Collect. Math. **42**, 3 (1991), 253–263

Σ-completeness and closed graph theorems

Santiago Díaz

Departamento de Matemática Aplicada, Universidad de Sevilla

E.S. Ingenieros Industriales, Avda. Reina Mercedes s/n, 41012-Sevilla (Spain)

Received July 9, 1991. Revised February 10, 1992

Abstract

A certain completeness concept, Σ–completeness, allows us to give some closed graph theorems between spaces whose dual space is weak* Σ–complete and some classes of Banach spaces in the range space, namely: reflexive Banach spaces, Banach spaces without copy of ℓ_∞ and dual spaces of Banach spaces with unconditional basis and without copy of ℓ_1.

1. Introduction

In his study of the Orlicz–Pettis topologies, P. Dierolf [2] introduced the following concept: a locally convex separated space (shortly space) E is said to be Σ–complete if every unconditionally Cauchy series in E is convergent.

Clearly, every sequentially complete space is a Σ–complete space. Non-trivial examples of Σ–complete spaces are given by $[E, \sigma(E, E')]$, where E is a Banach space not containing a copy of c_0, according to a well–known result of Bessaga and Pełczynski (see [3, p. 45]).

In the context of duality closed graph theorems, we have obtained results for mappings between spaces E such that $[E', \sigma(E', E)]$ is Σ–complete (in the sequel, we refer to this spaces as dual Σ–complete spaces) and some classes of Banach spaces in the range space. We have arranged these results in two sections: *usual* and *mixed* closed graph theorems. For *usual*, we mean that the topologies on the spaces which appear in the theorem belong to the same dual pair. However, for the *mixed* theorems, the involved topologies on the spaces can belong to *different* dual pairs.

In what follows, we refer to Jarchow's monograph [4] for terminology and notation. A wide and deep account on closed graph theorems as well as related barrelledness conditions can be found in Pérez Carreras–Bonet [7].

2. Usual Closed Graph Theorems

With the purpose to establish optimal closed graph theorems for mappings having barrelled spaces in the domain, Γ_r–spaces were introduced (see [7, ch. 7]). We recall that a space E is said to be a Γ_r–space if given any quasicomplete subspace G of $[E^*, \sigma(E^*, E)]$ such that $G \cap E'$ is dense in $[E', \sigma, (E', E)]$, then G contains E'.

A similar approach for local completeness was made by Valdivia [10] who considered the Λ_r–spaces (insert "quasicomplete" for "local complete" in last definition). He also gave the corresponding closed graph theorem between c_0–barrelled spaces and Λ_r–spaces [10]. In this line, we introduce the following concept: a space E is said to be a Σ_r–space if every Σ–complete subspace of $[E^*, \sigma(E^*, E)]$ intersecting E' in a dense subspace of $[E', \sigma(E', E)]$, contains E'.

First of all, a maximal closed graph theorem between dual Σ–complete spaces and Σ_r–spaces is given.

Theorem 1.

F is a Σ_r–space if and only if for every dual Σ–complete space and for every linear mapping T from E to F with closed graph, T is weakly continuous.

Proof. Let us consider the subspaces $L = \{f \in F^* : T^*(f) \in E'\}$ of F^* and $H = L \cap F'$ of F'. Since T has closed graph, H is weak* dense in F' [4, p. 197]. Bearing in mind that F is a Σ_r–space, if we prove that L is $\sigma(F^*, F)$–Σ–complete, we will have $H = F'$ and this will imply that T is weakly continuous [4, p. 161].

So, let $\sum f_n$ be an unconditionally $\sigma(F^*, F)$–Cauchy series in L. $[F^*, \sigma(F^*, F)]$ is a complete space, so there exists $f \in F^*$ such that f is the $\sigma(F^*, F)$–sum of $\sum f_n$. Since T^* is $\sigma(F^*, F)$–$\sigma(E^*, E)$ continuous, we have that the series $\sum T^*(f_n)$ is unconditionally $\sigma(E^*, E)$–convergent to $T^*(f)$. Moreover, since $T^*(f_n)$ belongs to E' for all $n \in \mathbb{N}$ and E is dual Σ–complete, we have that $\sum T^*(f_n)$ is unconditionally $\sigma(E', E)$–convergent to a certain $x \in E'$. Finally, by the uniqueness of the limit, $x = T^*(f)$ or equivalently $f \in L$.

Reciprocally, if F is not a Σ_r–space, there exists a $\sigma(E^*, E)$–Σ–complete subspace G such that $G \cap F'$ is weak* dense and $G \cap F' \neq F'$.

