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On the Composition Operator in $\mathcal{AC}[a,b]$

Nelson Merentes *

Central University of Venezuela, Caracas

L. Eötvös University, Budapest, Hungary

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Abstract

Denote by F the composition operator generated by a given function $f: \mathbb{R} \to \mathbb{R}$, acting on the space of absolutely continuous functions. In this paper we prove that the composition operator F maps the space $\mathcal{AC}[a,b]$ into itself if and only if f satisfies a local Lipschitz condition on $\mathbb{R}$.

1. Introduction

The well-known De La Vallée-Poussin's lemma (see [5]) states that: *If $f \in \mathcal{AC}[a,b]$, then the composition operator generated by f maps the space $\mathcal{AC}[a,b]$ into itself if and only if, for every $u \in \mathcal{AC}[a,b]$, the following holds:*

$$(f' \circ u)\, u' \in L_1([a,b]),$$

where the product is defined to be zero when $u'(t) = 0$, even if $f'(u(t))$ is not defined.

Here we shall give an example of an absolutely continuous function f, such that the composition operator F generated by f does not map $\mathcal{AC}[a,b]$ into itself. So, it is natural to look for necessary and sufficient conditions on f, in order that

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the corresponding composition operator maps $\mathcal{AC}[a,b]$ into itself. In this paper, we prove that this is so if and only if $f:\mathbb{R}\to\mathbb{R}$ satisfies a local Lipschitz condition. Moreover, the operator $F:\mathcal{AC}[a,b]\to\mathcal{AC}[a,b]$ is always bounded on bounded sets.

2. Preliminaries

In this section we give some definitions and results concerning the space$\mathcal{AC}[a,b]$, that we shall use in this work.

The space $\mathcal{AC}[a,b]$ consists of all absolutely continuous functions u defined on $[a,b]$, equipped with either of the norms:

$$\|u\| := |u(a)| + \|u'\|_{L_1[a,b]}$$

or

$$|u| := |u(a)| + \sup_{\tau}\sum_{j=1}^{m}|u(t_j) - u(t_{j-1})|,$$

where the supremum is taken over all partitions $\tau : a = t_0 < \ldots < t_m = b$ of the interval $[a,b]$.

Let $f:\mathbb{R}\to\mathbb{R}$ and $u:[a,b]\to\mathbb{R}$ be given functions. Let us define

$$\mathrm{F}\, u(t) := f(u(t)) \qquad (t\in[a,b]).$$

The operator F is usually called a composition operator, on Nemytskii's operator.

Let us remember some known facts on the composition operator on the space $\mathcal{AC}[a,b]$. First of all, we point out that f absolutely continuous does not necessarily imply F $(\mathcal{AC}[a,b])\subset\mathcal{AC}[a,b]$.

In this regard, we can consider the following

Example

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by:

$$f(u) := \begin{cases} 1 & \text{if } -\infty < u \le -1, \\ \sqrt{|u|} & \text{if } -1 < u < 1, \\ 1 & \text{if } 1 \le u < \infty. \end{cases}$$

This function f is absolutely continuous on $\mathbb{R}$ and so is the function

$$x(s) := s^2 \sin^2 \frac{1}{s},$$

but the composition function $f\circ x$ restricted to $[-1,1]$ does not belong to the space $\mathcal{AC}[-1,1]$.

As will be seen in the next section, the above statement about F is due to the fact that f does no satisfy a Lipschitz condition at $u = 0$.

3. Main result

In this section we shall present a characterization of functions $f : \mathbb{R} \to \mathbb{R}$ for which the composition operator F maps the space $\mathcal{AC}[a,b]$ into itself.

Theorem

The composition operator F generated by $f : \mathbb{R} \to \mathbb{R}$ maps the space $\mathcal{AC}[a,b]$ into itself if and only if f satisfies a local Lipschitz condition in $\mathbb{R}$; i.e., for every $r > 0$ there exists $k(r) > 0$ such that:

$$|f(u) - f(v)| \leq k(r)|u - v| \qquad (|u|, |v| \leq r) \tag{1}$$

Moreover, the composition operator $F : \mathcal{AC}[a,b] \to \mathcal{AC}[a,b]$ is always bounded on bounded sets.

Proof. It is well-known that, if f is locally Lipschitz then the composition operator F generated by f maps $\mathcal{AC}[a,b]$ into itself (see example [5] or [6]). Furthermore given $u \in \mathcal{AC}[a,b]$, $\|u\| \leq r$ the following inequality holds

$$\|\mathrm{F}\, u\| \leq |f(0)| + 2k(r)\|u\| \tag{2}$$

Since the function $x(s) \equiv s$ is absolutely continuous, the function f is bounded on $[-r, r]$, with a bound M. Without loss of generality, we can assume that $M = 1/2$. Suppose that f does not satisfy a local Lipschitz condition on $\mathbb{R}$, and hence there exists an interval $[-r, r]$ such that

$$\frac{|f(u) - f(v)|}{|u - v|}$$

is unbounded for $|u|, |v| \leq r$ $(u \neq v)$ and there exists sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ in $[-r, r]$ such that:

$$k_n|u_n - v_n| \leq |f(u_n) - f(v_n)| < 1 \tag{3}$$

By considering to subsequences, if necessary, we may assume that $u_n \to u^*$ as $n \to +\infty$ and $|u_n - u^*| \leq 1/k_n$ $(n = 1, 2, \ldots)$. From the inequality (3) it follows $v_n \to u^*$ as $n \to +\infty$. The analysis can be reduced to the following two cases:

1) u^* belongs only to finitely many intervals $[u_n, v_n]$.

