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On some geometric properties concerning closed convex sets

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Abstract

In this article, we consider the (weak) drop property, weak property (α), and property (ω) for closed convex sets. Here we give some relations between those properties. Particularly, we prove that C has (weak) property (α) if and only if the subdifferential mapping of C^o is (n-n) (respectively, (n-w)) upper semicontinuous and (weak) compact valued. This gives an extension of a therorem of Giles and the first author.

1. Introduction

Let C be a closed convex subset of a real Banach space X such that $0 \in C$. (In this article, we always assume that C is a closed convex set with $0 \in C$.) $F(C)$ denotes the set of all bounded linear functionals $x^* \in X^*$, $x^* \neq 0$ which are bounded above on C. For any $x^* \in F(C)$, and $\delta > 0$, the *slice* $S(x^*, C, \delta)$ is the set

$$\{x \in C : \langle x, x^* \rangle \geq M - \delta\},$$

where $M = \sup\{\langle x, x^* \rangle : x \in C\}$. For any subset C of X, the *Kuratowski measure* of C is the number

$$\alpha(C) = \inf\{\epsilon > 0 : C \text{ is covered by a finite number of sets of diameter less than } \epsilon\}.$$

(Note: if C is not bounded, then we define $\alpha(C) = \infty$). A closed convex set C is said to have *property* (α) if $\lim_{\delta\to 0} \alpha(S(f, C, \delta)) = 0$ for all $f \in F(C)$. It is easy to see that a nonempty closed convex set C has property (α) if and only if for every $x^* \in F(C)$ and every sequence $\{x_n\}$ with $x_n \in S(x^*, C, 1/n)$, $\{x_n\}$ contains a convergent subsequence. Hence, a closed convex nonempty set C is said to have *weak property* (α) if for every $x^* \in F(C)$ and $x_n \in S(x^*, C, 1/n)$, $\{x_n\}$ contains a (weakly) convergent sequence. It is known (see [5] and [7]) that if C has weak property (α), then for every $x^* \in F(C)$, $S(x^*, C, \delta)$ is bounded. In [5] and [7], Rolewicz and the authors proved the following theorem.

Theorem A.

Let C be a non-compact (respectively, non-weakly compact) closed convex subset of X. Then C has the (weak) drop property if and only if C has nonempty interior and C has (weak) property (α).

Let X be a Banach space and let B be the unit ball of X. For a closed convex set C, the *weak measure of noncompactness* of C ([1]), is defined by

$$\omega(C) = \inf\{t > 0 : \text{ there exists a weakly compact set } K \text{ such that } C \subseteq K + tB\}.$$

(If C is not bounded, then we define $\omega(C) = \infty$.) For a closed convex set C is said to have *property* (ω) if for every $x^* \in F(C)$,

$$\lim_{n\to\infty} w(S(x^*, C, 1/n)) = 0.$$

In [1], De Blasi showed (Theorem 3 [1]) that if C_n is a sequence of nonempty decreasing weakly closed subsets of X such that $\lim_{n\to\infty} \omega(C_n) = 0$, then any sequence $\{x_n\}$ with $x_n \in C_n$ has a weakly convergent subsequence. Hence, if C has property (ω), then C has weak property (α).

In section 2, we study weak property (α) and property (ω). We show if C has weak property (α), then property (ω). Hence, weak property (α) is equivalent to property (ω). On the other hand, we give an example which shows they are not equivalent locally.

Let C be a closed convex set of X with $0 \in C$. The *polar* C^o of C is the set

$$\{x^* \in X^* : \langle x, x^*\rangle \leq 1 \text{ for all } x \in C\}.$$

Since C^o is a convex subset of X^* with $0 \in C^o$, it induces a *gauge* p on X^* by

$$p(x^*) = \inf\{\lambda > 0 : x^* \in \lambda C^o\}.$$

(Note: if $x^* \notin \lambda C^o$ for all $\lambda > 0$, then we define $p(x^*) = \infty$.) Clearly,

(i) $p(x^*) = \sup_{x \in C}\langle x, x^*\rangle$, and $F(C) = \{x^* : p(x^*) < \infty\} \setminus \{0\}$;

(ii) if C is bounded, then $F(C) = X^* \setminus \{0\}$ and p is a continuous function on X^*.

For any $x_o^* \in F(C)$, a *subgradient* of p at x_o^* is a continuous linear functional x^{**} such that

$$p(x_o^*) + \langle x^* - x_o^*, x^{**}\rangle \leq p(x^*)$$

for all $x^* \in F(C)$. The *subdifferential* of p at x_o^*, denoted by $\partial p(x_o^*)$, is the set of all subgradients of p at x_o^*.