$[G, \sigma(G, F)]$ is a Σ–complete space and the identity map from $[F, \sigma(F, G)]$ to F has closed graph but it is not weakly continuous, so the sought contradiction is obtained. □

We provide a wide class of Σ_r–spaces in next theorem.

Theorem 2.

Let E be a space such that $[E', \mu(E', E)]$ is metrizable. Then E with any compatible topology for the pair dual (E, E') is a Σ_r–space.

Proof. Let H be a Σ–complete subspace of $[E^*, \sigma(E^*, E)]$ intersecting E' in a weak* dense subspace of E'. Given $z \in E'$, z also belongs to the $\mu(E', E)$–closure of $H \cap E'$. By the hypothesis of metrizability, there exists a F–norm q which generates the topology $\mu(E', E)$, so we can obtain a sequence (y_n) of elements of $H \cap E'$ $\mu(E', E)$–convergent to z such that

$$q(y_{n+1} - y_n) \leq \frac{1}{n^2}, \quad \text{for all } n \in \mathbb{N}.$$

If we set $z_n = y_{n+1} - y_n$, for any $n \in \mathbb{N}$, we have that $\sum z_n$ is unconditionally $\mu(E', E)$–convergent to $z - y_1$.

On the other hand, $\sum z_n$ is also unconditionally $\sigma(E^*, E)$–convergent, and since H is $\sigma(E^*, E)$–Σ–complete, we get that $z - y_1 \in H$ and therefore $z \in H$. □

Remark . A similar proof can be given to show that every B_r–complete Schwartz space is a Σ_r–space (see [4, p. 214]).

Corollary 1.

Every linear mapping from a dual Σ–complete space to a reflexive Banach space, having closed graph, is weakly continuous.

Proof. If E is a reflexive Banach space, then $\beta(E', E) = \mu(E', E)$ is metrizable. Applying theorems 1 and 2 yields the result. □

It is also possible to give necessary conditions for being a Σ_r–space. With these results, we are trying to characterize Σ_r–spaces in the context of locally convex spaces.

Theorem 3.

Let E be a dual Σ–complete space. If E is a Σ_r–space, then either E is semireflexive or $[E', \beta(E', E)]$ contains a copy of c_0.

Proof. Let us suppose that E is a Σ_r–space and $[E',\beta(E',E)]$ contains no copy of c_0. If E is not semireflexive, then there exists a $\beta(E',E)$–closed $\sigma(E',E)$–dense hyperplane F in E'.

Since E is a Σ_r–space, if we prove that F is $\sigma(E',E)$–Σ–complete, we will obtain that $F = E'$ and we will get a contradiction.

So, let $\sum x_n$ be an unconditionally $\sigma(E',E)$–Cauchy series with $x_n \in F$, $n \in \mathbb{N}$. We have that E is dual Σ–complete, therefore $\sum x_n$ is $\sigma(E',E)$–convergent to $x \in E'$. By the Bourbaki–Robertson's lemma, if we show that $\sum x_n$ is unconditionally $\beta(E',E)$–Cauchy, then that series will be $\beta(E',E)$–convergent to x. But F is $\beta(E',E)$–closed, so $x \in F$.

Let us suppose that $\sum x_n$ is not unconditionally $\beta(E',E)$–Cauchy. By induction, we can obtain a $\beta(E',E)$–neighbourhood U of 0 and a sequence of finite subsets α_n of positive integers with $\sup \alpha_n < \inf \alpha_{n+1}$ which verify that

$$z_n = \sum_{i\in\alpha_n} x_i \notin U, \text{ for all } n \in \mathbb{N}.$$

For every $(a_n) \in c_0$, the series $\sum a_n z_n$ is unconditionally $\sigma(E',E)$–Cauchy and, therefore $\sigma(E',E)$–convergent. This allows to define the following linear continuous mapping

$$T : [c_0, \sigma(c_0, \ell_1)] \to [E', \sigma(E',E)] \quad (a_n) \mapsto T(a_n) = \sum_{n=1}^{\infty} a_n z_n$$

T is also $\beta(c_0,\ell_1)$–$\beta(E',E)$–continuous. By [5], the sequence $(T(e_n) = z_n)_n$ must tend to zero for the $\beta(E',E)$ topology, and we get a contradiction. □

Our last theorem originates from a natural question: Is the Banach space ℓ_1 a Σ_r–space?. Unfortunately, the answer is negative as we show in the next example. However, we will see that it is possible to give closed graph theorems between dual Σ–complete spaces and ℓ_1 in *mixed* situation.

