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Simultaneous resolutions of the identity operator in normed spaces

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Abstract

We construct in this paper some simultaneous projective resolutions of the identity operator which we later use to obtain certain new results on quasi-complementation property and Markushevich bases.

All linear spaces mentioned throughout the following are assumed to be real. $\mathbb{Q}$ stands for the field of rationals. $\aleph_0$ denotes the first infinite cardinal and ω is the first ordinal of cardinality $\aleph_0$. The symbol $|A|$ denotes the cardinal of the set A. Similarly, for a given ordinal α, $|\alpha|$ represents its cardinal number.

X^* stands for the conjugate of a given normed linear space X. For a given subset A of X^*, by A_σ, or even $(A)_\sigma$, we mean the set A endowed with the topology induced by the weak-star topology of X^*; $A_\perp$ denotes the orthogonal subspace of A in X, and $\operatorname{lin} A$ is the linear hull of A. For a continuous linear operator T from X into X, T^* is its conjugate operator and $\ker T$ its kernel. B_X will denote the closed unit ball of X. For a given subset M of X, $M^\perp$ indicates the orthogonal subspace of M in X^*, $[M]$ is the closed linear hull of M, and $L(M)$ is the normed subspace of X given by the linear hull of M. If M is a closed absolutely convex bounded subset, then X_M will stand for the normed space on $\operatorname{lin} M$ with M as closed unit

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ball. Given a closed subspace Y of X, a closed subspace Z of X is said to be a quasicomplement of Y whenever $Y \cap Z = \{0\}$ and $Y + Z$ is dense in X. The symbol $\|\cdot\|$ denotes the norm of any normed space X. If x is in X, u is in X^* and A is a subset of X, then $\langle x, u\rangle$ is the value of u on x, and $d(x, A)$ is the distance from x to A.

The density character of a topological space E is defined as the first cardinal number λ such that there is a dense subset A of E with $|A| = \lambda$. We then write $\lambda = \text{dens } E$.

A Markushevich basis in a normed space X is a biorthogonal system

$$(x_i, u_i)_{i\in I}, \qquad x_i \in X,\ u_i \in X^*,$$

such that

$$X = \left[\{x_i : i \in I\}\right],$$

and

$$\text{lin } \{u_i : i \in I\}$$

is weak-star dense in X^*.

We shall then say, as Plicko [11], that $(x_i, u_i)_{i\in I}$ is countably 1-norming provided that the set of all elements u in B_{X^*} for which the set of indices

$$\{i \in I : \langle x_i, u\rangle \neq 0\}$$

is countable is weak-star dense in B_{X^*}.

A projective resolution of the identity operator in a normed space X, or simply a resolution of the identity in X, is a family

$$\{P_\alpha : \omega \leq \alpha \leq \mu\} \tag{1}$$

of continuous projections on X, with μ being the first ordinal of dens$X$, such that: P_μ is the identity operator in X,

$$\|P_\alpha\| = 1, \quad \text{dens } P_\alpha(x) \leq |\alpha|, \quad P_\alpha \circ P_\beta = P_\beta \circ P_\alpha, \quad \omega \leq \beta \leq \alpha \leq \mu,$$

and, for each limit ordinal α, the closure of

$$\bigcup \{P_\beta(X) : \omega \leq \beta < \alpha\}$$

in X coincides with $P_\alpha(X)$. A Markushevich basis $(x_i, u_i)_{i\in I}$ is said to be associated to the resolution of the identity (1) whenever there is a partition of the index set I,

$$I_\omega, I_{\alpha+1}, \qquad \omega \leq \alpha < \mu,$$

such that

$$\left(x_i, u_i\big|_{P_\omega(X)}\right)_{i\in I_\omega}$$

is a Markushevich basis of $P_\omega(X)$, and

$$\left(x_i, u_i\big|_{(P_{\alpha+1}-P_\alpha)(X)}\right)_{i\in I_{\alpha+1}}$$

is a Markushevich basis of $(P_{\alpha+1} - P_\alpha)(X)$, $\omega \le \alpha < \mu$.

We select a countable base $\mathcal{O}$ of the usual topology of $\mathbb{R}$.

Given a compact topological space K, $\mathcal{C}(K)$ is the Banach space of all continuous real functions f in K, with the norm

$$\|f\| = \sup\left\{|f(x)| \ : \ x \in K\right\}.$$

For a retraction r in K, T_r is the operator in $\mathcal{C}(K)$ such that

$$T_r f = f \circ r, \qquad f \in \mathcal{C}(K).$$

If $K_1, K_2, \ldots, K_n$ are closed subsets of K and $o_1, o_2, \ldots, o_n \in \mathcal{O}$, we write:

$$P(K_1, K_2, \ldots, K_n; o_1, o_2, \ldots, o_n) = \\ \left\{f \in \mathcal{C}(K) \ : \ f(K_1) \subset o_1, f(K_2) \subset o_2, \ldots, f(K_n) \subset o_n\right\}.$$

Assigning to each element x of a given normed space X its restriction to B_{X^*} we may consider X as a subspace of $\mathcal{C}((B_{X^*})_\sigma)$.

For a set Γ, given $\gamma \in \Gamma$ and $f \in \mathbb{R}^\Gamma$, we set $e_\gamma(f) = f(\gamma)$; e is the function with constant value one for every point of $\mathbb{R}^\Gamma$. If J is a subset of Γ, a mapping

$$r_J \colon \mathbb{R}^\Gamma \longrightarrow \mathbb{R}^\Gamma$$

is defined by setting, for each $x = (x_\gamma \ : \ \gamma \in \Gamma)$ in $\mathbb{R}^\Gamma$,

$$\begin{cases} r_J(x)_\gamma = 0 & \text{if } \gamma \notin J, \\ r_J(x)_\gamma = x_\gamma & \text{if } \gamma \in J, \end{cases}$$

If $z = (z_\gamma \ : \ \gamma \in \Gamma)$ is in $\mathbb{R}^\Gamma$, we define

$$\operatorname{supp} z := \{\gamma \in \Gamma \ : \ z_\gamma \neq 0\}.$$

If A is a subset of $\mathbb{R}^\Gamma$, then

$$\operatorname{supp} A := \bigcup\{\operatorname{supp} z \ : \ z \in A\}.$$

If K is a compact subset of $\mathbb{R}^\Gamma$, $K(\Gamma)$ is the subset of K formed by all points z such that $\operatorname{supp} z$ is countable. $\mathcal{C}_\sigma(K)$ denotes the linear space $\mathcal{C}(K)$ with the topology of the pointwise convergence respect to the points of $K(\Gamma)$.

