*) \qquad E, X \in (P) \quad \Longrightarrow \quad E(X) \in (P).$$

Here, P is a property of (isomorphism classes of) Banach spaces, X a Banach space, E a Banach ideal space (Köthe function space) over some measure space (Ω, Σ, μ), and $E(X)$ is the X-valued version of E (definitions see below). For the procedure to work it is necessary that the theorem is already known for $E = L^1(\mu)$, μ =finite measure, and one actually attempts a reduction of the general case to that special case using an appropriate semi-embedding.

Two basic examples of properties P for which the procedure, formalized as Theorem 2.2, yields exactly the Theorem $(*)$, are the analytic Radon-Nikodým property (ARNP) and the property not to contain (an isomorphic copy of) c_0. In both cases $(*)$ has been established by Bukhvalov [11, Theorem 7], [10]. However, the proof given here for ARNP is painless compared to that in [11] —being finished before the "delicate questions concerning measurability" [11, p. 55] arise. Note that the case $E = L^1(\mu)$, due independently to Dowling [20, Theorem 2] and this author [31, Satz 3.1] is easy, using the Fubini theorem. (I have had no access to [10] but was informed by Professor Bukhvalov that a proof of $(*)$ is contained therein for the property not to contain c_0 based on ideas of Bourgain [1], [2]. Here, the case $E = L^1(\mu)$ is due to Kwapień [35, Theorem], cf. also Bourgain [1, Theorem 2]. Independently of me, Dowling recently also reproved the c_0 result [23], the proof following Bukhvalov's arguments for the ARNP case.)

As a joint generalization of both these examples, $(*)$ also holds for the type II-Λ-Radon-Nikodým property if Λ is a Riesz subset of $\mathbb{Z}$ (2.6). For the type I-Λ-RNP a slightly weaker result is obtained. These two Λ-RNPs have been introduced and studied by Dowling [21–23] and Edgar [25]. I am grateful to Professor H. Jarchow for directing my attention towards reference [22].

Another example is the property (P) of Costé and Lust-Piquard [16]: If E and X have (P) and if X has the separable complementation property SCP, then $E(X)$ has (P) (and SCP) as well (2.8). In proving this, $(*)$ is also established for SCP (2.8 Remark 2). Finally, consider the near Radon-Nikodým property NRNP of Kaufman, Petrakis, Riddle and Uhl [34]: If E has NRNP and X has RNP then $E(X)$ has NRNP as well (2.9).

For more properties P, some of them isometric, fulfilling $(*)$ or some variant, see e.g. Day [17], Halperin [28], [29], [30], Bukhvalov [7], [9], Kamińska and Turrett [33]. More references on the spaces $E(X)$ are given in [13, 5.3].

For the remainder of §1, no originality is claimed, although the presentation is free from unnecessary restrictions often found in the literature.

The scalar field is always $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, unless indicated otherwise. X denotes a Banach space, X' its dual, $\langle x, x' \rangle := x'(x)$ the dual pairing ($x \in X$, $x' \in X'$), B_X the closed unit ball of X. The term "operator" includes "bounded linear", "isomorphism" includes surjectivity, whereas "isometry" does not. Terminology in Banach lattices follows [37, II], [42].

1.2 Vector-valued measurability

Let (Ω, Σ, μ) be a measure space. Since no restriction such as σ-finiteness is imposed on μ, precise definitions as regards measurability seem in order.

Let

$$St(\Sigma; X) := \left\{ \sum_{i=1}^{n} \chi_{A_i} x_i : n \in \mathbb{N},\ A_i \in \Sigma \text{ disjoint},\ x_i \in X \right\}$$

be the vector space of X-valued step functions. A function $f: \Omega \to X$ is called (strongly, Bochner) μ-measurable (the prefix μ- will be suppressed) if the following equivalent conditions are satisfied:

i) $\exists f_n \in St(\Sigma; X)$, $f_n \to f$ μ-a.e.

ii) f is a.e. Borel measurable and a.e. separably valued

iii) $\exists f_n \in St(\Sigma; X)$, $\|f_n(\cdot)\|_X \leq \|f(\cdot)\|_X$, $f_n \to f$ a.e.

(See [41,Lemma V-2-4] in which the definition of f_n must be slightly modified, however, in order to really achieve the inequality in iii).)