2) u^* belongs to infinitely many intervals $[u_n, v_n]$.

Suppose that we are in the 1st case, and that infinitely many intervals not containing u^* lie to the right of u^*. Let us define a subsequence of these intervals having the following properties:

a) $u^* < u_{n+1} < v_{n+1} < u_n < v_n$ $(n = 1, 2, \ldots)$,

b) $|v_n - u^*| \leq 1/2^n$ $(n = 1, 2, \ldots)$,

c) $k_n > 2^{2n+1}$ $(n = 1, 2, \ldots)$.

From (b), it follows that

$$|v_n - u_n| < \frac{1}{2^n} \quad \text{and} \quad |u_n - v_{n+1}| < \frac{1}{2^n} \qquad (n = 1, 2, \ldots).$$

Let us choose integers m_n (see [1, p. 433]) so that

$$\frac{1}{2^{n+1}} < m_n |v_n - u_n| \leq \frac{2}{2^n} \qquad (n = 1, 2, \ldots), \tag{4}$$

and groups of points in $[a, b]$ such that

$$\begin{gathered} t^1_{m_1} > \bar{t}^1_{m_1} > \ldots t^1_1 > \bar{t}^1_1 > t^2_{m_2} > \bar{t}^2_{m_2} > \ldots > t^2_1 > \bar{t}^2_1 > t^3_{m_3} > \bar{t}^3_{m_3} > \ldots \\ \ldots > t^3_1 > \bar{t}^3_1 > \ldots t^n_{m_n} > \bar{t}^n_{m_n} > \ldots > t^n_1 > \bar{t}^n_1 > t^{n+1}_{m_{n+1}} > \bar{t}^{n+1}_{m_{n+1}} > \ldots \\ \ldots > t^{n+1}_1 > \bar{t}^{n+1}_1 > \ldots \qquad (n = 1, 2, \ldots, \quad t^n_{m_n} \to a, \quad \bar{t}^n_{m_n} > a). \end{gathered}$$

Define the function u on $[a, b]$ in the following way: $u(a) = u^*$, $u(t) = v_1$ if $t^1_{m_1} \leq t \leq b$, and on the other intervals by:

$$u(t) := \begin{cases} \dfrac{v_n - u_n}{t^n_j - \bar{t}^n_j}(t - t^n_j) + v_n & \bar{t}^n_j \leq t \leq t^n_j \ (n = 1, 2, \ldots, \ j = 1, 2, \ldots, m_n), \\ \dfrac{v_n - u_n}{t^n_{j-1} - \bar{t}^n_j}(t - \bar{t}^n_j) + u_n & t^n_{j-1} \leq t \leq \bar{t}^n_j \ (n = 1, 2, \ldots, j = 2, \ldots, m_n), \\ \dfrac{u_n - v_{n+1}}{\bar{t}^n_1 - t^{n+1}_{m_{n+1}}}(t - \bar{t}^n_1) + u_n & t^{n+1}_{m_{n+1}} \leq t \leq \bar{t}^n_1 \ (n = 1, 2, \ldots). \end{cases}$$

We claim that $u \in \mathcal{AC}[a, b]$, but $\mathrm{F}\, u \notin \mathcal{AC}[a, b]$. Indeed, from the definition of the function u, it follows that u is continuous on $[a, b]$ and its derivative $u'(t)$ exists for all $t \in [a, b]$ except on a countable set, and u' is integrable, in fact the following inequality holds:

$$\|u'\|_{L_1[a,b]} \leq \sum_{n=1}^{\infty} \Big[2m_n |v_n - u_n| + |u_n - v_{n-1}| \Big] \leq 2 \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} + \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

Moreover, the function u is absolutely continuous on $[a,b]$ (see [3, p.183]). From the inequalities (3), (4) and the property (c), we get

$$V(\mathrm{F}\ u;[a,b]) \geq \sum_{n=1}^{\infty} m_n|f(v_n)-f(u_n)|$$
$$\geq \sum_{n=1}^{\infty} k_n m_n|v_n-u_n| \geq \sum_{n=1}^{\infty} \frac{2^{2n+1}}{2^{n+1}} = \sum_{n=1}^{\infty} 2^n = +\infty$$

Thus $u \in \mathcal{AC}[a,b]$ and $\mathrm{F}\ u \notin \mathcal{AC}[a,b]$ which is a contradiction. The same argument applies if infinitely many intervals lie to the left of u^*.