We shall say that a compact space K belongs to the class $\mathcal{A}$ whenever it is homeomorphic to a subspace of $\mathbb{R}^\Gamma$, for some set Γ, such that $K(\Gamma)$ is dense in K. In particular, if $K(\Gamma) = K$, is said to be a Corson compact.

Lemma 1

Let K be a compact subset of $\mathbb{R}^\Gamma$ such that $K(\Gamma)$ is dense in K. Let A_0 and B_0 be two infinite subsets of $\mathcal{C}(K)$ and $K(\Gamma)$, respectively, such that $|A_0| = |B_0|$. Then there is a retraction r in K satisfying

(a) $A_0 \subset T_r(\mathcal{C}(K))$, $B_0 \subset r(K)$.

(b) $\operatorname{dens} T_r(\mathcal{C}(K)) \leq |A_0|$.

Proof. Let g_γ and g denote the restrictions of e_γ and e to K, respectively, $\gamma \in \Gamma$. The algebra generated by the set

$$\{g_\gamma \ : \ \gamma \in \Gamma\} \cup \{g\}$$

separates points in K and contains the constant functions. Thus, for each continuous real function f, defined in K, there is a countable subset $\Gamma(f)$ of Γ such that f lies in the closure in $\mathcal{C}(K)$ of the algebra generated by

$$\{g_\gamma \ : \ \gamma \in \Gamma(f)\} \cup \{g\}.$$

We set

$$I_0 := \Big(\bigcup\{\Gamma(f) \ : \ f \in A_0\}\Big) \cup \operatorname{supp} B_0, \qquad \lambda := |A_0|.$$

The procedure we shall now follow in the construction of the retraction r contains a method already used by Gul'ko and Benjamini for Corson compacts [2]. We proceed by induction assuming that, for a non-negative integer n, we have already defined the subset I_n of Γ with $|I_n| \leq \lambda$. Since

$$\operatorname{dens} r_{I_n}(K) \leq \lambda,$$

there is a subset M_n of $K(\Gamma)$ such that $|M_n| \leq \lambda$ and $r_{I_n}(M_n)$ is dense in $r_{I_n}(K)$. We then define

$$I_{n+1} := I_n \cup \operatorname{supp} M_n.$$

Then $|I_{n+1}| \leq \lambda$. Now, let

$$I := \bigcup \{I_n \ : \ n = 1, 2, \dots\}.$$

We write r meaning the restriction of r_I to K. Let z be an element in K. By compactness of K, we may find, for each non-negative integer n, a point $u^{(n)}$ in the closure of M_n such that $r_{I_n}(u^{(n)}) = r_{I_n}(z)$. Let u be a cluster point of the sequence $(u^{(n)})$. Take an element γ in I. We then have a positive integer m such that γ is in I_m. Thus $u_\gamma^{(n)} = z_\gamma$, $n \geq m$, and, hence, $u_\gamma = z_\gamma$. But also we have

$$\operatorname{supp} u^{(n)} \subset I_{n+1} \subset I, \qquad n = 0, 1, 2, \dots ,$$

hence supp u is contained in I, i.e., $r_I(u) = u$, and r is a retraction in K.

Now, let x be in K and f in A_0. Since $\Gamma(f)$ is contained in I, we have that f vanishes in the vector $x - r(x)$ of $\mathcal{C}(X)^*$, and, consequently,

$$f(x) = f(r(x)) = (T_r\, f)(x),$$

thus f coincides with $T_r f$ and the final conclusion is now immediate. □

Lemma 2

Let K be a compact subset of $\mathbb{R}^\Gamma$ with $K(\Gamma)$ dense in K. Let (N_m) be a sequence of closed subsets of $\mathcal{C}_\sigma(K)$. Let A_0 and B_0 be two infinite subsets of $\mathcal{C}(K)$ and $K(\Gamma)$, respectively, such that $|A_0| = |B_0|$. If G is a subset of $\mathcal{C}(K)$ such that, for each x in $K(\Gamma)$,

$$\{f \in G \; : \; f(x) \neq 0\}$$

is countable, then there is a subset J of I such that the restriction s of r_J to K is a retraction in K and there is a subset G_1 of G with the following properties:

(a) $A_0 \subset T_s(\mathcal{C}(K))$, $B_0 \subset s(K)$.

(b) $\operatorname{dens} T_s(\mathcal{C}(K)) \leq |A_0|$.

(c) $T_s(N_m) \subset N_m, \quad m = 1, 2, \ldots$

(d) $G_1 \subset T_s(\mathcal{C}(K))$, $G \setminus G_1 \subset \ker T_s$.

Proof. As established in the previous lemma, we determine a subset I of Γ such that the restriction r of r_I to K be a retraction satisfying:

$$|I| \leq |A_0|, \quad A_0 \subset T_r(\mathcal{C}(K)), \quad B_0 \subset r(K).$$

Let us define

$$\lambda := |A_0|, \; J_0 := I, \; s_0 := r, \; Q_m := \mathcal{C}(K) \setminus N_m, \; m = 1, 2, \ldots .$$

Again, an inductive procedure allows us to assume that, for a non-negative integer n, we have a subset J_n of Γ such that $|J_n| \leq \lambda$ and the restriction s_n of r_{J_n} to K is a retraction, and we also have the sets

$$A_n \subset T_{s_n}(\mathcal{C}(K)), \quad B_n \subset s_n(K), \quad |A_n| = |B_n| = \lambda.$$