As a corollary, if $f_n, f: \Omega \to X$, the f_n are measurable, and $f_n \to f$ a.e., then f is measurable.

Let $S(\mu; X)$ be the vector space of X-valued μ-measurable functions modulo functions vanishing μ-a.e. In case $X = \mathbb{K}$, this letter is omitted from notation.

1.3 Vector-valued Banach ideal spaces

A Banach ideal space (Banach function space, Köthe function space) is a vector subspace $E \subset S(\mu)$, equipped with a complete norm $\|\cdot\|_E$, such that $f \in S(\mu)$, $g \in E$, $|f| \leq |g|$ a.e. implies $f \in E$ and $\|f\|_E \leq \|g\|_E$. In other words, E is an ideal of the vector lattice $S(\mu)$, endowed with a complete lattice norm. Obviously, E is then a σ-order complete Banach lattice.

Now let X be a Banach space. The "vector-valued ideal space" $E(X)$ is defined as $\{f \in S(\mu; X) : \|f(\cdot)\|_X \in E\}$ and carries the norm $\|f\|_{E(X)} := \big\| \|f(\cdot)\|_X \big\|_E$. Sometimes the notation $E(\mu; X)$ for $E(X)$ is used.

As the name indicates, $E(X)$ is a Banach space. The proof is a routine exercise, using the observation that if $f_n \leq f_{n+1} \to f$ in $(E; \|\cdot\|_E)$, then $f_n \to f$ a.e.; this follows from the triviality [42, Lemma II.5.8].

Lemma

If E is σ-order continuous, then the space

$$\begin{aligned} St_E(\Sigma; X) :&= \left\{ \sum_{i=1}^{n} \chi_{A_i} x_i \ : n \in \mathbb{N},\ A_i \in \Sigma \textit{ disjoint},\ \chi_{A_i} \in E,\ x_i \in X \right\} \\ &= St(\Sigma; X) \cap E(X) \end{aligned}$$

is dense in $E(X)$.

Proof. Let $f \in E(X)$. By 1.2, f can be assumed Borel measurable, and there exist $f_n \in St(\Sigma; X)$ such that $f_n \to f$ a.e. and $\|f_n(\cdot)\|_X \le \|f(\cdot)\|_X$. Define $g_1 := f_1$ and

$$g_{n+1}(\omega) := \begin{cases} f_{n+1}(\omega), & \text{if } \|f_{n+1}(\omega) - f(\omega)\| \le \|g_n(\omega) - f(\omega)\| \\ g_n(\omega), & \text{otherwise.} \end{cases}$$

Then $g_n \in St_E(\Sigma; X)$ and $\|g_n - f\|_{E(X)} \to 0$, by σ-order continuity of $\|\cdot\|_E$. □

Remarks. *(1°)* σ-order continuity is in this case the same as order continuity [37, II. 1.a. 8], E being σ-order complete.

(2°) The lemma implies in particular that $E \otimes X$ (where $\sum_{i=1}^n f_i \otimes x_i$ is identified with $\sum_{i=1}^n f_i(\cdot)x_i$) is dense in $E(X)$ if E is order continuous.

(3°) Conversely, if $E \otimes X$ is dense in $E(X)$ for one infinite-dimensional Banach space X then, under mild assumptions on the measure space, E is order continuous (Bukhvalov [8, Theorem 1]).

(4°) E, X separable implies $E(X)$ separable. This follows from *(2°)* in an obvious way, E being order continuous [37, II. p.7].

1.4 The tensor product $E\tilde{\otimes}X$

Let E be an arbitrary Banach lattice and X be a Banach space. Following Chaney [15] and Lotz [38], the norm $\|\cdot\|^{\sim}$ on $E \otimes X$ is defined as

$$\|u\|^{\sim} := \inf\left\{ \left\| \sum_{i=1}^n \|x_i\|_X\, f_i \right\|_E : f_i \ge 0,\ u = \sum_{i=1}^n f_i \otimes x_i \right\}.$$

Following Levin [36], the norm $\|\cdot\|_{\sim}$ on $E \otimes X$ is defined as

$$\|u\|_{\sim} := \inf\left\{ \left\| \sum_{i=1}^n \|x_i\|_X\, |f_i| \right\|_E : u = \sum_{i=1}^n f_i \otimes x_i \right\}.$$

It is easy to see that $\|u\|^{\sim} \ge \|u\|_{\sim} \ge \|u\|^{\vee}$, where $\|\cdot\|^{\vee}$ denotes the injective tensor norm, so that $\|\cdot\|^{\sim}$ and $\|\cdot\|_{\sim}$ are actually norms.

