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Non-removable ideals in commutative topological algebras with separately continuous multiplication

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ABSTRACT

An ideal in a commutative topological algebra with separately continuous multiplication is non-removable if and only if it consists locally of joint topological divisors of zero. Also, any family of non-removable ideals can be removed simultanously.

The notion of non-removable ideals in commutative Banach algebras was introduced by Arens [1] and further studied e.g. in [2], [3]. In [5] it was proved that an ideal in a commutative Banach algebra is non-removable if and only if it consists of joint topological divisors of zero. Any countable family of removable ideals can be removed simultanously [6] which is not true for non-countable families [3].

Non-removable ideals in locally convex and topological algebras were studied in [7], [8] and [4]. In [4] some partial results for topological algebras with separately continuous multiplication were obtained. In the present paper we continue the investigations of [4]. It turns out that there exists a nice characterization of non-removable ideals in the class of topological algebras with separately continuous multiplication.

All algebras in this paper will be commutative, complex and with units.

As in [4], by an s-algebra we shall mean a topological linear space with a separately continuous associative multiplication which makes of it an algebra. The topology of an s-algebra A can be given by means of a system $\nu(A)$ of zero-neighbourhoods which is closed under finite intersections and satisfies

(1) for every $U \in \nu(A)$ there exists $V \in \nu(A)$ such that $V + V \subset U$
(2) for every $U \in \nu(A)$ and complex number λ with $|\lambda| < 1$, $\lambda U \subset U$
(3) every $V \in \nu(A)$ is absorbent
(4) for every $U \in \nu(A)$ and $x \in A$ there exists $V \in \nu(A)$, $xV \subset U$.

Let A, B be commutative s-algebras with unit elements. We say that B is an extension of A if there exists a unit preserving algebra isomorphism $f : A \to B$ which is also a topological homomorphism. We shall identify A with its image $f(A)$ and write shortly $A \subset B$.

Let I be an ideal in a commutative s-algebra A with unit e. We say that I is removable if there exists an extension $B \supset A$ such that I is contained in no proper ideal of B. Equivalently, this means that there exist a finite number of elements $x_1, \dots, x_n \in I$ and $b_1, \dots, b_n \in B$ such that $\sum_{s=1}^n x_s b_s = e$. Otherwise we say that I is non-removable. Let $\{x_1, \dots, x_n\}$ be a finite subset of a commutative $s-$algebra A. We say that $x_1, \dots, x_n$ are joint topological divisors of zero if there exists a net $\{z_\alpha\} \subset A$ which does not tend to 0 but $\lim\limits_\alpha z_\alpha x_s = 0$ for $s = 1, \dots, n$.

If $\nu(A)$ is a system of zero-neighbourhoods giving the topology of A then $x_1, \dots, x_n$ are not joint topological divisors of zero if and only if for every $U \in \nu(A)$ there exists a neighbourhood $V \in \nu(A)$ such that

$$z \in A, \quad zx_s \in V \quad (s = 1, \dots, n) \qquad \text{implies } z \in U.$$

Let $m \geq 1$. It is easy to prove by induction on m that $x_1, \dots, x_n$ are not joint topological divisors of zero if and only if for every $U \in \nu(A)$ there exists a neighbourhood $V \in \nu(A)$ such that

$$(5) \quad z \in A, \quad zx_1^{q_1} \dots x_n^{q_n} \in V \text{ for every } q_1, \dots, q_n, \sum_{t=1}^n q_t = m \qquad \text{implies } z \in U.$$

Let I be an ideal of a commutative s-algebra A. We say that I consists locally of joint topological divisors of zero (cf. [8]) if every finite subset of I consists of joint topological divisors of zero. If I consists locally of joint topological divisors of zero then it is non-removable. Indeed, suppose on the contrary that there exist $B \supset A$, $x_1, \dots, x_n \in I$ and $b_1, \dots, b_n \in B$ such that $\sum_{s=1}^n x_s b_s = 1$. Let $\{u_\alpha\}$ be the net satisfying $z_\alpha x_s \to 0$ $(s = 1, \dots, n)$ and $z_\alpha \not\to 0$. Then

$$z_\alpha = z_\alpha \left(\sum_{s=1}^n x_s b_s \right) = \sum_{s=1}^n (z_\alpha x_s) b_s.$$

We have $(z_\alpha x_s) b_s \to 0$ $(s = 1, \dots, n)$ so $z_\alpha \to 0$, a contradiction.

