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# The Bade property and the $\lambda$-property in spaces of convergent sequences

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Abstract

In this note we study the Bade property in the $\mathcal{C}(K,X)$ and $c(X)$ spaces. We also characterize the spaces $X=\mathcal{C}(K,\mathbb{R})$ such that $c(X)$ has the uniform $\lambda$-property.

## 1. Introduction

Given a normed space $X$, $B_X$ denotes its closed unit ball, $S_X$ the closed unit sphere of $X$ and $\text{Ext}B_X$ the set of extreme points of $B_X$. $X$ is said to have the Bade property if $\overline{\text{Co}(\text{Ext } B_X)} = B_X$.

The following questions were developed by R.M. Aron and R.H. Lohman [2]: If $x \in B_X$, a triple $(e,y,\lambda)$ is said to be amenable to $x$ if $e \in \text{Ext } B_X$, $y \in B_X$, $0<\lambda\le 1$ and $x=\lambda\, e+(1-\lambda)\, y$. In this case, we define

$$\lambda(x)=\sup\{\lambda : (e,\lambda,y) \text{ is amenable to } x\},$$

and

$$\frac{1-\|x\|}{2}\le \lambda(x)\le \frac{1+\|x\|}{2}$$

is verified. $X$ is said to have the $\lambda$-property if each $x\in B_X$ admits an amenable triple. If, in addition,

$$\lambda(X)=\inf\{\lambda(x) : x\in B_X\}>0,$$

then $X$ is said to have the uniform $\lambda$-property.

In [3], it is shown that a Banach space $X$ has the $\lambda$-property if and only if $X$ has the convex series representation property, i.e. every point $x$ in $B_X$ can be written as an infinite series:

$$x = \sum_{n=1}^{\infty} \lambda_n \, e_n$$

where the points $e_n \in \mathrm{Ext} B_X$ and the scalars $\lambda_n$ satisfy

$$\lambda_n \geq 0 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \lambda_n = 1.$$

Let $K$ be a compact Hausdorff space and let $X$ be a normed space. $\mathcal{C}(K,X)$ denotes the Banach space of all $X$-valued continuous functions $f$ on $K$, with the uniform norm. As usual, $\mathcal{C}(K)$ denotes the $\mathcal{C}(K,X)$ space when $X = \mathbb{R}$. Bade's theorem states that $\mathcal{C}(K)$ has the Bade property if and only if $K$ is 0-dimensional (see [7] and [8]).

In [2], it is shown that if $X$ has the $\lambda$-property then $X$ has the Bade property, but it's also shown that the converse assertion is false by means of $\mathcal{C}(K,\mathbb{C})$ where $K$ is the unit ball of $\mathbb{C}$.

In [4] and [5] it's shown that if $K$ is a compact Hausdorff space, then $\mathcal{C}(K)$ has the $\lambda$-property if and only if $K$ is 0-dimensional and, in this particular case, $\mathcal{C}(K)$ has the uniform $\lambda$-property and $\lambda(\mathcal{C}(K)) = 1/2$. These results were also obtained independently by A. Suarez Granero.

Given a normed space $X$, the space of convergent sequences is denoted by $c(X)$, endowed with the supreme norm. In [2] it's shown that $c(X)$ has the uniform $\lambda$-property when $X$ is a strictly convex normed space. In [1] it's shown that if $K$ is a 0-dimensional Hausdorff compact space and $X$ is a strictly convex Banach space, then $\mathcal{C}(K,X)$ has the uniform $\lambda$-property and, as a particular case, when $K = \gamma\,\omega$ — Alexandroff's compactification of the discrete space $\omega$ — we get that $c(X)$ also has the uniform $\lambda$-property.

## 2. The Bade property in $\mathcal{C}(K,X)$ and $c(X)$ spaces

Let $X$ be a normed space. It is easy to prove that $X$ has the Bade property if and only if

$$\sup_{x \in B_X} f(x) = \sup_{x \in \mathrm{Ext}\, B_X} f(x)$$

for every $f\colon X \to \mathbb{R}$ continuous linear form.