Remarks. *(1°)* $\|u\|^{\sim}$ coincides with Schaefer's l-norm $\|u\|_l$ [42, IV.7]. One way to see this: It is straightforward and well-known [15, p. 2], [38, p. 89] that the dual of $(E \otimes X, \|\cdot\|^{\sim})$ is $(\mathcal{L}^l(E, X'), \|\cdot\|_l)$, the space of cone absolutely summing operators from E into X' as defined in [42, IV.3]. On the other hand, this is also the dual of $(E \otimes X, \|\cdot\|_l)$ [42, Theorem IV.7.4].

(2°) Levin [36] considers also, for $u = \sum_{i=1}^{n} f_i \otimes x_i \in E \otimes X$, the norm

$$n_E(u) := \inf\left\{ \|f\|_E : f \geq \left|\sum_{i=1}^{n} \langle x_i, x' \rangle f_i\right| \; \forall x' \in B_{X'} \right\}.$$

It is known that $n_E = \|\cdot\|_{\sim} = \|\cdot\|^{\sim}$ [43, Lemma 1.3] (thanks to Professor Bukhvalov for pointing out this reference), cf. [36, p. 56]. In case E is a Banach ideal space with order continuous norm, the proof is much easier and, in view of Lemma 1.3, actually contained in [36, p. 54 f.]

Now let E again be a Banach ideal space. The following representation of $E(X)$ as a topological tensor product will be crucial since it allows to compute the norm in $E(X)$ without recourse to the underlying measure space. Let $E\tilde{\otimes}X$ be the completion of $(E \otimes X, \|\cdot\|^{\sim})$.

Theorem (Levin)

The canonical map $f \otimes x \mapsto f(\cdot)x$ *defines an isometry* $E\tilde{\otimes}X \to E(X)$*, surjective if* E *has order continuous norm.*

Proof. [36, Theorem 1]. (Levin uses the norm n_E introduced above in his proof. It is also possible to give a direct proof, using only $\|\cdot\|^{\sim}$. —The last assertion follows from Remark 1.3, *(2°)*, of course.) □

For the study of $E(X)$ and its connection with $E\tilde{\otimes}X$ in much more general situations than considered here, see Bukhvalov [4].

1.5 Injectivity of $\tilde{\otimes}$

Let E_0, E be arbitrary Banach lattices, X_0, X Banach spaces, $S: E_0 \to E$, $T: X_0 \to X$ operators, where $S \geq 0$. It is straightforward to verify that

$$S \otimes T\colon\ E_0 \otimes X_0 \longrightarrow E \otimes X$$

is bounded for $\|\cdot\|^{\sim}$, thus extending to an operator

$$S\tilde{\otimes}T\colon\ E_0\tilde{\otimes}X_0 \longrightarrow E\tilde{\otimes}X.$$

Theorem (Lotz)

If S *is a lattice isometry and* T *is an isometry then* $S\tilde{\otimes}T$ *is an isometry.*

Proof. [38, Proposition 4.3]. (See [42, p. 194 f.] for the definition of lattice homomorphism in case $\mathbb{K} = \mathbb{C}$.) □

Remark. Levin proved the same if E is σ-order complete and $S(E_0)$ is an ideal in E [36, Proposition 6]. This would not suffice for later purposes, however.

2. The theorem, and some variants

2.1 Conditions on Banach space properties

Let P be a property of Banach spaces; the notation $X \in (P)$ stands for "X possesses the property P". Consider the following conditions on P:

(1) P is separably determined.

(2) If $S: X \to Y$ is a semi-embedding of Banach spaces, X is separable, and $Y \in (P)$ then $X \in (P)$.

(3) If ν is a finite measure and $X \in (P)$, then the space of Bochner integrable functions $L^1(\nu; X) \in (P)$.

(4) $c_0 \notin (P)$.