The aim of this paper is to prove the converse implication. Also we prove that any number of removable ideals can be removed simultanously. Thus the situation in the class of s-algebras differs from that of Banach algebras where only countable families of removable ideals can be removed simultanously (see [3], [6]).

Theorem 1

Let A be a commutative s-algebra with unit e. Let Λ be a set and p_l a positive integer for every $l \in \Lambda$. Let $u_{l,s}$ $(l \in \Lambda, 1 \le s \le p_l)$ be a system of elements of A such that, for each $l \in \Lambda$, $u_{l,1}, \dots, u_{l,p_l}$ are not joint topological divisors of zero. Then there exists an extension $B \supset A$ and elements $b_{l,s} \in B$ $(l \in \Lambda, 1 \le s \le p_l)$ such that $\sum_{s=1}^{p_l} u_{ls} b_{ls} = e$ for every $l \in \Lambda$.

Proof. We may assume that $p_l \ge 2$ for every $l \in \Lambda$ (if $p_l = 1$ for some $l \in \Lambda$ we can replace the element $u_{l,1}$ by the pair $u_{l,1}, u_{l,2} = u_{l,1}$).

Denote by N the set of all non-negative integers,

$$T = \{(l,s), l \in \Lambda, 1 \le s \le p_l\},$$
$$D = \{\mathbf{k} : T \to N, \mathbf{k}((l,s)) \ne 0 \text{ only for a finite number of } (l,s) \in T\}.$$

For $\mathbf{k},\mathbf{j} \in D$ and $(l,s) \in T$ denote $k_{ls} = \mathbf{k}((l,s))$, $|\mathbf{k}|_l = \sum_{s=1}^{p_l} k_{ls}$ and $(\mathbf{k}+\mathbf{j}) \in D$, $(k+j)_{ls} = k_{ls} + j_{ls}$. We write $\mathbf{k} \le \mathbf{j}$ if $k_{ls} \le j_{ls}$ for all $(l,s) \in T$.

Denote by $Q(A)$ the algebra of all polynomials with coefficients from A and with variables $b_\mathbf{j}$ $(\mathbf{j} \in D)$ i.e.

$$Q(A) = \left\{ \sum_{\mathbf{j} \in D} a_\mathbf{j} \mathbf{b}^\mathbf{j},\ a_\mathbf{j} \in A,\ a_\mathbf{j} \ne 0 \text{ for a finite number of } \mathbf{j} \in D \right\}.$$

Here $\mathbf{b}^\mathbf{j}$ stands for $\prod_{(l,s) \in T} b_{ls}^{j_{ls}}$.

The algebraic operations in $Q(A)$ are defined in the natural way.

Let $\nu(A)$ be a system of zero-neighbourhoods in A giving the topology of A which satisfies (1)-(4).

We define the topology in $Q(A)$ in the following way: for any mapping $d : D \to \nu(A)$ define a zero-neigbourhood V_d in $Q(A)$ by

$$V_d = \left\{ \sum_{\mathbf{j} \in D} a_\mathbf{j} \mathbf{b}^\mathbf{j} \in Q(A),\ a_\mathbf{j} \in d(\mathbf{j}) \text{ for every } \mathbf{j} \in D \right\}.$$

Clearly the system $\nu(Q(A)) = \{V_d,\ d : D \to \nu(A)\}$ satisfies conditions (1)-(4) (condition (4) is clear for every $a \in A$ and for $\mathbf{b}^\mathbf{j}$, $(\mathbf{j} \in D)$ and every $x \in Q(A)$ is a finite combination of these elements).

Therefore $Q(A)$ with the topology determined by this system is an s-algebra.