We choose a family of compact subsets of $s_n(K)$,

$$\{K_{nh} \; : \; h \in H_n\}, \quad |H_n| \leq \lambda, \tag{2}$$

such that their interiors K_{nh} in $s_n(K)$ are a base for the topology of that space and the closure of $\overset{\circ}{K}_{nh}$ coincides with K_{nh}. Given the positive integers

$$m, h_1, h_2, \ldots, h_j \in H_n, \quad o_{n_1}, o_{n_2}, \ldots, o_{n_j} \in \mathcal{O},$$

we select, if it exists, an open subset of $\mathcal{C}_\sigma(K)$ of the type

$$P\big(\{x_1\}, \{x_2\}, \ldots, \{x_i\}; o_{m_1}, o_{m_2}, \ldots, o_{m_i}\big), \qquad \begin{cases} x_1, x_2, \ldots, x_i \in K(\Gamma), \\ o_{m_1}, o_{m_2}, \ldots, o_{m_i} \in \mathcal{O} \end{cases}$$

such that it contains

$$P\big(s_n^{-1}(K_{nh_1}), s_n^{-1}(K_{nh_2}), \ldots, s_n^{-1}(K_{nh_j}); o_{n_1}, o_{n_2}, \ldots, o_{n_j}\big)$$

and at the same time contained in Q_m. We write D_n for the reunion of all sets $\{x_1, x_2, \ldots, x_i\}$ corresponding to all

$$m, j \in \mathbb{N}, \quad h_1, h_2, \ldots, h_j \in H_n, \quad o_{n_1}, o_{n_2}, \ldots, o_{n_j} \in \mathcal{O}.$$

Obviously, $|D_n| \leq \lambda$. Now, let F_n be a subset dense in $s_n(K) \cap K(\Gamma)$, with $|F_n| \leq \lambda$. Then we define

$$A_{n+1} := A_n \cup \{f \in G \ : \ f(x) \neq 0, x \in B_n\}$$
$$B_{n+1} := B_n \cup D_n \cup F_n.$$

Clearly, $|A_{n+1}| = |B_{n+1}| = \lambda$.

Applying the previous lemma with A_{n+1} and B_{n+1} instead of A_0 and B_0, respectively, we obtain a subset J_{n+1} in Γ such that the restriction s_{n+1} of $r_{J_{n+1}}$ to K is a retraction and

$$|J_{n+1}| \leq \lambda, \quad A_{n+1} \subset T_{s_{n+1}}(\mathcal{C}(K)), \quad B_{n+1} \subset s_{n+1}(K).$$

Now, let

$$J := \bigcup \{J_n \ : \ n = 1, 2, \ldots\}.$$

If x lies in K, it is clear that $r_J(x)$ is the limit of $(s_n(x))$ and, therefore the restriction s of r_J to K is a retraction which clearly satisfies the properties (a) and (b).

Take now an element f in N_m and assume that $T_s f$ is not in N_m. We may find

$$z_1, z_2, \ldots, z_k \in K(\Gamma), \ o_1, o_2, \ldots, o_k \in \mathcal{O},$$

such that

$$P(\{z_1\}, \{z_2\}, \ldots, \{z_k\}; o_1, o_2, \ldots, o_k) \tag{3}$$

be a neighbourhood of $T_s f$ contained in Q_m.

Let $\varepsilon > 0$ such that

$$[(T_s f)(z_i) - 3\varepsilon, (T_s f)(z_i) + 3\varepsilon] \subset o_i, \qquad i = 1, 2, \ldots, k.$$

Obviously $T_{s_j}(\mathcal{C}(K))$ is a linear algebra containing the constant functions. Also

$$T_{s_j}(\mathcal{C}(K)) \subset T_{s_{j+1}}(\mathcal{C}(K)), \qquad j = 1, 2, \ldots,$$

thus allowing

$$\bigcup\{T_{s_j}(\mathcal{C}(K)) \ : \ j = 1, 2, \ldots\} \tag{4}$$

to be a linear algebra containing constants. If

$$x = (x_\gamma \ : \ \gamma \in \Gamma), \quad y = (y_\gamma \ : \ \gamma \in \Gamma)$$

are two distinct points of $s(K)$, then there is a positive integer i and an index γ in J_i such that $x_\gamma \neq y_\gamma$. Then

$$e_\gamma(x) = x_\gamma \neq y_\gamma = e_\gamma(y)$$

and, since $e_\gamma|_K$ belongs to $T_{s_j}(\mathcal{C}(K))$, we have that the restriction of the functions in (4) to $s(K)$ are a dense subset of $\mathcal{C}(s(K))$. For this reason, we may find an element g in (4) such that

$$|g(x) - (T_s f)(x)| < \varepsilon, \qquad x \in s(K).$$

Set a positive integer n that g is in $T_{s_n}(\mathcal{C}(K))$. Then

$$|g(z_v) - (T_s f)(z_v)| = |g(s(z_v)) - (T_s f)(s(z_v))| < \varepsilon, \qquad v = 1, 2, \ldots, k.$$

For each $v = 1, 2, \ldots, k$, since the intersection of all the elements in (2) containing $s_n(z_v)$ is $\{s_n(z_v)\}$, there are $h_1, h_2, \ldots, h_k \in H_n$ such that

$$s_n(z_v) \in K_{nh_v}, \quad g(K_{nh_v}) \subset \Big[g(s_n(z_v)) - \varepsilon, g(s_n(z_v)) + \varepsilon\Big], \qquad v = 1, 2, \ldots, k.$$

Thus, if x is a point of $s_n^{-1}(K_{nh_v})$, with $1 \leq v \leq k$,

$$\begin{aligned}
|(T_s f)(x) - (T_s f)(z_v)| &\leq |(T_s f)(x) - g(x)| + |g(x) - g(z_v)| + |g(z_v) - (T_s f)(z_v)| \\
&\leq \varepsilon + |g(x) - g(z_v)| + \varepsilon = 2\varepsilon + |g(s_n(x)) - g(s_n(z_v))| \\
&\leq 3\varepsilon,
\end{aligned}$$

and we get

$$(T_s f)(x) \in \Big[(T_s f)(z_v) - 3\varepsilon, (T_s f)(z_v) + 3\varepsilon\Big] \subset o_v.$$

Hence, writing

$$Z := P(s_n^{-1}(K_{nh_1}), s_n^{-1}(K_{nh_2}), \ldots, s_n^{-1}(K_{nh_k}; o_1, o_2, \ldots, o_k),$$

we have that $T_s f$ is in Z and, since this latter set is contained in (3), there are $y_1, y_2, \ldots, y_q$ in D_n and $B_1, B_2, \ldots, B_q$ in $\mathcal{O}$, such that

$$Z \subset P(\{y_1\}, \{y_2\}, \ldots, \{y_q\}; B_1, B_2, \ldots, B_q) \subset Q_m.$$

Clearly, $y_1, y_2, \ldots, y_q$ are in $s_{n+1}(K)$, therefore

$$f(y_v) = f(s_{n+1}(y_v)) = f(s(y_v)) = (T_s f)(y_v) \in B_v, \qquad v = 1, 2, \ldots, q,$$

concluding that

$$f \in P(\{y_1\}, \{y_2\}, \ldots, \{y_q\}; B_1, B_2, \ldots, B_q) \subset Q_m,$$

which is a contradiction. Thus showing (c).