Here, $L^1(\nu; X)$ is nothing but $L^1(\nu)(X)$ in the sense of 1.3, of course. An operator $S: X \to Y$ is a semi-embedding iff S is injective and SB_X is closed in Y [39], [3]. For the sake of clarity I remark that if P fulfils (2) then P devolves upon separable subspaces, and if P fulfils (1), (2) then it is automatically an isomorphic invariant. More comments on these conditions will be made below (2.3).

Theorem 2.2

Let E be a Banach ideal space on an arbitrary measure space (Ω, Σ, μ) and X a Banach space.

(a) If P is a property of Banach spaces satisfying 2.1, (1)-(4), then

$$(*) \qquad E, X \in (P) \quad \textit{implies} \quad E(X) \in (P).$$

(b) If P satisfies only 2.1, (1)-(3), then

$$E \not\supset c_0, \quad X \in (P) \quad \textit{implies} \quad E(X) \in (P).$$

(The notation $E \not\supset c_0$ means that no Banach subspace of E is isomorphic to c_0, or, what amounts to the same [37, II, Remark p. 35], that no Banach sublattice of E is isomorphic, as a Banach lattice, to c_0).

(a) has been formulated for aesthetic reasons. Of course, it suffices to prove (b).

Proof. First, as in [9, 1.] a reduction to separable E and X is carried out. Since there is a minor inaccuracy in [loc.cit], I am giving the details. P being separably determined (1), it suffices to prove $Y \in (P)$ for all separable Banach subspaces of $E(X)$. Since E does not contain c_0, E has order continuous norm [42, Theorem II.5.14]. After Levin's theorem 1.4, $E(X)$ can and will be identified with $E\tilde{\otimes}X$. Now fix a separable subspace $Y \subset E\tilde{\otimes}X$, choose $D \subset Y$ countable dense, and for every $y \in D$ choose a sequence $(u_n(y))_{n\in\mathbb{N}}$ in $E \otimes X$ converging to y in $E\tilde{\otimes}X$. Write $u_n(y) = \sum_{i=1}^{k_n(y)} f_{in}(y) \otimes x_{in}(y)$.

Let E_0 be the Banach sublattice of E, generated by all $f_{in}(y) \in E$, and X_0 be the Banach subspace of X generated by all $x_{in}(y) \in X$ ($y \in D$, $n \in \mathbb{N}$, $i = 1, \dots, k_n(y)$). After Lotz's theorem 1.5, $E_0\tilde{\otimes}X_0$ can and will be identified with its canonical isometric image in $E\tilde{\otimes}X$. Then it is clear that $Y \subset E_0\tilde{\otimes}X_0$, and it remains to show that $E_0\tilde{\otimes}X_0 \in (P)$.

E_0 is separable [42, Exercise II. 5(e)] and order continuous, e.g. since E_0 does not contain c_0, too. After the representation theorem 1.b.14 in [37, II], E_0 is order isometric to a Banach ideal space $(F, \|\cdot\|_F)$ on some probability space (Ω', Σ', ν), where $F \subset L^1(\nu)$ (continuous inclusion, not closed in general). Since F does not contain c_0, F is a KB-space, i.e. norm bounded increasing sequences converge in F [37, Theorem II. 1. c.4]. From there, it is easy to see and well-known that the inclusion map $F \to L^1(\nu)$ is a semi-embedding [26, Proof of Proposition III.1.2), Remark p. 323], cf. also [40, pp. 758 f.]. Using Levin's theorem 1.4 in the other direction, $F\tilde{\otimes}X_0 (\cong E_0\tilde{\otimes}X_0)$ can and will be identified with the vector-valued Banach ideal space $F(X_0) = F(\nu; X_0)$. A verification shows that the induced inclusion map $F(\nu; X_0) \to L^1(\nu; X_0)$ is a semi-embedding, too. Since $X_0 \subset X \in (P)$ is separable, $X_0 \in (P)$, whence $L^1(\nu; X_0) \in (P)$ after condition (3) on P. Since $F(X_0)$ is separable after Remark 1.3, *(4°)*, it finally follows that $F(X_0) \in (P)$ by condition (2). □

Remark. If $\mathbb{K} = \mathbb{C}$, the above proof is valid; however, some care is in order. For instance, in order to conclude that E_0 is separable it helps to know that the complexification of a real Bananch sublattice of $Re\, E$ is a complex vector sublattice of E (see [42, p. 134] for definition), e.g. due to order continuity of E (that is, of $Re\, E$). Also the complex version of the representation theorem [37, II.1.b.14] is needed which, however, can be derived from the real one without difficulties.