Let $I \subset Q(A)$ be the ideal generated by the elements $\{e - \sum_{s=1}^{p_l} u_{ls} b_{ls}, l \in \Lambda\}$ and denote by $B = Q(A)|\bar{I}$. Then B is an s-algebra (see [4]).

Consider the mapping $f : A \to B$, $f = \pi f_0$ where $f_0 : A \to Q(A)$ is the natural identification of A with the constant polynomials and $\pi : Q(A) \to Q(A)|\bar{I}$ is the canonical projection. Clearly f is a continuous homomorphism and

$$\sum_{s=1}^{p_l} f(u_{ls})\pi(b_{ls}) = e_B \text{ for every } l \in \Lambda.$$

Therefore it is sufficient to show that f is open.

We must show that for every $U \in \nu(A)$ there exists a mapping $d' : D \to \nu(A)$, such that

$$(V_{d'} + \bar{I}) \cap A \subset U \tag{6}$$

(cf. [4]). In fact is sufficient to show that for every $U \in \nu(A)$ there exists a mapping $d : D \to \nu(A)$ such that

$$\big(V_d + I\big) \cap A \subset U. \tag{7}$$

Indeed, suppose $U \in \nu(A)$ and $V_d \in \nu(Q(A))$ satisfies (7). Find $V_{d'} \in \nu(Q(A))$ such that $V_{d'} + V_{d'} \subset V_d$.

Let $a \in A$ and $x \in \bar{I}$ satisfy $a - x \in V_{d'}$. Then there exists $x_0 \in I$ such that $x - x_0 \in V_{d'}$ and $a - x_0 = (a - x) + (x - x_0) \in V_{d'} + V_{d'} \subset V_d$. By (7), $a \in U$. Therefore (6) is also satisfied and $f : A \to B$ is open.

Denote by $G = \big\{g : \Lambda \to N,\ g(l) \neq 0 \text{ for a finite number of } l \in \Lambda\big\}$. For $g \in G$ put $|g| = \sum_{l \in \Lambda} g(l)$. For $\mathbf{j} \in D$ let $g(\mathbf{j}) \in G$ be defined by $(g(\mathbf{j}))(l) = |\mathbf{j}|_l$ $(l \in \Lambda)$.

Let $U \in \nu(A)$. We define the zero-neighbourhoods $U_g, U'_g \in \nu(A)$ for $g \in G$ inductively. Choose $U_{\bar{0}} \in \nu(A)$ such that $U_{\bar{0}} + U_{\bar{0}} \subset U$ (here $\bar{0}$ is the zero function $\Lambda \to N$). Suppose U_h is defined for all $h \in G$, $h \leq g$, $h \neq g$. Choose $U'_g \in \nu(A)$ such that

$$x u_{l1}^{q_1} \dots u_{lp_l}^{q_{p_l}} \in U'_g \text{ for every } q_1, \dots, q_l \in N,\ \sum_{s=1}^{p_l} q_s = p_l(g(l) - 1) + 1 \tag{8}$$

implies $x \in U_h$, whenever $l \in \Lambda$, $g(l) \neq 0$ and $h \in G$ is determined by $h(l) = g(l) - 1$, $h(m) = g(m)$ $(m \neq l)$.

This is possible because of (5) and $g(l) \neq 0$ only for a finite number of $l \in \Lambda$.

Choose further $U_g \in \nu(A)$ such that

$$\underbrace{U_g + U_g + \ldots + U_g}_{c} \subset U'_g \tag{9}$$

where $c = 2^{|g|} \cdot \prod_{l \in \Lambda} p_l^{g(l)^2}$.

For $\mathbf{j} \in D$ find now a zero-neigbourhood $V_{\mathbf{j}} \in \nu(A)$ such that

$$V_{\mathbf{j}} \mathbf{u}^{\mathbf{q}} \subset U_{g(\mathbf{j})} \text{ for every } \mathbf{q} \in D \text{ such that } |\mathbf{q}|_l \leq p_l |\mathbf{j}|_l^2. \tag{10}$$

Define $d : D \to \nu(A)$ by $d(\mathbf{j}) = V_{\mathbf{j}}$ ($\mathbf{j} \in D$). We prove that the corresponding zero-neighbourhood $V_d \in \nu(Q(A))$ satisfies (7).