**Lemma 2.1**

*Let $X$ be a normed space and let $n \in \mathbb{N}$, $n > 0$. Let's consider the space $X^n$, with the norm*

$$\|(x_1, \ldots, x_n)\| = \max_{1 \leq i \leq n} \|x_i\|.$$

*Then:*

*a)* $(x_1, \ldots, x_n) \in \text{Ext } B_{X^n}$ *if and only if* $x_i \in \text{Ext } B_X$ *for every* $i \in \{1, 2, \ldots, n\}$.

*b)* $X^n$ *has the Bade property if and only if* $X$ *has the Bade property.*

*Proof.* We just want to show that whenever $X$ has the Bade property, then $X^n$ also has it.

Let $f: X \to \mathbb{R}$ be a continuous linear form, and let $\varepsilon > 0$. Every $(x_1, \ldots, x_n) \in X^n$ verifies

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_i),$$

where, for every $i \in \{1, \ldots, n\}$, $f_i: X \to \mathbb{R}$ is defined by

$$f_i(x) = f(0, \ldots, \underset{i}{x}, \ldots, 0).$$

For every $i \in \{1, \ldots, n\}$ there exists an $e_i \in \text{Ext } B_X$ such that

$$f_i(e_i) + \frac{\varepsilon}{n} > \sup_{x \in B_X} f_i(x).$$

Hence we have that $(e_1, \ldots, e_n) \in \text{Ext } B_{X^n}$ and

$$f(e_1, \ldots, e_n) + \varepsilon > \sup_{(x_1, \ldots, x_n) \in B_{X^n}} f(x_1, \ldots, x_n). \square$$

**Proposition 2.2**

*Let $K$ be a compact Hausdorff space and let $X$ be a normed space.*

*a) If $C(K)$ and $X$ have the Bade property, then $C(K, X)$ has the Bade property.*

*b) If $K$ is non-perfect and $C(K, X)$ has the Bade property, then $X$ has the Bade property.*

*Proof.* a) If $K$ is 0-dimensional, it can be easily proved that the subspace of the finite-valued functions is dense in $\mathcal{C}(K,X)$. Let $f \in \mathcal{C}(K,X)$ and $\varepsilon > 0$. Since $K$ is 0-dimensional, there exist $\{x_1,\dots,x_n\} \subset B_X$ and a partition $\{A_1,\dots,A_n\}$ of $K$, where the $A_i$ are disjoint clopen subsets such that

$$\left\| f - \sum_{i=1}^{n} x_i \, \chi_{A_i} \right\| < \frac{\varepsilon}{2}.$$

By Lemma 2.1, $X^n$ has the Bade property. Hence there exist

$$\{\beta_1,\dots,\beta_m\} \subset [0,1] \qquad \text{and} \qquad (y_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}} \subset \text{Ext } B_X$$

such that

$$\sum_{i=1}^{m} \beta_i = 1 \qquad \text{and} \qquad \left\| (x_1,\dots,x_n) - \sum_{i=1}^{m} \beta_i \, (y_{i1},\dots,y_{in}) \right\| < \frac{\varepsilon}{2}.$$

For every $i \in \{1,\dots,m\}$ we define

$$g_i = \sum_{j=1}^{n} y_{ij} \, \chi_{A_j} \qquad \text{and} \qquad g = \sum_{i=1}^{m} \beta_i \, g_i.$$

It's clear that $g \in \text{Co}(\text{Ext } B_{\mathcal{C}(K,X)})$ and that $\|f - g\| < \varepsilon$.

b) Let $x \in B_X$ and $\varepsilon > 0$. We define $f\colon K \to X$ by $f(t) = x$ for every $t \in K$. Since $\mathcal{C}(K,X)$ has the Bade property there exist

$$\{\alpha_1,\dots,\alpha_n\} \subset [0,1] \qquad \text{and} \qquad \{e_1,\dots,e_n\} \subset \text{Ext } B_{\mathcal{C}(K,X)}$$

such that

$$\sum_{i=1}^{n} \alpha_i = 1 \qquad \text{and} \qquad \left\| f - \sum_{i=1}^{n} \alpha_i \, e_i \right\| < \varepsilon.$$