In [4] and [3], Gile, Sims, Yorke and the first author studied the (weak) drop property and the subdifferential mapping of p. Gile and the first author proved the following theorem.

Theorem B.

Let C be a noncompact (respectively, non-weakly compact) bounded closed convex set with $0 \in C$. C has the (weak) drop property if and only if the interior of C is nonempty and the subdifferential mapping ∂p is (n-n) (respectively, (n-w)) upper semicontinuous and norm (respectively, weak) compact nonempty valued on $F(C) = X^* \setminus \{0\}$ (for the definition of upper semicontinuous set valued mappings, see section 3).

In section 3, we study the subdifferential mapping ∂p for unbounded closed convex subsets of X and we show the above results is still true if C is an unbounded closed convex set.

2. Property (ω)

It is known that if C has weak property (α), then $S(x^*, C, \delta)$ is bounded for every $x^* \in F(C)$ and $\delta > 0$. In [7], the fourth author showed that if C has weak property (α), then C is w*-closed. Hence, $S(x^*, C, \delta)$ is weakly compact for every $x^* \in F(C)$ and $\delta > 0$. We have the following theorem.

Theorem 1.

Let C be a closed convex set. If C has weak property (α), then C has property (ω).

Proof. Suppose C has weak property (α). Then $S(x^*,C,\delta)$ is weakly compact for every $x^* \in F(C)$ and $\delta > 0$. So $\omega(S(x^*,C,\delta)) = 0$ for every $\delta > 0$. This implies C has property (ω). □

It is natural to ask whether these two properties are equivalent locally. The following example shows the answer is negative.

EXAMPLE 1: Let $\{N_i\}$ be a partition of $\mathbb{N} \setminus \{1\}$ such that each N_i is an infinite set. Let

$$\begin{aligned} N_i &= \{n_1^{(i)} < n_2^{(i)} < \cdots\} \\ K_i &= \left\{ x_m^{(i)} = (1 - 1/i)e_1 + \sum_{j=1}^{m} e_{n_j^{(i)}} : m \in \mathbb{N} \right\} \subseteq c_o \\ K &= \cup_{i=1}^{\infty} K_i \\ C &= \overline{\text{co}}(K \cup \{e_1\}). \end{aligned}$$

It is easy to see that $e_1^* \in F(C)$ and $\sup\{\langle e_1^*, c\rangle : c \in C\} = 1 = \langle e_1^*, e_1\rangle$. Since $\omega(K_i) = \omega(K_{i'}) > 0$ for every $i, i' \in \mathbb{N}$, and $K_i \subseteq S(e_1^*, C, 1/i)$,

$$\lim_{n\to\infty} \omega(S(e_1^*, C, 1/n)) \geq \omega(K_1) > 0.$$

We claim that if $x_n \in S(e_1^*, C, 1/n)$, then $\{x_n\}$ converges to e_1 weakly.

Without loss of generality, we assume that $x_n = \sum_i \sum_m p_{im}^{(n)} x_m^{(i)} (\in \text{co}(K))$ for some $p_{im}^{(n)} \geq 0$ with $\sum_i \sum_m p_{im}^{(n)} = 1$.

For any $x^* = \sum_{j=1}^{\infty} a_j e_j^* \in \ell_1$, and $\epsilon > 0$, let j_o be any natural number such that $\sum_{j=j_o}^{\infty} |a_j| < \epsilon$. Let h be a natural number such that $\{2, 3, \cdots, j_o\} \subseteq \cup_{i=1}^{h} N_i$ and let $n_o = h/\epsilon$. For each $n > n_o$, we have $x_n \in S(x_o^*, C, 1/n_o)$. Hence, if $n > n_o$, then

$$\begin{aligned} 1 - \frac{1}{n_o} &\leq \sum_i \sum_m (1 - 1/i) p_{im}^{(n)} \\ &\leq \sum_{i\leq h} \sum_m (1 - 1/h) p_{im}^{(n)} + \sum_{i>h} \sum_m p_{im}^{(n)} \\ &= 1 - \frac{1}{h} \sum_{i\leq h} \sum_m p_{im}^{(n)}. \end{aligned}$$

This implies $\sum_{i\leq h}\sum_m p_{im}^{(n)} \leq h/n_o = \epsilon$. Note: if $2 \leq j \leq j_o$ and $i > h$, then $\langle e_j^*, x_m^{(i)}\rangle = 0$. Therefore, if $n > n_o$, then

$$\begin{aligned}|\langle x^*, x_n - e_1\rangle| &\leq \frac{|a_1|}{n_o} + \sum_{j=2}^{\infty} |a_j||\langle e_j^*, x_n\rangle| \\ &\leq |a_1|\epsilon + \sum_{j=2}^{j_o} |a_j||\langle e_j^*, x_n\rangle| + \epsilon \\ &\leq |a_1|\epsilon + \sum_{j=2}^{j_o} |a_j| \sum_{i\leq h}\sum_m p_{im}^{(n)} + \epsilon \leq 2\epsilon.\end{aligned}$$

So $\{x_n\}$ converges to e_1 weakly. And we prove our claim.