2.3 Comments on conditions 2.1, (1)–(4)

(1°) In presence of (1) and (2), condition (3) may be replaced by the formally weaker (3') if λ is Lebesgue-Borel measure on $[0, 1]$, and $X \in (P)$ is separable, then $L^1(\lambda; X) \in (P)$.

For, let (1), (2), (3') be satisfied and let any finite measure ν and Banach space X be given. After (1), (2) and [24, Lemma III.8.5], one can assume that $L^1(\nu)$ and X are separable. Then $L^1(\nu)$ is order isometric to a closed sublattice F of $L^1(\lambda)$ [27, Theorem 41.C], cf. [40, p. 759]. Then, by Theorems 1.4 and 1.5, $L^1(\nu; X) \cong L^1(\nu)\tilde{\otimes}X \cong F\tilde{\otimes}X \subset L^1(\lambda)\tilde{\otimes}X \cong L^1(\lambda; X)$. Now P for $L^1(\nu; X)$ follows from (2) and (3').

(2°) Assume there is some non-zero Banach space with P, and that P fulfils (1)–(3). Then, obviously, $\mathbb{K} \in (P)$, whence $L^1(\mu) \in (P)$ for all finite measures μ. By the techniques of proof of Theorem 2.2, every Banach lattice not containing c_0 must share P. In particular, l^2 has P, and thus all separable dual spaces share P because they semi-embed into l^2 [3, Proposition 1.2]. If, in addittion, P fulfils (4), then P is equivalent to not containing c_0 in the realm of Banach lattices.

EXAMPLE 2.4 (ARNP)

Let $\mathbb{K} = \mathbb{C}$. The analytic Radon-Nikodým property ARNP introduced by Bukhvalov and Danilevich [6], [12] fulfils (1)–(4): (1) is trivial; for (2) see [3, p. 153], [31, Corollar (3.4)]; for (3), look at section 1.1; (4) is trivial. It follows that $(*)$ holds for ARNP, reproving a result of Bukhvalov [11, Theorem 7], cf. 1.1.

EXAMPLE 2.5 ($\not\supset c_0$)

The property P not to contain (an isomorphic copy of) c_0 fulfils (1)–(4): (1) is trivial. In the situation of (2), assume there is a Banach subspace $X_0 \subset X$, $X_0 \cong c_0$. Since X is separable, $S: X \to Y$ is a G_δ-embedding [3, Proposition 1.8], whence also $S|X_0 \to Y$ is a G_δ-embedding and even an isomorphic embedding [3, Proposition 2.2(a)], contradiction. For (3), see introduction 1.1. (4) is trivial. Consequently, $(*)$ holds for this property, reproving again a result of Bukhvalov [10], cf. 1.1.

EXAMPLE 2.6 (Λ-RNPS)

Let G be a metrizable compact abelian group and Λ a Riesz subset of the dual group Γ (i.e. every measure on G with Fourier-Stieltjes coefficients vanishing off Λ is absolutely continuous). As introduced by Dowling [21–23] and Edgar [26], a complex Banach space X is said to possess the type I- (resp., type II-) Λ-RNP if every X-valued measure on G with bounded average range (resp., with bounded variation) and Fourier-Stieltjes coefficients vanishing off Λ admits of a Bochner integrable Radon-Nikodým derivative (all notions with respect to Haar measure on G). As is easily seen, both properties satisfy conditions 2.1 (1) and (4) (if Λ is infinite), cf. [22, §3, Remarks 1) and 4)].

Dowling [private communication] proved that type II-Λ-RNP fulfils 2.1 (2) and, if $\Gamma = \mathbb{Z}$, also 2.1 (3) [22, Proposition 8] (the latter proof is only notationally different from the case of ARNP, i.e. $\Lambda = \mathbb{N}_0$, treated in [20] and [31]). It follows that $(*)$ *holds for the type II-Λ-RNP provided Λ is a Riesz subset of $\mathbb{Z}$.*

Example 2.4 (ARNP) is the special case $\Lambda = \mathbb{N}_0$. On the other hand, if Λ is an infinite Sidon set then the assertion reduces to example 2.5 ($\not\supset c_0$) [21, Corollary 7].