Let G_1 be the subset of G whose elements are in $T_s(\mathcal{C}(K))$. It is easily seen from the construction just given that $G \setminus G_1$ is contained in $\ker T_s$, property (d) is then satisfied. □

Theorem 1

Let K be a compact subset of $\mathbb{R}^\Gamma$ with $K(\Gamma)$ dense in K. Let (N_m) be a sequence of closed absolutely convex subsets of $\mathcal{C}_\sigma(K)$. If μ and μ_m are the first ordinals of dens $\mathcal{C}(K)$ *and* dens N_m, *respectively, and $|\mu_m| \geq \aleph_0$, $m = 1, 2, \ldots$, then there is a resolution of the identity*

$$\{T_\alpha \ : \ \omega \leq \alpha \leq \mu\}$$

in $\mathcal{C}(K)$ with the following properties:

(a) $\{T_\alpha|_{L(N_m)} \ : \ \omega \leq \alpha \leq \mu_m\}$ is a resolution of the identity in $L(N_m)$, $m = 1, 2, \ldots$.

(b) In $\mathcal{C}_\sigma(K)$, T_α is continuous and $T_\alpha(\mathcal{C}(K))$ is closed, $\omega \leq \alpha \leq \mu$.

(c) $T_\alpha(N_m) \subset N_m$, $m = 1, 2, \ldots$, $\omega \leq \alpha \leq \mu$.

Proof. We may clearly assume $|\Gamma| =$ dens K. Let

$$\{f_\alpha \ : \ 0 \leq \alpha < \mu\} \quad \text{and} \quad \{f_{m\alpha} \ : \ 0 \leq \alpha < \mu_m\}$$

be dense subsets of $\mathcal{C}(K)$ anf $L(N_m)$, respectively, $m = 1, 2, \ldots$. We apply our former lemma for the particular case

$$A_0 = \{f_\alpha \ : 0 \leq \alpha \leq \omega\} \cup \Big(\bigcup_{m=1}^{\infty} \{f_{m\alpha} \ : \ 0 \leq \alpha \leq \omega\} \Big)$$

and B_0 a countably infinite subset of $K(\Gamma)$ and we obtain a subset Γ_0 of Γ such that the restriction s of r_{Γ_0} to K satisfies properties (a), (b) and (c) there stated. We set now T_ω, Γ_ω and s_ω instead of T_s, Γ_0 and s, respectively, and we proceed by transfinite induction. Let α be an ordinal number, $\omega < \alpha \leq \mu$, such that, for each ordinal β, with $\omega \leq \beta < \alpha$, a subset Γ_β of Γ has been defined so that

$$|\Gamma_\beta| \leq |\beta|, \quad \Gamma_\eta \subset \Gamma_\xi, \quad \omega \leq \eta \leq \xi < \alpha,$$

and the restriction of r_{Γ_β} to K is a retraction s_β. We define $T_\beta := T_{s_\beta}$. Let us assume first that α is not a limit ordinal. Let $\nu := \alpha - 1$. We choose a dense subset A_ν in $T_\nu(\mathcal{C}(K))$ such that $|A_\nu| \leq |\nu|$. If f is an element in A_ν and m, n are positive integers with $\nu < \mu_m$, we select in N_m an element $f_{\nu_{mn}}$ such that

$$\|f - f_{\nu_{mn}}\| < d(f, N_m) + \frac{1}{n}.$$

We define

$$\begin{cases} F_{\nu_m} := \{f_{m\alpha}\} & \text{if } \nu < \mu_m, \\ F_{\nu_m} := \emptyset & \text{if } \mu_m \leq \nu \end{cases} \qquad m = 1, 2, \dots .$$

We choose now a dense subset B_ν of $s_\nu(K)$, with $|B_\nu| = |A_\nu|$. Apply again the previous lemma for the particular case

$$\begin{aligned} A_0 &:= A_\nu \cup \{f_\alpha\} \cup \{f_{\nu_{mn}} \ : \ \nu < \mu_m, n = 1, 2, \dots\} \cup \left(\bigcup \{F_{\nu_m} \ : \ m = 1, 2, \dots\} \right), \\ B_0 &:= B_\nu \end{aligned}$$

and thus we get a subset Γ_α of Γ for which the restriction s_α of r_{Γ_α} to K is a retraction with the properties (a), (b) and (c) stated in the mentioned lemma with s_α instead of s. Then $\Gamma_\nu \subset \Gamma_\alpha$. Let $T_\alpha := T_{s_\alpha}$. If α is a limit ordinal, we write

$$\Gamma_\alpha := \bigcup \{\Gamma_\beta \ : \ \omega \leq \beta < \alpha\},$$

and $T_\alpha := T_{s_\alpha}$, s_α being the restriction of r_{Γ_α} to K.

We show next that

$$\{T_\alpha \ : \ \omega \leq \alpha \leq \mu\}$$

is a resolution ofthe identity in $\mathcal{C}(K)$. Evidently,

$$\|T_\alpha\| = 1, \quad \operatorname{dens} T_\alpha(\mathcal{C}(K)) \leq |\alpha|, \quad T_\alpha \circ T_\beta = T_\beta = T_\beta \circ T_\alpha, \quad \omega \leq \beta \leq \alpha \leq \mu.$$

Also, if α is a limit ordinal, $\omega < \alpha \leq \mu$, it is simple to see that

$$\bigcup \{T_\beta(\mathcal{C}(K)) \ : \ \omega \leq \beta < \alpha\} \tag{5}$$

is a subalgebra of $\mathcal{C}(K)$ containing constants that separates points in $s_\alpha(K)$. Therefore, for a given $\varepsilon > 0$ and an element f of $T_\alpha(\mathcal{C}(K))$, we may find in (5) an element g such that

$$|f(x) - g(x)| < \varepsilon, \qquad x \in s_\alpha(K).$$

Now, if z is an arbitrary element of K, we have

$$|f(z) - g(z)| = |(T_\alpha f)(z) - (T_\alpha g)(z)| = |f(s_\alpha(z)) - g(s_\alpha(z))| < \varepsilon,$$

thus the closure of (5) in $\mathcal{C}(K)$ coincides with $T_\alpha(\mathcal{C}(K))$. Finally, if $\alpha = \mu$, (5) is dense in $\mathcal{C}(K)$, concluding that T_μ is the identity operator.