Let $t_0 \in K$ be such that $\{t_0\}$ is a clopen subset of $K$. Every $e \in \text{Ext } B_{\mathcal{C}(K,X)}$ verifies that $e(t_0) \in \text{Ext } B_X$. Therefore

$$\sum_{i=1}^{n} \alpha_i \, e_i(t_0) \in \text{Co}(\text{Ext } B_X) \qquad \text{and} \qquad \left\| x - \sum_{i=1}^{n} \alpha_i \, e_i(t_0) \right\| < \varepsilon. \ \square$$

*Remark 2.3.* Aron and Lohman ([2], Th. 1.6) proved that if $K$ is a compact metric space and $X$ is a strictly convex normed space then $\mathcal{C}(K,X)$ has the uniform $\lambda$-property (and, hence, the Bade property). As a consequence, it may happen that $\mathcal{C}(K,X)$ has the Bade property but $\mathcal{C}(K)$ does not have it (this occurs, for instance, when $K = [0,1]$). Proposition 2.2 a) gives us a sufficient condition for $\mathcal{C}(K,X)$ to have the Bade property, if $\mathcal{C}(K)$ and $X$ have it (in this case $K$ is 0-dimensional). We don't know if there exist spaces $\mathcal{C}(K,X)$ with the Bade property such that neither $\mathcal{C}(K)$ nor $X$ have that property. Proposition 2.2 b) tells us that this cannot occur if $K$ is non-perfect.

As a consequence of Proposition 2.2 we obtain:

**Corollary 2.4**

*Let $X$ be a normed space. Then $X$ has the Bade property if and only if $c(X)$ has the Bade property.*

## 3. The $\lambda$-property in $c(X)$ when $X = \mathcal{C}(K)$

If $c(X)$ has the $\lambda$-property (resp. the uniform $\lambda$-property), then $X$ has the $\lambda$-property (resp. the uniform $\lambda$-property). J.C. Navarro [6] has obtained a Banach space $X$, in fact a 3-dimensional space, with the uniform $\lambda$-property such that the corresponding $c(X)$ space has not the $\lambda$-property.

Nevertheless, as a consequence of 2.4, $c(X)$ has the Bade property. This raises the question about geometric conditions, additional to the $\lambda$-property, on $X$ that are necessary for $c(X)$ to have the $\lambda$-property (or, the uniform $\lambda$-property).

**Proposition 3.1**

*Let $X$ be a normed space with the $\lambda$-property. If*

$$x = (x_n)_{n\in\mathbb{N}} \in B_{c(X)}, \quad x_\infty = \lim_{n\to\infty}(x_n) \quad \textit{and} \quad \|x_\infty\| < 1,$$

*then $x$ has an amenable triple.*

*Proof.* Let $\alpha \in \mathbb{R}$ be such that $\|x_\infty\| < \alpha < 1$, there exists a $n_0 \in \mathbb{N}$ such that $\|x_n\| < \alpha$ for every $n \geq n_0$. Let

$$\lambda < \min\Big\{\frac{1-\alpha}{2}, \lambda(x_1), \dots, \lambda(x_{n_0})\Big\}.$$

For every $n \in \mathbb{N}$, $\lambda < \lambda(x_n)$ and also $\lambda < \lambda(x_\infty)$. Hence, there exists an amenable triple $(e, y, \lambda)$ for $x_\infty$. Since

$$\lim_{n\to\infty} \|x_n - \lambda\, e\| = \|x_\infty - \lambda\, e\| \leq \|x_\infty\| + \lambda < 1,$$

there exists a $n_1 \in \mathbb{N}$ such that $\|x_n - \lambda e\| < 1 - \lambda$ for every $n \geq n_1$.

