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Representation of operators by kernels

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ABSTRACT

We prove that differences of order–continuous operators acting between function spaces can be represented with a pseudo–kernel, proved the underlying measure spaces satisfy certain (rather weak) conditions.

To see that part of these conditions are necessary, we show that the strict localizibility of a measure space can be characterized by the existence of a pseudo–kernel for a certain operator.

1. Introduction

The representation of classes of operators between function spaces by kernels is a widely used tool in operator theory and functional analysis with an impressive list of applications. See [1], [3], [4], [7], [8], [10], [11] and [14]. The aim of the present note is to prove a representation theorem for certain operators between function spaces under very weak conditions concerning the underlying measure spaces.

Such a generalization seems to be worthwhile for the following reason: The results available in the literature usually require that one of the measure spaces be Standard Borel, the Borel space of a second countable topological space or something similar which, at least, implies that the Borel σ–algebra is countably generated.

Our Theorem 1.1 shows that such a condition is not necessary.

In the second section we use the Gelfand isomorphism of $L_\infty(X,\mathfrak{A},\mu)$ to show that the assumption of strict localizability of $(X,\mathfrak{A},\mu)$ in Theorem 1.1 is indispensable. To be more precise, we construct an operator which has a representation with a pseudo–kernel iff $\mathfrak{L}_\infty(X,\mathfrak{A},\mu)$ has a lifting.

1. The Representation Theorem

In this section we are going to show that linear operators between spaces of functions which satisfy a certain "continuity property" can be represented by means of pseudo–kernels.

First, we have to recall some measure theoretic notions for which there seems to be no standard terminology. (For detailed information we refer the reader to [5].) Starting with a measure space $(X,\mathfrak{A},\mu)$, a function $f: X \to [-\infty,\infty]$ is called $\mathfrak{A}$*–measurable*, if $\{f \le \alpha\} \in \mathfrak{A}$ for every $\alpha \in \mathbb{R}$ (here $\{f \le \alpha\} = \{x \in X; f(x) \le \alpha\}$). We weaken this condition and call $f: X \to [-\infty,\infty]$ *locally* μ*–measurable*, if, for every $A \in \mathfrak{A}^\mu_{fin} := \{B \in \mathfrak{A}; \mu(B) < \infty\}$ there is an $\mathfrak{A}$–measurable function which coincides with f μ–a.e. on A (i.e. on $A \setminus N$ for some $N \in \mathfrak{A}$ with $\mu(N) = 0$). $N \subset X$ is said to be a *local* μ*–null set* if, for every $A \in \mathfrak{A}^\mu_{fin}$ there is an $N_A \in \mathfrak{A}$ with $\mu(N_A) = 0$ such that $N \cap A \subset N_A$. By $\mathfrak{L}_0(X,\mathfrak{A},\mu)$ we denote the space of all locally μ–measurable functions which are finite *locally* μ*–a.e.* (i.e. outside some local μ–null set).

For every $f \in \mathfrak{L}_0(X,\mathfrak{A},\mu)$ its equivalence class is defined as $\tilde{f} := \{g \in \mathfrak{L}_0(X,\mathfrak{A},\mu); f = g$ locally μ–a.e.$\}$ and the symbol $L_0(X,\mathfrak{A},\mu)$ is used for the corresponding space of equivalence classes.

An element $f \in \mathfrak{L}_0(X,\mathfrak{A},\mu)$ is said to be *essentially bounded*, if $\|f\|_\infty := \inf\{c; |f| \le c$ locally μ–a.e.$\}$ is finite (where we use the convention $\inf\emptyset = \infty$). By $\mathfrak{L}_\infty(X,\mathfrak{A},\mu)$ we denote the set of essentially bounded functions in $\mathfrak{L}_0(X,\mathfrak{A},\mu)$, while $L_\infty(X,\mathfrak{A},\mu)$ stands for the corresponding space of equivalence classes.

Given a Hausdorff space Y we say that ν is a *Radon measure* on Y, if ν is locally finite (i.e. each point has a ν–integrable neighborhood) and ν is inner regular, i.e. $\nu(B) = \sup\{\nu(K); K \subset B, K$ compact$\}$ for every $B \in \mathfrak{B}_Y$, where the latter denotes the Borel σ–algebra of Y. We write $M_R(Y)$ for the space of finite signed Radon measures on Y.