Condition (c) clearly holds. Fix now the positive integer m. Condition (c) guarantees that $T_\alpha|_{L(N_m)}$ is an operator in $L(N_m)$. If α is a limit ordinal, $\omega < \alpha \leq \mu_m$, and f belongs to $T_\alpha(\mathcal{C}(K)) \cap L(N_m)$, we may find a real number $b > 0$ such that bf lies in N_m. Also, we may determine a sequence (f_n) in

$$\bigcup \{T_\beta(\mathcal{C}(K)) \ : \ \omega \leq \beta < \alpha\}$$

convergent to bf in $\mathcal{C}(K)$. Hence, we find

$$\omega \leq \alpha_1 < \alpha_2 < \ldots < \alpha_n < \ldots < \alpha$$

such that f_n is in $T_{\alpha_n}(\mathcal{C}(K))$. By the preceding construction, there are

$$f_{nn} \in N_m, \quad f_{nn} \in T_{\alpha_{n+1}}(\mathcal{C}(K))$$

such that

$$\|f_n - f_{nn}\| < d(f_n, N_m) + \frac{1}{n} \qquad n = 1, 2, \ldots$$

Then, (f_{nn}) is in N_m and converges to bf in $\mathcal{C}(K)$, so that the closure of

$$\bigcup \left\{ (T_\beta|_{L(N_m)})(L(N_m)) \ : \ \omega \leq \beta < \alpha \right\}$$

in $L(N_m)$ equals $T_\alpha|_{L(N_m)}(L(N_m))$. The remaining properties to complete the proof of (a) are immediate. Condition (b) is straghtforward. □

Note 1. Let X be a Banach space. We identity (B_{X^*}) with a subspace K of $\mathbb{R}^\Gamma$. Suppose that there is a linear subspace Y of X^* such that $Y \cap K$ is dense in K and $K(\Gamma)$ contains $Y \cap K$. Then X is closed in $C_\sigma(K)$ and applying Theorem 1 a resolution of the identity operator is obtained in X. Using a method introduced in [15] this result can be proved also, [10].

Corollary 1.1

Let X be a Banach space such that the closed unit ball of X^*, with the weak-star topology, is a Corson compact. Let (N_m) be a sequence of closed absolutely convex subsets of X. If μ and μ_m are the first ordinal numbers of dens X *and* dens N_m, *respectively, and $|\mu_m| \geq \aleph_0$, $m = 1, 2, , \ldots$, then there is a resolution of the identity*

$$\{T_\alpha \ : \ \omega \leq \alpha \leq \mu\}$$

in X, such that, for each positive integer m

$$\{T_\alpha|_{L(N_m)} \ : \ \omega \leq \alpha \leq \mu_m\}$$

is a resolution of the identity in $L(N_m)$ and $T_\alpha(N_m)$ is contained in $N_m, \omega \leq \alpha \leq \mu_m$.

Proof. Set $K := (B_{X^*})_\sigma$. We may certainly assume that K is a compact subset of $\mathbb{R}^\Gamma$, for a convenient Γ, such that $K(\Gamma) = K$. We then have $X, N_1, N_2, \ldots, N_m, \ldots$ are closed absolutely convex subsets of $c_\sigma(K)$. An application of the previous theorem leads to the desired conclusion. □

Theorem 2

Let K be a compact subset of $\mathbb{R}^\Gamma$ with $K(\Gamma)$ dense in K. Let M be a closed absolutely convex subset of $C_\sigma(K)$, such that $L(M)$ has infinite dimension. If ν is the first ordinal number of dens $L(M)$, then there is a resolution of the identity

$$\{S_\alpha \ : \ \omega \leq \alpha \leq \nu\}$$

in $L(M)$ and associated Markushevich basis $(f_i, u_i)_{i \in I}$ such that

$$S_\alpha(M) \subset M, \qquad \omega \leq \alpha \leq \nu,$$

and for each x in $K(\Gamma)$, the set

$$\{i \in I \ : \ f_i(X) \neq 0\}$$

is countable.

Proof. We write g_γ and g for the restrictions of e_γ and e to K, respectively, $\gamma \in \Gamma$. Let

$$G := A \cup \{g\}$$

where A denotes the subset of $\mathcal{C}(K)$ formed by all finite products of the type

$$g_{\gamma_1} g_{\gamma_2} \cdots g_{\gamma_n}, \quad \gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma.$$

The linear hull of G is an algebra containing the constant functions that separates points in K. Thus, $[G] = \mathcal{C}(K)$. Besides, for each x in $K(\Gamma)$,

$$\{f \in G : f(x) \neq 0\}$$

is clearly countable.

We base our discussion on the density character of $L(M)$. If $|\nu| = \aleph_0$, the assertion of the theorem is then obvious [7, Prop. 1. f.3]. Suppose that $|\nu| > \aleph_0$ and for each closed absolutely convex subset B of $C_\sigma(K)$, with $L(B)$ infinite dimensional, such that dens $L(B) < |\nu|$, there is a resolution of the identity in $L(B)$ and associated Markushevich basis $(g_j, v_j)_{j \in J}$ such that, for each x in $K(\Gamma)$, the set

$$\{j \in J \ : \ g_j(x) \neq 0\}$$

is countable.