EXAMPLE : Let $(p_k)_k$ be the increasing sequence of primes, i.e., (2,3,5,...). And let us consider the following sequence $(x_k)_{k\geq 1}$ in ℓ_∞:

$$x_k(j) = \begin{cases} 1 & j = k - 2^n \\ 0 & 0 \leq j < 2^n, j \neq k - 2^n \\ 1 & j \geq 2^n,\ j \text{ is multiple of } p_k \\ -1 & j \geq 2^n,\ j \text{ is not multiple of } p_k \end{cases} \qquad 2^n \leq k < 2^{n+1},\ n = 0,1,2,\ldots$$

First of all, we are going to show that (x_k) is a basic sequence, in fact, it is a unit vector basis of ℓ_1. If $c_1, ...c_p$ are real scalars, then

$$\left\| \sum_{k=1}^{p} c_k x_k \right\| \leq \sum_{k=1}^{p} |c_k| \|x_k\| = \sum_{k=1}^{p} |c_k|.$$

Moreover, we can obtain a sufficiently large natural number j_0 (we mean that $x_k(j) = \pm 1, j \geq j_0, 1 \leq k \leq p$) whose prime factorization uses only primes from $\{p_k : 1 \leq k \leq p, c_k > 0\}$. Therefore,

$$\left\| \sum_{k=1}^{p} c_k x_k \right\| \geq \left| \sum_{k=1}^{p} c_k x_k(j_0) \right| = \sum_{k=1}^{p} |c_k|.$$

If we denote by F the closed linear span of (x_k) in ℓ_∞, then F is a closed subspace of ℓ_∞ isomorphic to ℓ_1. So, F has no copy of c_0 and according to Bessaga–Pełczyinski's theorem F is $\sigma(\ell_\infty, (\ell_\infty)')$–$\Sigma$–complete. Bearing in mind that, in any Banach space E, a series is unconditionally $\sigma(E', E)$–Cauchy if and only if it is unconditionally $\sigma(E', E'')$–Cauchy, we can conclude that F is also $\sigma(\ell_\infty, \ell_1)$–$\Sigma$–complete.

In order to prove that ℓ_1 is not a Σ_r–space, it is enough to show that F is $\sigma(\ell_\infty, \ell_1)$–dense.

If $(a_i) \in \ell_1 \setminus \{0\}$, then there exists (without loss of generality) $n_0 \in \mathbb{N}$ such that $a_{n_0} > 0$. We can also obtain $m_0 \in \mathbb{N}$ such that

$$\sum_{i=2^{m_0}}^{\infty} |a_i| < a_{n_0}, \quad \text{and } 2^{m_0} > n_0.$$

Let us denote $s = 2^{m_0} + n_0$. Since $2^{m_0} + n_0 < 2^{m_0} + 2^{m_0} = 2^{m_0+1}$, the definition of the sequence (x_k) tells us that $x_s(n_0) = 1$, $x_s(j) = \pm 1, j \geq 2^{m_0}$ and the other coordinates are equal to zero. Therefore $(\alpha_k = \pm 1)$,

$$\langle x_s, a \rangle = a_{n_0} + \sum_{k=2^{m_0}}^{\infty} \alpha_k a_k \geq a_{n_0} - \sum_{k=2^{m_0}}^{\infty} |a_k| > 0.$$

If we restrict to dual Σ–complete sequence spaces, we can establish other closed graph theorems. As we see, these results show that Σ–completeness is closer to sequential completeness for sequence spaces. We begin by proving a lemma which can be of independent interest.

Theorem 4.

Let λ be a perfect sequence space and $[E,\tau]$ a B_r–complete space with no copy of ℓ_∞. Then every linear closed graph map from $[\lambda,\mu(\lambda,\lambda^\alpha)]$ to E is continuous.

Proof. Let T a mapping under the hypothesis of the theorem. By Bennett–Kalton [1], we know that T maps subseries summable sequences of λ in subseries summable sequences of E.

Let $OP(\tau)$ be the finest locally convex topology which has the same subseries summable sequences as the topology τ (see [9]). According to Dierolf [2], last paragraph can be rewritten in this way: T is $OP(\mu(\lambda,\lambda^\alpha))$–$\tau$ continuous.