An important condition in our Representation Theorem is the *strict localizability* of a measure space $(X,\mathfrak{A},\mu)$. This means that there is a family $\mathfrak{D} \subset \mathfrak{A}^\mu_{fin}$ of disjoint sets of nonzero measure which satisfies: $\forall A \in \mathfrak{A}^\mu_{fin}, \mu(A) > 0\ \exists D \in \mathfrak{D} : \mu(A \cap D) > 0$. Such a family $\mathfrak{D}$ is called a μ*–decomposition.* We are going to use that every strictly localizable $(X, \mathfrak{A}, \mu)$ admits a *linear lifting* of $\mathfrak{L}_\infty(X, \mathfrak{A}, \mu)$, i.e. a positive $\Lambda : L_\infty(X, \mathfrak{A}, \mu) \to \mathfrak{L}_\infty(E, \mathfrak{A}, \mu)$ with $\Lambda f \in f$ for all $f \in L_\infty(X, \mathfrak{A}, \mu)$ and $\Lambda \tilde{1} = 1$ (see [5], [6]).

We use $M_\infty(X, \mathfrak{A}, \mu)$ for the space of equivalence classes of bounded measurable functions.

1.1. Representation Theorem

Let $(X, \mathfrak{A}, \mu)$ be strictly localizable and ν be a Radon measure on the Hausdorff space Y with $\mathfrak{B} = \mathfrak{B}_Y$. Let $T : M_\infty(Y, \mathfrak{B}, \nu) \to L_0(X, \mathfrak{A}, \mu)$ be linear and $T = T_+ - T_-$, with order continuous T_+ and T_-. Then there exists $\tau : X \to M_R(Y)$ such that

$$\int f(y)\tau(\cdot)(dy) \in T\tilde{f}$$

for every bounded, $\mathfrak{B}$-measurable f.

We express this relation between τ and T by saying that τ is a *pseudo-kernel* for T.

Under more restrictive conditions concerning the underlying measure spaces, similar results have been proved in [1], [3], [4], [7], [8], [11], [12] and [15]. The last reference contains more information concerning the relevant literature and an excellent study of the consequences of theorems of the above kind. We call an operator T *order-continuous*, if $T(\sup \mathfrak{F}) = \sup T(\mathfrak{F})$ for any set $\mathfrak{F}$ of functions which is directed under $\leq$. Note that if μ is finite (as is the case in [15]), it is sufficient to check this condition for sequences.

1.2. Corollary

With $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ as in 1.1, let $T : L_1(X, \mathfrak{A}, \mu) \to L_1(Y, \mathfrak{B}, \nu)$ be linear and bounded. Then there exists a mapping $\tau : X \to M_R(Y)$ such that

$$\int_B Tf \, d\nu = \int f(x)\, \tau(x)(B)\, \mu(dx)$$

for every $f \in L_1(X, \mathfrak{A}, \mu)$ and every $B \in \mathfrak{B}$.

Proof. Apply 1.1 to $T' : L_\infty(Y, \mathfrak{B}, \nu) \to L_\infty(X, \mathfrak{A}, \mu)$ and use that the dual of a positive operator on L_1 is order-continuous. □

Note that in all of the above quoted references the σ-algebra $\mathfrak{B}$ has to be countably generated.

Since in [13] we wanted to apply a result like 1.2 in the case where Y is an uncountable product, we were forced to prove the above generalization.

Proof of 1.1. Without restriction we may assume $T \geq 0$. Since $T\tilde{1} \in L_0(X, \mathfrak{A}, \mu)$, we find a finite representative $g \in T\tilde{1}$. The locally μ–measurable sets $X_n := \{g \in [n-1, n)\}$ cover X as n runs through $\mathbb{N}$. Note that $0 \leq Tf \leq n\|f\|_\infty$ on X_n for every $n \in \mathbb{N}$ and $f \in L_\infty(Y, \mathfrak{B}, \nu)$.

Let $\Lambda : L_\infty(X, \mathfrak{A}, \mu) \to \mathfrak{L}_\infty(X, \mathfrak{A}, \mu)$ be a linear lifting. Denote the set of compact subsets of Y by $\mathfrak{K}$.

For $K \in \mathfrak{K}$ and $\varphi \in C(K)$ let φ^0 be the function which is 0 on $Y \backslash K$ and equals φ on K.

Set

$$\langle \tau(x), \varphi \rangle := \Lambda\left((T\tilde{\varphi}^0)\chi_{X_n} \right)(x)$$

for $n \in \mathbb{N}, x \in X_n, K \in \mathfrak{K}$ and $\varphi \in C(K)$.