We proceed now as in the proof of Theorem 1, making use also of condition (d) of Lemma 2, and thus we obtain a projective resolution of the identity in $\mathcal{C}(K)$,

$$\{T_\alpha \ : \ \omega \leq \alpha \leq \mu\},$$

where μ is the first ordinal of dens $\mathcal{C}(K)$, and a partition of G,

$$G_\omega, G_{\alpha+1}, \qquad \omega \leq \alpha < \mu,$$

such that

$$\{T_\alpha|_{L(M)} \ : \ \omega \leq \alpha \leq \nu\}$$

is a resolution of the identity in $L(M)$,

$$T_\alpha(M) \subset M, \qquad \omega \leq \alpha \leq \mu,$$

$$G_\omega \subset T_\omega(\mathcal{C}(K)), \quad G_{\alpha+1} \subset \big(T_{\alpha+1} - T_\alpha\big)(\mathcal{C}(K)), \qquad \omega \leq \alpha < \mu,$$

and, also, in $C_\sigma(K)$, T_α is continuous and $T_\alpha(\mathcal{C}(K))$ is closed, $\omega \le \alpha \le \mu$. Clearly,

$$[G_\omega] = T_\omega(\mathcal{C}(K)), \quad [G_{\alpha+1}] = \big(T_{\alpha+1} - T_\alpha\big)(\mathcal{C}(K)), \qquad \omega \le \alpha < \mu.$$

Now we write

$$S_\alpha := T_\alpha|_{L(M)}, \qquad \omega \le \alpha \le \nu.$$

Since $S_\omega(L(M))$ is separable, there is a biorthogonal system $(f_i, u_i)_{i\in I_\omega}$ in $L(M)$ such that

$$\big[\{f_1 \ : \ i \in I_\omega\}\big] \cap L(M) = S_\omega(L(M))$$

and lin $\{u_i \ : \ i \in I_\omega\}$ is weak-star dense in $S^*_\omega(L(M)^*)$. Obviously, for each x in $K(\Gamma)$, the set

$$\{i \in I_\omega \ : \ f_i(x) \ne 0\}$$

is countable. Given an ordinal number α, $\omega \le \alpha < \mu$, we have that

$$M_{\alpha+1} := M \cap \big(S_{\alpha+1} - S_\alpha\big)(L(M))$$

is closed in $C_\sigma(K)$ and

$$L(M_{\alpha+1}) = \big(S_{\alpha+1} - S_\alpha\big)(L(M)), \quad \text{dens } L(M) < |\nu|,$$

thus, there is a Markushevich basis $(f_i, v_i)_{i\in I_{\alpha+1}}$ in $L(M_{\alpha+1})$ such that, for each x in $K(\Gamma)$, the set

$$\{i \in I_{\alpha+1} \ : \ f_i(x) \ne 0\}$$

is countable. We may therefore find a biorthogonal system $(f_i, u_i)_{i\in I_{\alpha+1}}$ in $L(M)$ such that

$$\big[\{f_i \ : \ i \in I_{\alpha+1}\}\big] \cap L(M) = \big(S_{\alpha+1} - S_\alpha\big)(L(M))$$

and lin $\{u_i \ : \ i \in I_{\alpha+1}\}$ is weak-star dense in $\big(S^*_{\alpha+1} - S^*_\alpha\big)(L(M)^*)$. If we take

$$I_\omega, I_{\alpha+1}, \qquad \omega \le \alpha < \nu,$$

pairwise disjoint and define

$$I := I_\omega \cup \{I_{\alpha+1}, \omega \le \alpha < \nu\}$$

we clearly have that $(f_i, u_i)_{i\in I}$ is a Markushevich basis in $L(M)$. Choose an arbitrary point x of $K(\Gamma)$ and suppose that

$$\{i \in I \ : \ f_i(x) \ne 0\} \tag{6}$$

is not countable. Then, there is an uncountable subset A in the interval $[\omega, \nu]$ and an element f^δ in $G_{\delta+1}$ such that

$$f^\delta(x) \ne 0, \qquad \delta \in A$$

which is a contradiction. Thus, the set (6) is countable. □

Corollary 2.1

Let K be a compact subset of $\mathbb{R}^\Gamma$ with $K(\Gamma)$ dense in K. Let φ be a continuous mapping from K onto a compact H accomplishing the following conditions:

(a) The restriction ψ of φ to $K(\Gamma)$ is a quotient mapping of $K(\Gamma)$ onto $\psi(K(\Gamma))$.

(b) If x is a point of H such that $\varphi^{-1}(x)$ has more than a point, then $\varphi^{-1}(x) \cap K(\Gamma)$ is dense in $\varphi^{-1}(x)$.

Then H belongs to the class $\mathcal{A}$.

Proof. We identity $C(H)$ with a subspace of $C(K)$ simply assigning to each f in $C(H)$ the function $f \circ \varphi$.

We now take a function g of $C(K)$ belongs to the closure of $C(H)$ in $C_\sigma(K)$. It is not hard then to see that there is a function f in $C(H)$ such that $g = f \circ \varphi$, thus having that $C(H)$ is closed in $C_\sigma(K)$. The previous theorem applies to obtain a Markushevich basis $(f_i, u_i)_{i \in I}$ in $C(H)$ such that, for each x in $K(\Gamma)$, the set

$$\{i \in I \ : \ f_i(x) \neq 0\}$$

is countable. For each z in H, we choose x in K such that $\varphi(x) = z$, and write

$$\psi(z) := \big(f_i(x) \ : \ i \in I\big).$$

Then ψ is a continuous injection of H in $\mathbb{R}^I$ such that $\psi\big(\varphi(K(\Gamma))\big)$ is dense in $\psi(H)$. The conclusion now follows. □

In the above corollary, if K is a Corson compact, then $K = K(\Gamma)$, thus conditions (a) and (b) of the corollary are obviously satisfied, and H belongs to class $\mathcal{A}$. It is easily verified that H is angelic and, hence, H is a Corson compact. This result is due to Gul'ko [4] and to Michael and Rudin [9].

Corollary 2.2

Let K be a compact subset of $\mathbb{R}^\Gamma$ with $K(\Gamma)$ dense in K. Let M be a closed absolutely convex subset of $C_\sigma(K)$, such that $L(M)$ has infinite dimension. Then there is a resolution of the identity in $L(M)$ and an associated countably 1-norming Markushevich basis.

Proof. Applying Theorem 2 we obtain a resolution of the identity in $L(M)$ and an associated Markushevich basis $(f_i, u_i)_{i\in I}$ such that, for each x in $K(\Gamma)$, the set

$$\{i \in I \ : \ f_i(x) \neq 0\}$$

is countable.

We write D for the absolutely convex hull of $K(\Gamma)$ in $\mathcal{C}(K)^*$. Let v be in $L(M)^*$, $\|v\| \leq 1$. Hahn-Banach's theorem provides an element w of $\mathcal{C}(K)^*$ such that $\|w\| = \|v\|$, $w|_{L(M)} = v$. We find a net

$$\{w_j \ : \ j \in J, \ \geq\}$$

in D weak-star convergent to w.