Let $a \in A$, $x \in I$, $a + x \in V_d$,

$$x = \sum_{l \in \Lambda} \sum_{\mathbf{j} \in D} a_{\mathbf{j}}^{(l)} (e - u_{l,1} b_{l,1} - \ldots - u_{l,p_l} b_{l,p_l})$$

where only a finite number of elements $a_{\mathbf{j}}^{(l)} \in A$ are non-zero. The condition $a + x \in V_d$ may be rewritten as follows:

$$a + \sum_{l \in \Lambda} a_0^{(l)} \in V_0$$

$$f_{\mathbf{i}} \in V_{\mathbf{i}} \quad (\mathbf{i} \in D, \mathbf{i} \neq 0) \tag{11}$$

where

$$f_{\mathbf{i}} = \sum_{l \in \Lambda} a_{\mathbf{i}}^{(l)} - \sum_{l \in \Lambda} \sum_{\substack{1 \leq t \leq p_l \\ i_{lt} \neq 0}} a_{\mathbf{j}}^{(l)} u_{lt}$$

and $j_{lt} = i_{lt} - 1$, $j_{ms} = i_{ms}$ for $(m, s) \neq (l, t)$.

Suppose that elements $a, a_{\mathbf{j}}$ ($\mathbf{j} \in D$) satisfying (11) are fixed. It is sufficient to show that (11) implies $a \in U$.

In the following we shall need some notations and results of [6].

It is convenient to consider linear combinations of $a_{\mathbf{j}}^{(l)}$'s as formal expressions. Therefore we denote by W the free additive group with generators $\hat{a}_{\mathbf{j}}^{(l)} \hat{\mathbf{u}}^{\mathbf{k}}$ ($\mathbf{j}, \mathbf{k} \in D$, $l \in \Lambda$). Here we consider $\hat{a}_{\mathbf{j}}^{(l)} \hat{\mathbf{u}}^{\mathbf{k}}$ as one symbol; there is no multiplication in W. Define the additive mapping $P : W \to A$ by $P \hat{a}_{\mathbf{j}}^{(l)} \hat{\mathbf{u}}^{\mathbf{k}} = a_j^{(l)} u^k$.

Define the following additive mappings acting in W:

Let $\mathbf{i}, \mathbf{k} \in D$, $\mathbf{k} \geq \mathbf{i}$, $l, m \in \Lambda$, $d \in N$. Put

$$H_{md}(\hat{a}_{\mathbf{i}}^{(l)} \hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}}) = \begin{cases} \hat{a}_{\mathbf{i}}(l) \hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}} & \text{if } |\mathbf{i}|_m = d \\ 0 & \text{otherwise,} \end{cases}$$

$$\pi_{lm}(\hat{a}_{\mathbf{i}}^{(l)} \hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}}) = \hat{a}_{\mathbf{i}}^{(m)} \hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}}, \quad \pi_{lm}(\hat{a}_{\mathbf{i}}^{(r)} \hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}}) = 0 \qquad \text{for } r \neq l,$$

$$F_{lm}(\hat{a}_{\mathbf{i}}^{(l)} \hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}}) = \sum_{\mathbf{j} \in M_{i,m}} \hat{a}_{\mathbf{j}}^{(l)} \hat{\mathbf{u}}^{\mathbf{k}-\mathbf{j}}$$

where $M_{i,m} = \{\mathbf{j} \in D$, there exists t, $1 \le t \le p_m$ such that $j_{mt} = i_{mt} - 1$, $j_{rs} = i_{rs}$ for $(r,s) \ne (m,t)\}$,

$$F_{lm}\big(\hat{a}_{\mathbf{i}}^{(r)}\hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}}\big) = 0 \qquad \text{for } r \ne l.$$