In Tweddle [9] can be seen that $OP(\sigma(E,E')) = OP(\mu(E,E')) = \mu(E,G_E)$, where G_E is the subspace of E^* such that $y \in G_E$ if

$$\left\langle \sigma(E,E')-\sum_{n=1}^{\infty} x_n, y\right\rangle = \sum_{n=1}^{\infty}\langle x_n, y\rangle$$

for all series $\sum x_n$ weakly subseries convergent in E.

So if we prove that $OP(\mu(\lambda,\lambda^\alpha)) = \mu(\lambda,\lambda^\alpha)$, the proof will be finished. This last happens, since (λ,λ^α), (λ,G_λ) are isomorphic dual pairs.

In fact, given $x \in \lambda$ and $g \in G_\lambda$, we have

$$\langle x,g\rangle_{(\lambda,G_\lambda)} = \left\langle \sum_{n=1}^{\infty} x_n e_n, g\right\rangle = \sum_{n=1}^{\infty}\langle x_n e_n, g\rangle = \langle x, (\langle e_n, g\rangle)_n\rangle_{(\lambda,\lambda^\alpha)}. \square$$

Theorem 5.

Let λ be a sequence space containing ϕ. Then the following are equivalent:

(1) Every linear map from $[\lambda^\alpha,\mu(\lambda^\alpha,\lambda)]$ to E, having closed graph, is continuous, where E is a separable B_r–complete space.

(2) Every linear map from $[\lambda^\alpha,\mu(\lambda^\alpha,\lambda)]$ to E, having closed graph, is continuous, where E is a weakly compactly generated B_r–complete space.

(3) Every linear map from $[\lambda^\alpha,\mu(\lambda^\alpha,\lambda)]$ to E, having closed graph, is continuous, where E is a B_r–complete space not containing a copy of ℓ_∞.

(4) Every linear map from $[\lambda^\alpha,\mu(\lambda^\alpha,\lambda)]$ to ℓ_1, having closed graph, is continuous.

(5) Every linear map from $[\lambda^\alpha,\mu(\lambda^\alpha,\lambda)]$ to c_0, having closed graph, is continuous.

(6) λ is a perfect sequence space.

(7) $[\lambda,\sigma(\lambda,\lambda^\alpha)]$ is Σ–complete.

Proof. (5) $\Leftrightarrow$ (6). Since λ is a perfect sequence space if and only if $[\lambda, \sigma(\lambda, \lambda^\alpha)]$ is sequentially complete, and Kalton [6].
(6) $\Rightarrow$ (7). Since sequential completeness implies Σ–completeness.
(7) $\Rightarrow$ (6). Let (x_n) be a $\sigma(\lambda, \lambda^\alpha)$–Cauchy sequence. Since $[\lambda^{\alpha\alpha}, \sigma(\lambda^{\alpha\alpha}, \lambda^\alpha)]$ is sequentially complete, (x_n) is weak* convergent to $z = (z_n) \in \lambda^{\alpha\alpha}$.

On the other hand, the series $\sum z_n e_n$ is unconditionally $\sigma(\lambda^{\alpha\alpha}, \lambda^\alpha)$–convergent to z. Since $z_n e_n \in \phi \subset \lambda$ and (7), we deduce that $z \in \lambda$.
(4) $\Rightarrow$ (7). Let $\sum x_n$ be a weakly unconditionally Cauchy series in λ. This means

$$\sum_{n=1}^{\infty} |\langle y, x_n \rangle| < +\infty, \quad \text{for all } \; y \in \lambda^\alpha$$

So we can define the following linear map

$$T : \lambda^\alpha \to \ell_1, \quad y \mapsto T(y) = (\langle y, x_n \rangle)_n$$

Moreover, T has $\mu(\lambda^\alpha, \lambda)$–$\mu(\ell_1, \ell_\infty)$–closed graph. Therefore, by (4) T is continuous, so the adjoint $T^* : \ell_\infty \to \lambda$ is weakly* continuous and the series $\sum x_n$ is bounded multiplier convergent in $[\lambda, \sigma(\lambda, \lambda^\alpha)]$.
(6) $\Rightarrow$ (3). By theorem 4.
(3) $\Rightarrow$ (2). Since the copies of ℓ_∞ are always complemented [4, p. 133], and ℓ_∞ is not weakly compactly generated.
(3) $\Rightarrow$ (1). By Kalton [5].
(1) $\Rightarrow$ (4), (2) $\Rightarrow$ (4). Since ℓ_1 is separable and therefore weakly compactly generated. $\square$

Remarks . (1) We notice that in theorem 5, since c_0 is not perfect we can not consider sequence spaces in the way $[\lambda, \mu(\lambda, \lambda^\alpha)]$.
(2) The third sentence improves Kalton's results on closed graph theorems for dual sequentially complete spaces (see [6]).