Then

$$\{w_j|_{L(M)} \ : \ j \in J \ \ \geq\}$$

is a net in the closed unit ball of $L(M)^*$ weak-star convergent to v. From the equalities

$$\langle f_i, w_j|_{L(M)}\rangle = \langle f_i, w_j\rangle, \qquad i \in I, j \in J,$$

we deduce that the set

$$\{i \in I \ : \ \langle f_i, w_j|_{L(M)}\rangle \neq 0\}$$

is countable, $j \in J$, and, therefore, the basis $(f_i, u_i)_{i\in I}$ is countably 1-norming. □

Theorem 3

Let K be a compact subset of R^Γ with $K(\Gamma)$ dense in K. Let X be a subspace of $\mathcal{C}(K)$ closed in $C_\sigma(K)$. Let M be a bounded absolutely convex subset of X, closed in $C_\sigma(K)$ and such that $L(M)$ has infinite dimension. If the weak topologies of X and X_M coincide on M, then there are a countably 1-norming Markushevich basis $(f_i, u_i)_{i\in I}$ in X and a subset I_1 of I such that $(f_i, u_i|_{L(M)})_{i\in I_1}$ is a contably 1-norming Markushevich basis in X_M.

Proof. Our discussion is based on the density character of X. Suppose first that dens $X \leq \aleph_0$. If $L(M)$ is dense in X, the result is obvious, considering that X_M is separable and by virtue of [7, Prop 1.f.3]. If $L(M)$ is not dense in X, we find a quasicomplement Y of the closure of $L(M)$ in X, [8]. We also find two biorthogonal systems in X, $(f_i, u_i)_{i\in I_1}$ and $(f_i, u_i)_{i\in I_2}$ with I_1 and I_2 disjoint, such that

$$[\{f_i \ : \ i \in I_2\}] = Y,$$

$\operatorname{lin}\{f_i : i \in I_1\}$ is dense in X_M, $\operatorname{lin}\{u_i : i \in I_2\}$, $\operatorname{lin}\{u_i : i \in I_2\}$ are weak-star dense subsets of $Y^{\perp}$ and $L(M)^{\perp}$, respectively. If $I := I_1 \cup I_2$ then $(f_i, u_I)_{i\in I}$ is the basis stated in the theorem. Clearly, for each x in $K(\Gamma)$,

$$\{i \in I : f_i(x) \neq 0\}$$

is countable. Let us assume now that

$$\lambda := \text{dens } X > \aleph_0$$

and, for each subspace F of $\mathcal{C}(K)$, closed in $C_\sigma(K)$, dens $F < \lambda$, and each bounded absolutely convex subset P of F, closed in $C_\sigma(K)$, with $L(P)$ infinite dimensional, such that the weak topologies of F and F_P coincide in P, there is a Markushevich basis $(g_j, v_j)_{j\in J}$ in X, and a subset J_1 of J such that $\left(g_j, v_j|_{L(P)}\right)_{j\in J_1}$ is a countably 1-norming Markushevich basis in F_P, and for each x in $K(\Gamma)$, the set

$$\{j \in J : g_j(x) \neq 0\}$$

is countable.

Let G be as in the proof of Theorem 2. Let μ, ν and ρ be the first ordinal numbers of dens $\mathcal{C}(K)$, λ and dens$L(M)$ = densX_M, respectively. By a similar argument to that of Theorem 2, making use also of condition (d) of Lemma 2, we obtain a projective resolution of the identity in $\mathcal{C}(K)$,

$$\{T_\alpha : \omega \leq \alpha \leq \mu\}$$

and a partition of G,

$$G_\omega, G_{\alpha+1}, \qquad \omega \leq \alpha < \mu,$$

such that

$$T_\alpha(M) \subset M, \quad G_\omega \subset T_\omega(\mathcal{C}(K)), \quad G_{\alpha+1} \subset \left(T_{\alpha+1} - T_\alpha\right)(\mathcal{C}(K)), \qquad \omega \leq \alpha < \mu,$$

and

$$\{T_\alpha|_X : \omega \leq \alpha \leq \nu\} \quad \text{and} \quad \{T_\alpha|_{L(M)} : \omega \leq \alpha \leq \rho\}$$

are projective resolution of the identity in X and $L(M)$, respectively. Also, in $C_\sigma(K)$, T_α is continuous and $T_\alpha(\mathcal{C}(K))$ is closed, $\omega \leq \alpha \leq \mu$. Clearly,

$$[G_\omega] = T_\omega(\mathcal{C}(K)) \quad \text{and} \quad [G_{\alpha+1}] = \left(T_{\alpha+1} - T_\alpha\right)(\mathcal{C}(K)), \qquad \omega \leq \alpha < \mu.$$

We choose now a limit ordinal β, $\omega < \beta \leq \rho$, and an element f of $T_\beta(\mathcal{C}(K) \cap L(M)$. We then determine a non-zero real number b for wich bf lies in M. Using a similar process in the construction of the resolution of the identity to that of Theorem 1, there is a sequence (f_n) in

$$M \cap \big(U\{T_\alpha(\mathcal{C}(K)) \,:\, \omega \leq \alpha < \beta\}\big)$$

norm convergent to bf. Then (f_n) converges to bf respect to the weak topology of X_M, hence concluding that

$$\{T_\alpha|_{L(M)} \,:\, \omega \leq \alpha \leq \rho\}$$

is a resolution of the identity in X_M. We define

$$S_\alpha := T_\alpha|_X, \qquad \omega \leq \alpha < \nu$$

Assume first that $\rho = \nu$. Since $S_\omega(X)$ and $M_\omega := S_\omega(X) \cap M$ are closed in $\mathcal{C}_\sigma(K)$, $S_\omega(X)$ is separable and the weak topologies of $S_\omega(x)$ and X_{M_ω} coincide in M_ω, we may find a biorthogonal system $(f_i, u_i)_{i \in I_\omega}$ in X and a subset I^1_ω of I_ω such that

$$\big[\{f_i \,:\, i \in I_\omega\}\big] = S_\omega(X),$$

lin $\{f_i : i \in I^1_\omega\}$ is a dense subset of X_{M_ω}, and $\{u_i : i \in I_\omega\}$ is a weak-star dense subset of $S^*_\omega(X^*)$. Clearly, for each x in $K(\Gamma)$,

$$\{i \in I_\omega \,:\, f_i(x) \neq 0\}$$

is countable. For a given ordinal α, $\omega \leq \alpha < \mu$, we have that

$$\big(S_{\alpha+1} - S_\alpha\big)(X) \qquad \text{and} \qquad M_{\alpha+1} := M \cap \big(S_{\alpha+1} - S_\alpha\big)(X)$$