For $1 \le s \le p_l$, $k_{ls} \ge i_{ls} + |i|_l + 1$ put

$$G_{ls}\big(\hat{a}_{\mathbf{i}}^{(l)}\hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}}\big) = \sum_{\mathbf{j}\in J_1}(-1)^{j_{ls}-i_{ls}-1}\frac{(j_{ls}-i_{ls}-1)!}{\prod_{\substack{t\ne s\\ 1\le t\le p_l}}(i_{lt}-j_{lt})!}\hat{a}_{\mathbf{j}}^{(l)}\hat{\mathbf{u}}^{\mathbf{k}-\mathbf{j}},$$

where

$$J_1 = \{j \in D, j_{rt} = i_{rt} \text{ for } r \ne l, j_{lt} \le i_{lt} \text{ for } t \ne s \text{ and } |\mathbf{j}|_l = |\mathbf{i}|_l + 1\}.$$

We put $G_{ls}(\hat{a}_{\mathbf{i}}^{(r)}\hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}}) = 0$ if either $r \ne l$ or $k_{ls} < i_{ls} + |\mathbf{i}|_l + 1$.

For $\hat{v} = \sum_{l\in\Lambda}\sum_{\mathbf{i},\mathbf{j}\in D}\gamma_{\mathbf{i},\mathbf{j}}^{(l)}\hat{a}_{\mathbf{i}}^{(l)}\hat{\mathbf{u}}^{\mathbf{j}} \in W$ (a finite sum with integer coefficients $\gamma_{\mathbf{i},\mathbf{j}}^{(l)}$) define

$$|\hat{v}| = \max_{l\in\Lambda}\sum_{\mathbf{i},\mathbf{j}\in D}|\gamma_{\mathbf{i},\mathbf{j}}^{(l)}|.$$

By the definition of G_{ls} we have

$$|G_{ls}\hat{a}_{\mathbf{i}}^{(l)}\hat{\mathbf{u}}^{\mathbf{j}}| \le \sum_{t=1}^{|\mathbf{i}|_l}\sum_{\substack{n_1\dots n_{p_l-1}\\ n_1+\dots+n_{p_l-1}=t}}\frac{t!}{n_1!\dots n_{p_l-1}!} = \sum_{t=1}^{|\mathbf{i}|_l}(p_l-1)^t \le p_l^{|\mathbf{i}|_l}. \tag{12}$$

(see Lemma 2 of [5]).

Further put $Z_{ls} = G_{ls} + \sum_{m\ne l}(\pi_{lm}G_{ls} - \pi_{lm}F_{lm}G_{ls} + \pi_{mm})$.

The properties of these mappings can be found in [6]. We shall need the following lemma (Lemma 5.1 of [6]).

Lemma 2

Let $\mathbf{k} \in D$, $g \in G$, $\hat{v} = \sum_{l\in\Lambda}\sum_{\mathbf{i}\in D}\gamma_{\mathbf{i}}^{(l)}\hat{a}_{\mathbf{i}}^{(l)}\hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}} \in W$ (finite sums with integer coefficients $\gamma_{\mathbf{i}}^{(l)}$). Let $\Lambda_0 = \{l \in \Lambda, \gamma_{\mathbf{i}}^{(l)} \ne 0$ for some $\mathbf{i} \in D\}$. Let $l \in \Lambda_0$, $1 \le s \le p_l$, $k_{ls} \ge i_{ls} + |\mathbf{i}|_l + 1$ whenever $\gamma_{\mathbf{i}}^{(l)} \ne 0$. Suppose that

(i) $|\mathbf{i}|_m \le g(m)$ whenever $\gamma_{\mathbf{i}}^{(m')} \ne 0$ for some $m' \in \Lambda$ and $|\mathbf{i}|_m = g(m)$ if $\gamma_{\mathbf{i}}^{(m)} \ne 0$

(ii) $H_{m',g(m')}\pi_{mm}\hat{v} = \pi_{mm}\hat{v} - F_{mm'}\pi_{mm}\hat{v}$ for every $m, m' \in \Lambda_0$, $m \ne m'$

(iii) $H_{m,g(m)}\pi_{m'm'}\hat{v} = \pi_{mm'}H_{m',g(m')}\pi_{mm}\hat{v}$ for every $m, m' \in \Lambda$ (i.e. $\gamma_{\mathbf{i}}^{(m)} = \gamma_{\mathbf{i}}^{(m')}$ for every $m, m' \in \Lambda_0$, $\mathbf{i} \in D$, $|\mathbf{i}|_m = g(m)$, $|\mathbf{i}|_{m'} = g(m')$).