Type I-Λ-RNP (for general Γ) clearly satisfies 2.1 (2) [22, §3, Remark 2], but I can prove only the weaker version of (3) [32]: If ν is a finite measure and X has type II-Λ-RNP then $L^1(\nu; X)$ has type I-Λ-RNP. The procedure of proof of Theorem 2.2 yields:

Remark. Let Λ be a Riesz subset of Γ. If E has type I-Λ-RNP and X has type II-Λ-RNP then $E(X)$ has type I-Λ-RNP.

EXAMPLE 2.7 (P)

Consider the property (P) introduced by Costé and Lust-Piquard [16]. It fulfils (2)–(4) [16, Proposition 4, Théorème 4, Corollaire 2a], but only the following weaker version of (1) [16, p. 55]:

(1') In spaces with the separable complementation property SCP, P is separably determined.

(A Banach space X has SCP iff every separable Banach subspace $X_0 \subset X$ is contained in a separable and complemented Banach subspace $X_1 \subset X$).

The following variant of Theorem 2.2 therefore applies to Costé's and Lust-Piquard's property (P).

Proposition 2.8

(a) If P satisfies (1'), (2), (3), (4), then

$$E, X \in (P) \text{ and } X \in (SCP) \quad \text{implies} \quad E(X) \in (P) \qquad (\text{and } E(X) \in (SCP)).$$

(b) If P satisfies only (1'), (2), (3), then

$$E \not\supset c_0,\, X \in (P),\, X \in (SCP) \quad \text{implies} \quad E(X) \in (P) \qquad (\text{and } E(X) \in (SCP)).$$

Lemma

If E is an order continuous Banach lattice and X a Banach space with SCP, then $E\tilde{\otimes}X$ has SCP as well.

Remarks. *(1°)* It is known that a Banach lattice E is order continuous if and only if E is σ-order complete and has SCP. For the "only if" part, see [5, Proposition 2.1, 1)], [26, Proof of Proposition IV. 1, 2)]. The other implication is stated without proof in [5, Proposition 2.1, 3)], [14, §2, Remark 3h)], seemingly only for order complete Banach lattices. Let me give an argument to prove the "if" part. After [37, II. 1.a.8] only σ-order continuity is to be shown. Assuming the contrary, then $l^\infty \hookrightarrow E$ as a Banach subspace [37, II. 1.a.7]. This, however, is impossible even for Banach *spaces*

E with SCP: Suppose, in somewhat sloppy notation, $l^\infty \subset E \Rightarrow c_0 \subset l^\infty \subset E \Rightarrow \exists X_0 \subset E$ separable and complemented in E s.t. $c_0 \subset X_0$. c_0 is complemented in X_0 by Sobczyk's theorem [37, I.2.f.5] $\Rightarrow c_0$ complemented in $E \Rightarrow c_0$ complemented in l^∞, contradiction.

(2°) Because of *(1°)*, in the realm of Banach ideal spaces E the lemma takes on the nice form: $E, X \in$(SCP) implies $E(X) \in$(SCP), that is, $(*)$ *holds for SCP.*

Proof of lemma. First I remark that after the famous Amir-Lindenstrauss theorem "WCG $\Rightarrow$ SCP" [18, Theorem V.2.3], in order to prove SCP for a Banach space Y it suffices to find, for every separable $Y_0 \subset Y$, a complemented *WCG* subspace $Y_1 \subset Y$ such that $Y_0 \subset Y_1$. Let $Y \subset E\tilde{\otimes}X$ be a separable Banach subspace. As in the proof of Theorem 2.2, there exists a separable Banach sublattice $E_0 \subset E$ and a separable Banach subspace $X_0 \subset X$ such that $Y \subset E_0\tilde{\otimes}X_0 \subset E\tilde{\otimes}X$. Following [26, Proof of Proposition IV. 1, 2)], there exists a Banach sublattice E_1 of E such that $E_0 \subset E_1$, E_1 is WCG and complemented in E by a *positive* projection $P: E \twoheadrightarrow E_1$. By hypothesis, there exists a Banach subspace X_1 of X such that $X_0 \subset X_1$, X_1 is separable (hence WCG) and complemented in X by a projection $Q: X \twoheadrightarrow X_1$. By a result of Bukhvalov [7, Theorem 2, Proof], $E_1\tilde{\otimes}X_1$ is WCG together with E_1 and X_1. Now $P\tilde{\otimes}Q: E\tilde{\otimes}X \twoheadrightarrow E_1\tilde{\otimes}X_1$ is a projection and $Y \subset E_0\tilde{\otimes}X_0 \subset E_1\tilde{\otimes}X_1 \subset E\tilde{\otimes}X$ (Theorem 1.5). By the initial remark, this proves the lemma.