3. Subdifferential

Let C be a closed convex set with $0 \in C$, and let p be the gauge induced by C^o. It is easy to see that

(iii) if $x_o^* \in F(C)$, and $x^{**} \in \partial p(x_o^*)$, then $\langle x_o^*, x^{**}\rangle = p(x_o^*)$;

(iv) if $x_o^* \in F(C)$, $x \in C$, and $\langle x, x_o^*\rangle = \sup_{y\in C}\langle y, x_o^*\rangle$, then $x \in \partial p(x_o^*)$.

The following Proposition shows if $x_o^* \in F(C)$, then $\partial p(x_o^*)$ is a w*-closed subset of the weak* closure of C in X^{**}.

Proposition 2.

Let C be a closed convex set with $0 \in C$. For any $x_o^* \in F(C)$,

$$\partial p(x^*) = \cap_{n\in\mathbb{N}} \overline{S(x_o^*, C, 1/n)}^{w^*}.$$

Proof. If $x^{**} \in \cap_{n\in\mathbb{N}} \overline{S(x_o^*, C, 1/n)}^{w^*}$, then for any $n \in \mathbb{N}$ and $x^* \in F(C)$,

$$\langle x^* - x_o^*, x^{**}\rangle = \langle x^*, x^{**}\rangle - \langle x_o^*, x^{**}\rangle \leq p(x^*) - p(x_o^*) + \frac{1}{n}.$$

So $\langle x^* - x_o^*, x^{**}\rangle \leq p(x^*) - p(x_o^*)$. This implies $x^{**} \in \partial p(x_o^*)$.

Suppose that $x^{**} \in \partial p(x_o^*)$. Then $\langle x_o^*, x^{**}\rangle = p(x_o^*)$. So we only need to show that $x^{**} \in \overline{C}^{w^*}$. If $x^{**} \notin \overline{C}^{w^*}$, then by Hahn-Banach Theorem there is $x^* \in X^*$ such that

$$\langle x^*, x^{**}\rangle > \sup_{x\in C}\langle x, x^*\rangle = p(x^*).$$

This implies $\langle x^* - x_o^*, x^{**}\rangle > p(x^*) - p(x_o^*)$, and it is impossible. We prove our proposition. □

Recall a set-valued mapping T from a subset $D \subseteq X^*$ into the collection of subsets of X^{**} is said to be (n-n) (respectively, (n-w)) *upper semicontinuous* at the point $x^* \in D$ if for, every (weak) open set V in X^{**} containing $T(x)$, there is an open neighborhood U of x^* such that $T(U \cap D) \subseteq V$. Let C be a closed convex set with $0 \in C$. We say the subdifferential mapping $x^* \to \partial p(x^*)$ for the gauge p of the polar C^o mapping from $F(C)$ into subsets of X^{**} is *usco* (respectively, *w-usco*) (for (n-n) (respectively, (n-w)) upper semicontinuous and (weak) compact valued) if it is (n-n) (respectively, (n-w)) upper semicontinuous and norm (respectively, weak) compact nonempty valued on $F(C)$.

Remark 1. Proposition 2 shows that

(a) $\partial p(x^*)$ is a convex and weak* closed of C^{oo};

(b) if C is bounded, then $\partial p(x^*)$ is a nonempty, and weak* compact subset of X^{**} for every $x^* \in F(C)$;

(c) if C has weak property (α) and if $x^* \in F(C)$, then $\partial p(x^*)$ is nonempty weakly compact set.

Remark 2. It is natural to require the domain of an upper semicontinuous to be an open set. In the proof of Theorem 3, we shall prove that

(a) if C has weak property (α), then $F(C)$ is open;

(b) if the subdifferential mapping ∂p (associated with C) is w-usco, then C has weak property (α).

So we did not require the domain must be an open set.

Remark 3. Some authors define the subdifferential of p at x_o^* is all linear functionals (not necessary to be continuous) such that

$$p(x_o^*) + \langle x^* - x_o^*, x^{**} \rangle \leq p(x^*)$$

for all $x^* \in F(C)$. It is known that if x_o^* is an interior of $F(C)$, then every element of the subdifferential (under the above definition) of p is a contiouous function. Hence, those two definitions coincide if C has weak property (α).