Then $\hat{w} = Z_{ls}\hat{v}$ satisfies conditions (i)–(iii) for $g' \in G$ determined by

$$g'(l) = g(l) + 1, \quad g'(l') = g(l') \qquad \text{for } l' \ne l.$$

Further $|\hat{w}| \le 2p_l^{g(l)}|\hat{v}|$.

Proof. The statements (i)–(iii) are proved in Lemma 5.1 of [6].

Further

$$
\begin{aligned}
|Z_{ls}\hat{v}| &= \Big|G_{ls}\hat{v} + \sum_{m\neq l}(\pi_{lm}G_{ls} - \pi_{lm}F_{lm}G_{ls} + \pi_{mm})\hat{v}\Big| \\
&= \Big|\sum_{m\neq l}\pi_{mm}\hat{v}\Big| + \Big|G_{ls}\hat{v} + \sum_{m\neq l}\pi_{lm}H_{m,g'(m)}G_{ls}\hat{v}\Big| \\
&\leq |\hat{v}| + |G_{ls}\hat{v}| \leq |\hat{v}|(1+p_l^{g(l)}) \leq 2p_l^{g(l)}|\hat{v}|.
\end{aligned}
$$

(We used the fact that $G_{ls} = \pi_{ll}G_{ls}$ and property (ii)). □

Suppose now that elements $a_{\mathbf{j}}(l) \in A$ ($\mathbf{j} \in D$, $l \in \Lambda$) satisfying (11) are given such that only finite number of them are non-zero. Put $\Lambda_0 = \{l \in \Lambda,\, a_j^{(l)} \neq 0$ for some $\mathbf{j} \in D\}$. Choose a sequence $g_0, g_1, g_2, \ldots \in G$ such that $g_0 = \bar{0} \in G$, $g_{n+1} \geq g_n$, $|g_n| = n$ $(n = 0, 1, \ldots)$, $g_n(l) = 0$ for $l \notin \Lambda_0$ and such that, for n sufficiently large,

$$g_n(l) > |\mathbf{j}|_l \qquad \text{whenever } \mathbf{j} \in D \text{ and } a_j^{(l)} \neq 0.$$

Put $M_0 = \left\{\sum_{l\in\Lambda_0} \hat{a}_0^l\right\} \subset W$ and define inductively sets $M_n \subset W$ $(n = 1, 2, \ldots)$ by

$$M_{n+1} = \left\{Z_{ls}(\hat{x}u_{l_1}^{q_1} \ldots u_{lp_l}^{q_{p_l}})\right\},$$

where $\hat{x} \in M_n$, $l \in \Lambda$ is determined by $g_{n+1}(l) = g_n(l) + 1$, $\sum_{t=1}^{p_l} q_t = g_n(l)p_l + 1$, $1 \leq s \leq p_l$ and $q_s \geq g_n(l) + 1$.

Lemma 3

Let $\hat{x} \in M_n$. Then $\hat{x}$ satisfies conditions (i)–(iii) of Lemma 2 for $g = g_n \in G$ and for some $\mathbf{k} \in D$, $|\mathbf{k}|_m \leq p_m g_n(m)^2$ $(m \in \Lambda)$.