are closed in $C_\sigma(K)$, the weak topologies of $(S_{\alpha+1} - S_\alpha)(X)$ and $(S_{\alpha+1} - S_\alpha)(X_{M_{\alpha+1}})$ coincide in $M_{\alpha+1}$ and dens $(S_{\alpha+1} - S_\alpha)(X) < \lambda$. Therefore, there is a Markushevich basis $(f_i, \omega_i)_{i \in I_{\alpha+1}}$ in $\big(S_{\alpha+1} - S_\alpha\big)(X)$ and a subset $I^1_{\alpha+1}$ of $I_{\alpha+1}$ such that, for each x in $K(\Gamma)$,

$$\{i \in I_{\alpha+1} \,:\, f_i(x) \neq 0\}$$

is countable and lin $\{f_i : I^1_{\alpha+1}\}$ is a dense subset of $X_{M_{\alpha+1}}$. We take a subset $\{u_i : i \in I_{\alpha+1}\}$ of $\big(S^*_{\alpha+1} - S^*_\alpha\big)(X^*)$ whose linear hull is weak-star dense in that space and such that $(f_i, u_i)_{i \in I_{\alpha+1}}$ is a biorthogonal system in X.

If we set

$$I_\omega, I_{\alpha+1}, \qquad \omega \le \alpha < \nu,$$

pairwise disjoint, and define

$$I_1 := I^1_\omega \cup \left(\bigcup\{I^1_{\alpha+1} : \omega \le \alpha < \nu\}\right), \qquad I := I_\omega \cup \left(\bigcup\{I_{\alpha+1} : \omega \le \alpha < \nu\}\right)$$

we have that $(f_i, u_i)_{i\in I}$ is a Markushevich basis in X such that, for each x in $K(\Gamma)$,

$$\{i \in I : f_i(x) \neq 0\}$$

is countable and, proceeding as in the proof of Corollary 2.2, it happens that $(f_i, u_i)_{i\in I}$ and $(f_i, u_i|_{L(M)})_{i\in I_1}$ are countable 1-norming Markushevich bases in X and $L(M)$, respectively. It is then quite easy to see that $(f_i, u_i|_{L(M)})_{i\in I_1}$ is a countably 1-norming Markushevich basis in X_M.

Suppose now that $\rho < \nu$. Lemma 2 applies to obtain two subspaces Y and Z of X, closed in $C_\sigma(X)$, $M \subset Y$, $\text{dens}Y = |\rho|$, such that Z is a topological complement of Y in X. We find a Markushevich basis $(f_j, v_j)_{j\in J}$ in Y and a subset I_1 of J such that $(f_i, v_i|_{L(M)})_{i\in I}$ is a countably 1-norming Markushevich basis in $Y_M = X_M$ and, for each x in $K(\Gamma)$, the set

$$\{j \in J : f_j(x) \neq 0\}$$

is countable. We find now in Z a Markushevich basisi $(f_h, v_h)_{h\in H}$, with H being disjoint with J, such that, for each x in $K(\Gamma)$

$$\{h \in H : f_h(x) \neq 0\}$$

is countable. Let $Y^\perp$ and $Z^\perp$ be the subspaces of X^* orthogonal to Y and Z, respectively. We may take

$$u_j \in Z^\perp, \quad j \in J, \qquad u_h \in Y^\perp, \quad h \in H,$$

so that $(f_j, u_j)_{j\in J}$ and $(f_h, u_h)_{h\in H}$, are biorthogonal systems in X. Writing $I := J \cup H$, then $(f_i, u_i)_{i\in I}$ is the desired Markushevich basis in X stated in the theorem. □

Corollary 3.1

Let X be a Banach space such that $(B_{X^*})_\sigma$ is a Corson compact. Let M be a closed bounded absolutely convex subset of X such that $L(M)$ has infinite dimension. If the weak topologies of X and X_M coincide in M, then the following properties hold:

(a) There is a Markushevich basis $(x_i, u_i)_{i\in I}$ in X and a subset I_1 of I such that $(x_i, u_i|_{X_M})_{i\in I_1}$ is a Markushevich basis in X_M.

(b) The closed unit ball of $(X_M)^*$, with the weak-star topology, is a Corson compact.

Proof. We write K to mean $(B_X{}^*)_\sigma$. We may assume that K is a compact subset of $\mathbb{R}^\Gamma$, for a convenient Γ, such that $K(\Gamma) = K$. Then M and X are closed in $C_\sigma(K)$ and property (a) obtains directly from Theorem 3. Since $L(M)^*$ is a dense subspace of $(X_M)^*$ and, for each v in $L(M)^*$, the set

$$\{i \in I_1 : \langle x_i, v\rangle \neq 0\}$$

is countable, we get that, for each u in $(X_M)^*$, the set

$$\{i \in I_1 : \langle x_i, u\rangle \neq 0\}$$

is countable, and property (b) is thus satisfied. □

The next corollaries are simple consequences of Theorem 3.

Corollary 3.2

If X is a Banach space such that the closed unit ball of X^* is a Corson compact for the weak-star topology, then every closed subspace of X admits a quasicomplement in X.

Corollary 3.3

If X is a Banach space such that $(B_{X^*})_\sigma$ is a Corson compact, then X admits a quasicomplement in $C((B_{X^*})_\sigma)$.

If X is a reflexive Banach space, then it admits a resolution of the identity [5]. This property is extend in [1] for the case of X being weakly compactly generated, and in [16] when X is a weakly countably determined Banach space.

The former results, changing the term Banach for Frèchet, are shown in [12] and [13] by a rather simple method.

Resolutions of the identity may be of interest to show that certain Banach subspaces admit Markushevich bases [5].

Note 2. Let K be a compact of the class $\mathcal{A}$. It is shown in [14] that there is in $\mathcal{C}(K)$ a resolution of the identity formed by extension operators. In [3] a resolution of the identity in $\mathcal{C}(K)$ is constructed in such a way that permit to prove that if H is the continuous image of a compact of the class $\mathcal{A}$ then H has Namioka's property, i.e., for each Baire topological space E and each mapping $g: E \times H \to \mathbb{R}$ separatedly continuous there is a residual subset Ω of E such that g is continuous in every point of $\Omega \times K$.

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