Clearly, $\tau(x) \in C(K)'$ for every $x \in X, K \in \mathfrak{K}$ and $\tau(x)$ is a measure in the sense of [2; §1, $N^0 3$, déf. 5]; see [2; §3, $N^0 2$, Thm 2] for the relation with the notion of a Radon measure as given above. Note that we simply write τ instead of Bourbaki's $\tau^\bullet$.

From the properties of Λ, we have

$$\langle \tau(\cdot), \varphi \rangle \in T\tilde{\varphi}^0$$

for all $K \in \mathfrak{K}, \varphi \in C(K)$.

It follows that

$$\tau(\cdot)(U) \in T\tilde{\chi}_U$$

for all open $U \subset Y$.

By its very definition,

$$\tau(x)(U) = \sup\left\{ \langle \tau(x), \varphi \rangle ; K \in \mathfrak{K}, K \subset U, \varphi \in C(\mathfrak{K}) 0 \leq \varphi \leq \chi_U \right\}$$

For $B \subset X_n, \mu(B) < \infty$,

$$\begin{aligned} \int_B \tau(x)(U)\mu(dx) &= \int_B \sup_{\cdots} \langle \tau(x), \varphi \rangle \mu(dx) \\ &= \sup_{\cdots} \int_B \Lambda\left[(T\tilde{\varphi}^0)\chi_{X_n} \right](x)\mu(dx) \\ &= \sup_{\cdots} \int_B T\tilde{\varphi}^0 d\mu = \int_B T\tilde{\chi}_n d\mu \ , \end{aligned}$$

where we used [6; Chap. III, Thm. 3, p. 40] for the crucial second equality sign and the order continuity of T for the last equation.

Now consider $\mathfrak{F} := \{g \text{ bounded, measurable}; \int g(y)\tau(\cdot)(dy) \in T\tilde{g}\}$. Then $\chi_U \in \mathfrak{F}$ for all open $U \subset Y$ by the last inclusion and the continuity of T implies that $\mathfrak{F}$ is closed under monotone limits.

An appeal to a monotone class theorem (eg. [9; App. 1, Lemma 3, p.241]) finishes the proof. □

Some of the ideas in the above proof were inspired by [10].

2. Pseudo–Kernels related to the Gelfand isomorphism

In this section we prove that the use of a lifting in Theorem 1.1 was, in fact, necessary. To be more precise, we show in 2.2 that a certain operator defined with the help of the Gelfand isomorphism of $L_\infty(X, \mathfrak{A}, \mu)$ admits a pseudo-kernel iff $(X, \mathfrak{A}, \mu)$ is strictly localizable.

By its very definition $L_\infty(X, \mathfrak{A}, \mu)$ is isometrically imbedded in $L_1(X, \mathfrak{A}, \mu)'$. From now on we assume that the measure space $(X, \mathfrak{A}, \mu)$ is *localizable*, i.e. this canonical mapping is surjective. As $L_\infty(X, \mathfrak{A}, \mu)$ is an abelian C^*–algebra, there is an isomorphism

$$\Gamma : L_\infty(X, \mathfrak{A}, \mu) \longrightarrow C(\hat{X}) ,$$

called the *Gelfand isomorphism*, where $\hat{X}$ denotes the maximal ideal space of $L_\infty(X, \mathfrak{A}, \mu)$.

To simplify notation, we sometimes write $\hat{f}$ instead of Γf. For $A \in \mathfrak{A}^\mu_{loc} := \{B \subset X : \chi_B \in \mathfrak{L}_\infty(X, \mathfrak{A}, \mu)\}$ the function $\Gamma\tilde{\chi}_A$ is a continuous idempotent; hence there is a compact open $\hat{A} \subset \hat{X}$ such that

$$\Gamma\hat{\chi}_A = \chi_{\hat{A}} \ .$$

2.1. Lemma

Let $A \in \mathfrak{A}^\mu_{fin}$ *and set*

$$\langle \hat{\mu}_A, \varphi \rangle := \int_A \Gamma^{-1}\varphi \, d\mu \ .$$

Then $\hat{\mu}_A$ *defines a measure on* $\hat{X}, \hat{\mu}_A(X) = \mu(A)$ *and the embedding*

$$C(\hat{A}) \hookrightarrow L_\infty(\hat{A}, \mathfrak{B}_{\hat{A}}, \hat{\mu}_A)$$

is surjective.