Further

$$|\hat{x}| \leq 2^n \cdot \prod_{m\in\Lambda} p_m^{g_n(m)^2}.$$

Proof. By induction on n:

Suppose $\hat{x} \in M_n$ satisfies conditions (i)–(iii) of Lemma 2 for $g = g_n$ and for some $\mathbf{k} \in D$, $|\mathbf{k}|_m \leq p_m g_n(m)^2$ $(m \in \Lambda)$ and let $|\hat{x}| \leq 2^n \prod_{m\in\Lambda} p_m^{g_n(m)^2}$. Let $l \in \Lambda$ be determined by $g_{n+1}(l) = g_n(l) + 1$ and let $q_1, \ldots, q_{p_l} \in N$, $\sum_{t=1}^{p_l} q_t = g_m(l)p_l + 1$. Let $1 \leq s \leq p_l$ and $q_s \geq g_n(l) + 1$. Put $\hat{y} = \hat{x}\hat{\mathbf{u}}_{l1}^{q_1} \ldots \hat{\mathbf{u}}_{lp_l}^{q_{p_l}}$.

Then $\hat{y}$ satisfies conditions (i)–(iii) of Lemma 2 for $g = g_n$ and for $\mathbf{k}' \in D$ where $|\mathbf{k}'|_m = |\mathbf{k}|_m \le p_m g_n(m)^2 = p_m g_{m+1}(m)^2$ for $m \ne l$ and

$$\begin{aligned} |\mathbf{k}'|_l = |\mathbf{k}|_l + p_l g_n(l) + 1 &\le p_l\big(g_n(l)^2 + g_n(l)\big) + 1 \\ \le p_l\big(g_n(l)^2 + 1\big)^2 &= p_l g_{n+1}(l)^2. \end{aligned}$$

Hence $Z_{ls}\hat{y}$ satisfies conditions (i)–(iii) of Lemma 2 for $g = g_{n+1}$ and for $\mathbf{k}' \in D$. Further

$$\begin{aligned} |Z_{ls}\hat{y}| \le 2p_l^{g_n(l)}|\hat{y}| &= 2p_l^{g_n(l)}|\hat{x}| \\ \le 2p_l^{g_n(l)} \cdot 2^n \prod_{m\in\Lambda} p_m^{g_n(m)^2} &\le 2^{n+1} \prod_{m\in\Lambda} p_m^{g_{n+1}(m)^2}. \ \square \end{aligned}$$

We prove $PM_n \subset U_{g_n}$ $(n = 0, 1, 2, \ldots)$.

By the previous lemma $PM_n = \{0\}$ for n sufficiently large as only finite number of the elements $a_{\mathbf{j}}^{(m)}$ are non-zero. Therefore it is sufficient to prove

$$PM_{n+1} \subset U_{g_{n+1}} \Longrightarrow PM_n \subset U_{g_n} \qquad (n = 0, 1, \ldots).$$

Let $\hat{x} \in M_n$ and let $l \in \Lambda$ be determined by $g_{n+1}(l) = g_n(l) + 1$. To prove $P\hat{x} \in U_{g_n}$ it is sufficient to show

$$P(\hat{x}\hat{\mathbf{u}}_{l1}^{q_1} \ldots \hat{\mathbf{u}}_{lp_l}^{q_l}) \in U'_{g_{n+1}} \qquad \text{for every } q_1, \ldots, q_{p_l} \in N,\ \sum_{t=1}^{p_l} q_t = p_l g_n(l) + 1$$

(see (8)). Fix $q_1, \ldots, q_{p_l}$ and put $\hat{y} = \hat{x}\hat{\mathbf{u}}_{l1}^{q_1} \ldots \hat{\mathbf{u}}_{lp_l}^{q_{p_l}}$. Then there exists s, $1 \le s \le p_l$ such that $q_s \ge g_n(l) + 1$. Then $Z_{ls}\hat{y} \in M_{n+1}$ by Lemma 2, so $PZ_{ls}\hat{y} \in U_{g_{n+1}}$.

We shall need the following lemma:

Lemma 4

Let $\mathbf{i}, \mathbf{k} \in D$, $g(\mathbf{i}) = g_n$, *let* $|\mathbf{k}|_m \le p_m(g_{n+1}(m))^2$ *for every* $m \in \Lambda$, *let* $k_{ls} \ge i_{ls} + |\mathbf{i}|_l + 1$.