For $n \leq n_1$, let $(e_n, y_n, \lambda)$ be an amenable triple for $x_n$. We consider the sequences $(e_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, where for $n > n_1$,

$$e_n = e \qquad \text{and} \qquad y_n = \frac{x_n - \lambda\, e}{1-\lambda},$$

then $\Big((e_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, \lambda\Big)$ is an amenable triple for $x$. □

*Remark 3.2.* Let's recall the fact that if $K$ is a Hausdorff compact space then $e \in \text{Ext } B_{\mathcal{C}(K)}$ if and only if $e = \chi_A - \chi_{A^c}$, where $A$ is a clopen subset of $K$.

**Proposition 3.3**

*Let $K$ be a 0-dimensional Hausdorff compact space and let $X = \mathcal{C}(K)$. Then $c(X)$ has the $\lambda$-property and $\lambda(c(X)) = 1/2$.*

*Proof.* Let

$$x = (x_n)_{n\in\mathbb{N}} \in B_{c(X)} \qquad \text{and} \qquad x_\infty = \lim_{n\to\infty} (x_n)_{n\in\mathbb{N}}.$$

For every $\lambda \in (0, 1/2)$ and $\alpha \in (\lambda, 1/2)$ there exists an amenable triple $(e, y_\infty, \alpha)$ for $x_\infty$. Let $A$ be a clopen subset of $K$ such that $e = \chi_a - \chi_{A^c}$. Since $\|x_\infty - \alpha\, e\| \le 1 - \alpha$, we obtain:

a) If $t \in A$, then $|x_\infty(t) - \alpha| \le 1 - \alpha \implies -1 + 2\,\alpha \le x_\infty(t) \le 1$.

b) If $t \in A^c$, then $|x_\infty(t) + \alpha| \le 1 - \alpha \implies -1 \le x_\infty(t) \le 1 - 2\,\alpha$

Since $\alpha - \lambda \ge 0$, there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_\infty\| \le \alpha - \lambda$ for every $n > n_0$. Therefore it follows that:

a) If $t \in A$, then $1 \ge x_n(t) \ge -\alpha + \lambda + x_\infty(t) \ge -1 + 2\,\lambda$, and hence $|x_n(t) - \lambda\, e(t)| \le 1 - \lambda$.

b) If $t \in A^c$, then $-1 \le x_n(t) \le \alpha - \lambda + x_\infty(t) \le 1 - 2\,\lambda$, and hence $|x_n(t) - \lambda\, e(t)| \le 1 - \lambda$.

For every $n \le n_0$, we choose an amenable triple $(e_n, y_n, \lambda)$ for $x_n$. Let's consider the sequences $(e_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, where $e_n = e$ and

$$y_n = \frac{x_n - \lambda\, e}{1 - \lambda}$$

for every $n > n_0$. Then $\Big((e_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, \lambda\Big)$ is an amenable triple for $x$. □

*Remark 3.4.* An immediate consequence of the former proposition is that $c(c)$ and $c(\ell_\infty)$ have the uniform $\lambda$-property.

As a consequence of corollary 2.4, the proposition 3.3 and the results obtained in [4] and [5] it's quite apparent that:

**Corollary 3.5**

*Let $K$ be a Hausdorff compact space, then the following statements are equivalent:*

a) *$K$ is 0-dimensional.*
b) *$\mathcal{C}(K)$ has the Bade property.*
c) *$c(\mathcal{C}(K))$ has the Bade property.*
d) *$\mathcal{C}(K)$ has the $\lambda$-property.*
e) *$\mathcal{C}(K)$ has the uniform $\lambda$-property and $\lambda(\mathcal{C}(K)) = 1/2$.*
f) *$c(\mathcal{C}(K))$ has the $\lambda$-property.*
g) *$c(\mathcal{C}(K))$ has the uniform $\lambda$-property and $\lambda(C(\mathcal{C}(K))) = 1/2$.*

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