Proof. See e.g. [14; Prop. 1.12, p.107]. □

By Zorn's lemma we find a family $\mathfrak{F} \subset \mathfrak{A}^{\mu}_{fin}$ such that $\mu(E \cap F) = 0$ for different $E, F \in \mathfrak{F}$ which is maximal with respect to the order given by $\sup\{\tilde{\chi}_F; F \in \mathfrak{F}\}$, $(L_\infty(X, \mathfrak{A}, \mu)$ is order complete as $(X, \mathfrak{A}, \mu)$ is localizable, cf [6; 16.6.4, p.282]).

Let $X^\# := \bigcup\{\hat{F}; F \in \mathfrak{F}\}$, which is open and dense in $\hat{X}$. For every $F \in \mathfrak{F}$ the measure $\hat{\mu}_F$ from 2.1 defines a finite Radon measure on $X^\#$ with support $\hat{F}$. Hence

$$\mu^\# := \sum_{F \in \mathfrak{F}} \hat{\mu}_F$$

defines a Radon measure on $X^\#$.

Let $\mathfrak{B}^\#$ denote the Borel σ–algebra and set

$$G : M_\infty(X^\#, \mathfrak{B}^\#, \mu^\#) \longrightarrow L_0(X, \mathfrak{A}, \mu)$$
$$Gf := \sup_{F \in \mathfrak{F}} \Gamma^{-1}(f\chi_{\hat{F}})$$

which is defined since, by 2.1, $f\chi_{\hat{F}} \in L_\infty(\hat{F})$ has a continuous representative.

$$G\chi_{\hat{A}} = \sup_{F \in \mathfrak{F}} \Gamma^{-1}(\chi_{\hat{A} \cap \hat{F}}) = \sup_{F \in \mathfrak{F}}(\chi_{A \cap F}) = \chi_A \ .$$

Again, the localizability of $(X, \mathfrak{A}, \mu)$ ensures the existence of this supremum.

It remains to check the order continuity of G. To this end it suffices to note that

$$G_* : L_1(X, \mathfrak{A}, \mu) \longrightarrow L_1(X^\#, \mathfrak{B}^\#, \mu^\#),$$
$$G_* f := (Tf)^\sim$$

is a positive contraction with $(G_*)' \supset G$.

Assume that G has a pseudo–kernel $\gamma : X \to M_R(X^\#)$. Changing γ on a local μ–null set, if necessary, we may restrict ourselves to the case that $\gamma(x, X^\#) = 1$ for all $x \in X$. For $f \in L_\infty(X, \mathfrak{A}, \mu)$ set

$$\Lambda f(x) := \langle \gamma(x), \Gamma f \rangle \ .$$

Then $\Lambda\tilde{1} = 1$ by what we just assumed and

$$\Lambda\tilde{\chi}_A = \chi_A \text{ locally } \mu\text{–a.e.}$$

for all locally measurable $A \subset X$.

To prove this last claim, observe that

$$G\chi_{\hat{A}} = \sup\{\tilde{\chi}_{A \cap F}; F \in \mathfrak{F}\} = \tilde{\chi}_A \ ,$$

where we used the maximality of $\mathfrak{F}$ for the last equation.

Linearity and density imply

$$\Lambda f \in f \text{ for all } f \in L_\infty(X, \mathfrak{A}, \mu) \, .$$

As Λ is clearly positive, it defines a linear lifting. Thus we have proven $(iii) \Rightarrow (i)$ of the following:

2.2. Theorem

Let $(X, \mathfrak{A}, \mu)$ be complete and localizable. Let $X^{\#}, \mu^{\#}$ and G be as above. Then the following conditions on $(C, \mathfrak{A}, \mu)$ are equivalent:

(i) $(X, \mathfrak{A}, \mu)$ *is strictly localizable.*

(ii) *The assertion of Theorem 1.1 holds for* $(X, \mathfrak{A}, \mu)$.

(iii) *The operator* $G : M_{\infty}(X^{\#}, \mathfrak{B}^{\#}, \mu^{\#}) \to L_{\infty}(X, \mathfrak{A}, \mu)$ *is pseudo-integral.*

The other implications are clear.

In [13] we also studied the assumptions (in 1.1) concerning $(Y, \mathfrak{B}, \nu)$ in detail. With arguments similar to those given above we could show the following:

Measure spaces $(Y, \mathfrak{B}, \nu)$ with countably generated $\mathfrak{B}$ for which the assertion of Theorem 1.1 holds are in an appropriate sense, isomorphic to compact spaces with a Radon measure.

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