Let us consider the second case. Suppose that u^* is contained in infinitely many intervals $[u_n, v_n]$. Considering, if necessary, to a subsequence, one can assume the following:

$$u_n \leq u^* \leq v_n \qquad (n=1,2,\ldots)$$

and

$$2n(n+1)|v_n-u_n| < |f(v_n)-f(u_n)| \leq 1 \qquad (n=1,2,\ldots) \tag{5}$$

From the inequality (5), we have

$$\frac{1}{2n(n+1)|v_n-u_n|} > 1 \qquad (n=1,2,\ldots).$$

Defining the numbers m_n and m'_n by:

$$m_n := \frac{1}{2n(n+1)|v_n-u_n|} \quad \text{and} \quad m'_n := [m_n] \qquad (n=1,2,\ldots),$$

where $[m_n]$ denotes the integer part of m_n. Hence

$$m'_n \leq m_n \quad \text{and} \quad \frac{m_n}{2} \leq m'_n \qquad (n=1,2,\ldots). \tag{6}$$

Without loss of generality we can assume that the interval $[a,b]$ is the interval $[0,1]$.

Now, for each $n=1,2,\ldots$, let I_n denote the interval

$$I_n := \left[\frac{1}{n+1}, \frac{1}{n}\right],$$

and let τ^n denote the partition of I_n defined by:

$$\tau^n := \frac{1}{n+1} = t_0^n < t_1^n < \ldots < t_{2m'_n}^n < t_{2m'_n+1}^n = \frac{1}{n}$$

where

$$t_0^n := \frac{1}{n+1}, \; t_j^n := \frac{1}{n+1} + \frac{j}{2}|u_n - v_n| \; (j = 1, 2, \ldots, 2m_n') \; \text{ and } \; t_{2m_n'+1}^n := \frac{1}{n}.$$

Let us define the function u on $[0,1]$ by: $u(0) = u^*$, $u(1) = u_1$ and on the intervals l_n by:

$$u(t) := \begin{cases} \frac{u_n - u_{n+1}}{t_1^n - t_0^n}(t - t_0^n) + u_{n+1} & \text{if } t_0^n \le t \le t_1^n, \\ \frac{v_n - u_n}{t_{j+1}^n - t_j^n}(t - t_j^n) + u_n & \text{if } t_j^n \le t \le t_{j+1}^n \; (j = 1, 3, \ldots, 2m_n' - 1), \\ \frac{u_n - v_n}{t_{j+1}^n - t_j^n}(t - t_j^n) + v_n & \text{if } t_j^n \le t \le t_{j+1}^n \; (j = 2, 4, \ldots, 2m_n'). \end{cases}$$

We claim that $u \in \mathcal{AC}[a,b]$, but F $u \notin \mathcal{AC}[a,b]$. Indeed, from the definition of the function u we have that is absolutely continuous on $[0,1]$. Moreover, the following estimation holds:

$$\|u'\|_{L_1[a,b]} \le \sum_{n=1}^{\infty} \Big[2m_n'|v_n - u_n| + |u_n - u_{n+1}|\Big] \le 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \infty.$$

From the inequalities (5) and (6), we have

$$\begin{aligned} V(\mathrm{F}\, u; [0,1]) &\ge \sum_{n=1}^{\infty} 2m_n'|f(v_n) - f(u_n)| \ge \sum_{n=1}^{\infty} 4m_n' n(n+1)|v_n - u_n| \\ &\ge \sum_{n=1}^{\infty} 2m_n n(n+1)|v_n - u_n| \ge \sum_{n=1}^{\infty} 1 = +\infty. \end{aligned}$$

Thus $u \in \mathcal{AC}[a,b]$ and F $u \notin \mathcal{AC}[a,b]$, which is a contradiction. □

Remarks. 1) If we suppose that F maps $\mathcal{AC}[a,b]$ into itself, then since $u(s) \equiv s$ is absolutely continuous, we know that f is locally absolutely continuous. In the present situation we prove that in addition f is locally Lipschitz on $\mathbb{R}$.
2) For $1 < p < \infty$, the inclusions

$$\mathrm{BV}_p[a,b] \subset \mathcal{AC}[a,b] \subset \mathrm{BV}[a,b]$$

are known, where $\mathrm{BV}_p[a,b]$ and $\mathrm{BV}[a,b]$ are the spaces of all functions of bounded p-variation ($1 < p < \infty$) and bounded variation, respectively. In 1981 M. Josephy

[2] proved that F maps the space $\mathrm{BV}[a,b]$ into itself if and only if $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz.

Recently, the present author [4] obtained a similar result for the space $\mathrm{BV}_p[a,b]$ $(1 < p < \infty)$. The above Theorem gives an analogous result for a space *intermediate* with respect to the inclusion.

3) From the above remark the following questions arise:

a) Suppose that X is any Banach space with the norm of $\mathrm{BV}[a,b]$, and

$$\mathrm{BV}_p[a,b] \subset X \subset \mathrm{BV}[a,b] \tag{7}$$

for some $1 < p < \infty$. Is it true that F maps X into itself if and only if $f:\mathbb{R}\to\mathbb{R}$ satisfies a local Lipschitz condition?

b) If the answer to a) is negative, characterize those *intermediate* spaces X for which the above type of result is true.

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