Remarks. *(3°)* For the convenience of the reader, let me sketch an (alternative) proof of the assertion: E WCG Banach lattice, X WCG Banach space implies $E\tilde{\otimes}X$ is WCG. As in [loc.cit], it suffices to show that if $f_n \to 0$ weakly in E, $x_n \to 0$ weakly in X then $f_n \otimes x_n \to 0$ weakly in $E\tilde{\otimes}X$. As indicated already in Remark 1.4, *(1°)*, an element of $(E\tilde{\otimes}X)'$ is given by a cone absolutely summing operator $T: E \to X'$ according to the formula $\langle f \otimes x, T\rangle = \langle x, Tf\rangle$. By [42, Proposition IV.3.3], T admits of a factoring $E \xrightarrow{T_1} L \xrightarrow{T_2} X'$, where L is an AL-space, so that

$$\langle x_n, Tf_n\rangle = \langle x_n, T_2T_1f_n\rangle = \langle T_1f_n, T_2'x_n\rangle \longrightarrow 0,$$

L possessing the Dunford-Pettis property [42, Theorems II.9.7, II.9.9].

(4°) In connection with the proof of the lemma, the following interesting problem arises. I say that a Banach lattice E has the **positive** *SCP* iff every separable Banach sublattice E_0 of E is contained in a separable Banach **sublattice** E_1 of E which is complemented by a **positive** projection. Does order continuity imply the positive SCP? By conditional expectation operators, this is true for $L^1(\nu)$-spaces, ν finite measure. Professor Bukhvalov has informed me that, building on this, it is also true for rearrangement invariant Banach function spaces.

Proof of proposition. Again, only (b) is to be shown. By hypotheses and lemma, $E(X) = E\tilde{\otimes}X \in$ (SCP), so that P for $E(X)$ is separably determined (1'). Now the proof proceeds as given for Theorem 2.2.

EXAMPLE 2.9 (NRNP)

As a final example, I consider the near Radon-Nikodým property introduced by Kaufman, Petrakis, Riddle and Uhl [34]. A Banach space X has NRNP iff every near Radon-Nikodým operator $T: L^1(\lambda) \to X$ is Riesz representable, where T is called a near Radon-Nikodým operator iff TD is Riesz representable for all Dunford-Pettis operators $D: L^1(\lambda) \to L^1(\lambda)$. By a routine argument (e.g. [19, Proof of Theorem III.3.2], this property fulfils condition 2.1 (1), and also (2) [34, Proofs of Corollaries 19, 22], and (4) [loc.cit., Example 10]. The main result of [loc.cit.] is that $L^1(\lambda)$ has the NRNP. Unfortunately, I do not know whether $L^1(\lambda; X)$ has NRNP together with X, that is, whether NRNP fulfils condition 2.3 (3'). (If so, then $(*)$ is true for NRNP (2.2, 2.3).) However, I can prove that $L^1(\mu; X)$ has NRNP if X has RNP (and μ is any measure) [32]. Inspecting the proof of Theorem 2.2 yields

Proposition

$E \in$(NRNP), $X \in$(RNP) implies $E(X) \in$(NRNP).

NOTE ADDED IN PROOF: The answer to the question raised in Remark 2.8 (4^o) is positive. This follows by an adaptation of Valdivia's stunningly simple proof of the Amir-Lindenstrauss theorem [M. Valdivia, Espacios de Fréchet de generación débilmente compacta, *Collect. Math.* **38** (1987), 17–25, Lemma 1] to the lattice situation. The proof of Lemma 2.8 simplifies accordingly.

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