3. Mixed Closed Graph Theorems

For our next closed graph theorem, we use a mixed dual pair condition on the spaces E' which appear in the range space. Namely, the topology used for the graph is related to the dual pair (E', E) and the topology used for the continuity is related to the dual pair (E', E''). These kind of conditions were already employed by Poppola–Tweddle to give closed graph theorems for ℓ_∞–barrelled spaces (see [7, ch. 8]).

To formulate the theorem, we need the so called property (u) for Banach spaces: a Banach space has this property if for each weakly Cauchy sequence (z_n) in E, there exists a weakly unconditionally Cuachy series $\sum x_n$ in E such that

$$z_n - \sum_{k=1}^{n} x_k \longrightarrow 0 \quad (\sigma(E, E')).$$

Theorem 6.

Let E be a space. The following conditions are equivalent:

(1) E is dual Σ–complete.

(2) Every linear mapping from E to $[\ell_1, \mu(\ell_1, c_o)]$, having closed graph, is $\sigma(E, E')$–$\sigma(\ell_1, \ell_\infty)$–continuous.

(3) Every linear mapping from E to $[\ell_1, \mu(\ell_1, H)]$, having closed graph, is $\sigma(E, E')$–$\sigma(\ell_1, \ell_\infty)$–continuous, for each dense subspace H of c_0.

(4) Every linear mapping from E to $[F', \mu(F', F)]$, having closed graph, is weakly continuous, where F is a separable Banach space which has the property (u) of Pełczinsky and does not contain a copy of ℓ_1.

(5) Every linear mapping from E to $[F', \mu(F', F)]$, having closed graph, is weakly continuous, where F is a Banach space with unconditional basis and which does not contain a copy of ℓ_1.

Proof. (1) $\Rightarrow$ (2). Let T be a map satisfying the corresponding hypothesis. Since $[c_0, \mu(c_0, \ell_1)]$ is metrizable, we obtain from theorems 1 and 2 that T is $\sigma(E, E')$–$\sigma(\ell_1, c_0)$–continuous.

In particular, this implies that $T^*(e_n) \in E'$ for all $n \in \mathbb{N}$, where e_n is the sequence which vanishes except in the n–th position. Since, for every $x \in E$, we can write $T(x) = (\langle T(x), e_n \rangle)_n$, the series $\sum T^*(e_n)$ is unconditionally $\sigma(E', E)$–Cauchy. In fact, $\sum \alpha_n T^*(e_n)$ is weakly* unconditionally Cauchy for every $(\alpha_n) \in \ell_\infty$.

Finally, since E is dual Σ–complete the series $\sum T^*(e_n)$ is weakly* bounded multiplier convergent in E', that is to say, $T^*(\ell_\infty) \subset E'$ and T is $\sigma(E, E')$–$\sigma(\ell_1, \ell_\infty)$–continuous.

(2) $\Rightarrow$ (1). Let $\sum x_n$ be a weakly* unconditionally Cauchy series in E'. This means

$$\sum_{n=1}^{\infty} |\langle y, x_n \rangle| < +\infty, \qquad \text{for all } \ y \in E$$

So we can define the following linear map

$$T : E \to \ell_1, \quad y \mapsto T(y) = (\langle y, x_n \rangle)_n$$

Moreover, T has $\mu(\ell_1, c_0)$–closed graph, since $x_n \in E'$ and the sequence e_n belongs to c_0.

Therefore, by (2) T is weakly continuous, so the adjoint $T^* : [\ell_\infty, \sigma(\ell_\infty, \ell_1)] \to [E', \sigma(E', E)]$ is continuous and the series $\sum x_n$ is weakly* bounded multiplier convergent.

(1) $\Rightarrow$ (3). Let T be a map satisfying the corresponding hypothesis and H an arbitrary dense subspace of c_0. Since $[H, \mu(H, \ell_1)]$ is metrizable, from theorems 1 and 2 we obtain that T is $\sigma(E, E')$–$\sigma(\ell_1, H)$–continuous.

Moreover, for all $n \in \mathbb{N}$ we can express e_n as the sum of an unconditionally $\sigma(c_0, \ell_1)$–Cauchy series $\sum_k h_k^n$ of elements of H.