Then

$$P\Big(\hat{a}_{\mathbf{i}}^{(l)}\hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}} - Z_{ls}\hat{a}_{\mathbf{i}}^{(l)}\hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}}\Big) \in \underbrace{U_{g_{n+1}} + \ldots + U_{g_{n+1}}}_{p_l^{g_n(l)}-\text{times}}.$$

Proof. By [6], Lemmas 3.1 and 3.2 we have

$$P\Big(\hat{a}_{\mathbf{i}}^{(l)}\hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}} - Z_{ls}\hat{a}_{\mathbf{i}}^{(l)}\hat{\mathbf{u}}^{\mathbf{k}-\mathbf{i}}\Big) = \sum_{\mathbf{j}\in J_1}(-1)^{j_{ls}-i_{ls}}\frac{(j_{ls}-i_{ls}-1)!}{\prod_{\substack{t\neq s\\1\le t\le p_l}}(i_{lt}-j_{lt})!}f_{\mathbf{j}}\mathbf{u}^{\mathbf{k}-\mathbf{j}}.$$

For every $\mathbf{j}\in J_1$ we have $f_{\mathbf{j}}\in V_{\mathbf{j}}$ so $f_{\mathbf{j}}\mathbf{u}^{\mathbf{k}-\mathbf{j}}\in U_{g_{n+1}}$ by (10) (note that $g(\mathbf{j}) = g_{n+1}$ for $\mathbf{j}\in J_1$).

The rest follows from the estimation

$$\sum_{\mathbf{j}\in J_1}\frac{(j_{ls}-i_{ls}-1)!}{\prod_{\substack{t\neq s\\1\le t\le p_l}}(i_{lt}-j_{lt})!}\le p_l^{g_n(l)}$$

(see (12)). □

(Continuation of the proof of Theorem 1):
We have

$$\hat{y} = (\hat{y}-Z_{ls}\hat{y}) + Z_{ls}\hat{y}\in\underbrace{U_{g_{n+1}}+\ldots+U_{g_{n+1}}}_{c-\text{times}}$$

where

$$c\le p_l^{g_n(l)}\cdot 2^n\cdot\prod_{m\in\Lambda}p_m^{g_n(m)^2}+1\le 2^{n+1}\prod_{m\in\Lambda}p_m^{g_{n+1}(m)^2}$$

hence $\hat{y}\in U'_{g_{n+1}}$ by (9).

We have proved that $PM_n\subset U_{g_n}$ for every n. In particular,

$$\sum_{l\in\Lambda_0}a_0^{(l)}\in PM_0\subset U_{g_0}=U_{\bar{0}}$$

and by (11),

$$a=\Big(a+\sum_{l\in\Lambda_0}a_0^{(l)}\Big)-\sum_{l\in\Lambda}a_0^{(l)}\in U_{\bar{0}}+U_{\bar{0}}\subset U.$$

This finishes the proof of Theorem 1. □

Corollary 5

An ideal I in a commutative s-algebra A with unit e is non-removable if and only if it consists locally of joint topological divisors of zero.

Proof. If I does not consist locally of joint topological divisors of zero then there exist elements $u_1,\ldots,u_n\in I$ whish are not joint topological divisors of zero. Theorem 1 for card $\Lambda = 1$ gives the existence of an extension $B\supset A$ and elements $b_1,\ldots,b_n\in B$ such that $\sum_{t=1}^n u_tb_t = l$, i.e. I is removable. □

Corollary 6

Let $\{T_l\}_{l\in\Lambda}$ be any family of removable ideals in a commutative s-algebra A with unit e. Then there exists an extension $B \supset A$ such that, for every $l \in I$, I_l is not contained in a proper ideal of B.

Proof. I_l does not consist locally of joint topological divisors of zero so there exists a finite number of elements $u_{l1}, \dots, u_{lp_l} \in I_l$ which are not joint topological divisors of zero. Apply Theorem 1. □

Corollary 7

Let $\{u_\alpha\}_{\alpha\in\Lambda}$ be any family of elements of a commutative s-algebra A with unit e. Suppose that u_α is not a topological divisor of zero for every $\alpha \in \Lambda$. Then there exists an extension $B \supset A$ such that all u_α's are invertible in B.

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