E is dual Σ–complete, so $\sum_k T^*(h_k^n)$ is unconditionally $\sigma(E', E)$–convergent to z_n. On the other hand, these series are unconditionally $\sigma(E^*, E)$–convergent to $T^*(e_n)$. By the uniqueness of the limit, we deduce that $T^*(e_n) \in E'$. After that, we continue as in implication (1) $\Rightarrow$ (2).

(3) $\Rightarrow$ (2). It is a particular case.

(1) $\Rightarrow$ (4). Let T be a map satisfying the corresponding hypothesis of (4). Since $[F', \mu(F', F)]$ is a Σ_r–space, we obtain that T is $\sigma(E, E')$–$\sigma(F', F)$–continuous.

Let us consider the former mapping in the following way

$$T : [E, \sigma(E, E')] \to [F', \sigma(F', F'')]$$

By the continuity of T which was proved before, we can deduce that T has $\sigma(E, E')$–$\sigma(F', F'')$–closed graph. So, by [4, p. 197] the subspace of F''

$$L = \{ f \in F'' : T^*(f) \in E' \}$$

is weak* dense in F''. The continuity also shows that $F \subset L \subset F''$, where we have identified F with the corresponding normed subspace of F''.

Given $z \in F''$, by Rosenthal's theorem [3, p. 215] there is a sequence (x_n) in F which is $\sigma(F'', F')$–convergent to z. Moreover, F has property (u), so there exists an unconditionally $\sigma(F, F')$–Cauchy series $\sum y_n$ in E such that

$$x_n - \sum_{k=1}^{n} y_k \xrightarrow{n} 0 \quad (\sigma(F, F'))$$

Since (x_n) is convergent, we really have that the series $\sum y_n$ is unconditionally $\sigma(F'', F')$-convergent to z.

If we prove that for every unconditionally $\sigma(F'', F')$-convergent series with elements in L, the sum of these series also belong to L, we will get $F'' \subset L$ and T will be $\sigma(E, E')$-$\sigma(F', F'')$-continuous and the theorem will be proved.

So, let (f_n) a sequence in L and let us suppose that the series $\sum f_n$ is unconditionally $\sigma(F'', F')$-convergent to $f \in F''$.

Since T^* from F'' to E^* is weakly* continuous, we have that the series $\sum T^*(f_n)$ is unconditionally $\sigma(E^*, E)$-convergent to $T^*(f)$.

But, for all $n \in \mathbb{N}$ $T^*(f_n)$ belongs to E', and E is dual Σ-complete. So $\sum T^*(f_n)$ is unconditionally $\sigma(E', E)$-convergent to a certain $x \in E'$. By the uniqueness of the limit, $x = T^*(f)$ or equivalently $f \in L$.

(4) $\Rightarrow$ (5). Since every Banach space with unconditional basis has property (u) [8, pp. 442–446].

(5) $\Rightarrow$ (2). Since c_0 is a separable Banach space which has unconditional basis, property (u) [8, pp. 442–446], no copy of ℓ_1 [4, p. 310] and a strong dual separable. □

Remark . The classes of Banach spaces which appear in (4) and (5) have been considered in the context of when a separable Banach space has a separable dual (see [3, p. 214]).

References

1. G. Bennet, N. J. Kalton, Addendum to "FK–spaces containing c_0", *Duke Math. J.* **39** (1972), 819–821.
2. P. Dierolf, Summable sequences and associated Orlicz–Pettis topologies, *Comment. Math.* T. S. in honorem W. Orlicz, **2** (1977), 71–88.
3. J. Diestel, *Sequences and Series in Banach Spaces*, Springer–Verlag, New York, Berlin, Heidelberg , 1984.
4. H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart, 1981.
5. N. J. Kalton, Exhaustive operators and vector measures, *Proc. Edinburgh Math. Soc.* **19** (1975), 291–300.
6. N. J. Kalton, Some forms of the closed graph theorem, *Math. Proc. Cambridge Philos. Soc.*, **70** (1971), 401–408.

7. P. Perez Carreras, J. Bonet, *Barrelled Locally Convex Spaces*, North Holland Math. Studies 131, Amsterdam, 1987.
8. I. Singer, *Bases in Banach Spaces I*, Springer–Verlag, Berlin, Heidelberg, 1970.
9. I. Tweddle, Unconditional convergence and vector valued measures, *J. London Math. Soc.* **2** (1970), 603–610.
10. M. Valdivia, Mackey convergence and the closed graph theorem, *Arch. Math.* **26** (1974), 649–656.