For more detail about the subdifferential of a convex function, see Chapter 1 [8].

First, let us state the main result in this section.

Theorem 3.

(1) Let C be a closed convex set with $0 \in C$. Then C has property (α) if and only if the subdifferential mapping $x^* \to \partial p(x^*)$ from $F(C)$ into subsets of X^{**} is usco.

(2) Let C be a closed convex set with $0 \in C$. Then C has weak property (α) if and only if the subdifferential mapping $x^* \to \partial p(x^*)$ from $F(C)$ into subsets of X^{**} is w-usco.

Before proving Theorem 3, we give some examples.

EXAMPLE 2: Let $\{e_i : i \in \mathbb{N}\}$ be the natural basis of ℓ_2, and let $C = \text{co}\{e_1, -e_1\}$. Then C is compact and C has the drop property. It is easy to see that

(a) $F(C) = X^* \setminus \{0\}$;

(b) $C^o = \{\sum_{i=1}^{\infty} a_i e_i \in \ell_2 : -1 \leq a_1 \leq 1\}$;

(c) for any $\sum_{i=1}^{\infty} a_i e_i \in F(C)$,

$$\partial p\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = \begin{cases} e_1, & \text{for } a_1 > 0 \\ C & \text{for } a_1 = 0 \\ -e_1 & \text{for } a_1 < 0. \end{cases}$$

So ∂p is usco.

EXAMPLE 3: Let C be the unit ball of c_o. Since C is a bounded set and C is not weakly compact, C does not have the weak drop property. It is easy to see that

(a) C^o is the unit ball of ℓ_1;

(b) for any $x^* = \sum_{i=1}^{\infty} a_i e_i^* \in F(C)$, (where (e_i^*) is the natural basis of ℓ_1)

$$\partial p(x^*) = \Big\{\sum_{i=1}^{\infty} b_i e_i \in \ell_\infty : b_i = \text{sgn}\, a_i \text{ if } a_i \neq 0; \text{ otherwise, } |b_i| \leq 1\Big\}.$$

Since $\partial p(e_1^*)$ is not weakly compact and ∂p is not upper semicontinuous at $\sum_{n=1}^{\infty} \frac{e_n^*}{n^2}$, ∂p is not w-usco.

In the following three examples, $\{e_i : i \in \mathbb{N}\}$ denotes the natural basis of ℓ_2, and Y denotes the subspace $\overline{\text{span}}\{e_i : i \geq 2\}$.

EXAMPLE 4: In [5], S. Rolewicz and the first author proved the set

$$C = \{ae_1 + y : a \geq 0,\ \|y\|^2 \leq a, \text{ and } y \in Y\}$$

has the drop property. One can show that

(a) $C^o \setminus \{0\} = \{ae_1 + y : a < 0,\ y \in Y \text{ and } -\|y\|^2/4a \leq 1\}$, and $F(C) = \{ae_1 + y : a < 0, \text{ and } y \in Y\}$;

(b) if $y \in Y$ and $ae_1 + y \in F(C)$, then

$$\partial p(ae_1 + y) = \left\{ \left(\frac{\|y\|}{2a}\right)^2 e_1 + \frac{-y}{2a} \right\}.$$

So ∂p is usco.

Let C be a closed convex subset of X. A ray $r = \{x + \lambda y : \lambda > 0\}$ is said to be an *asymptote* if $r \cap C = \emptyset$, and for any $\epsilon > 0$ there is $N > 0$ such that $\lambda > N$ implies $d(x + \lambda y, C) = \inf\{\|x + \lambda y - c\| : c \in C\} < \epsilon$.

EXAMPLE 5: Let $C = \{ae_1 + y : a \geq 0,\ y \in Y \text{ and } \|y\| \leq a/(a+1)\}$. One can easily show that

(a) $\{e_2 + \lambda e_1 : \lambda \geq 0\}$ is an asymptote of C;

(b) $F(C) = \{ae_1 + y : a \leq 0 \text{ and } y \in Y\} \setminus \{0\}$;

(c) if $x^* = ae_1 + y \in F(C)$, then

$$\partial p(x^*) = \begin{cases} \emptyset, & \text{if } a = 0 \\ 0, & \text{if } \sqrt{\frac{\|y\|}{-a}} \leq 1 \\ \left(\sqrt{\frac{\|y\|}{-a}} - 1\right)e_1 + \left(1 - \sqrt{\frac{-a}{\|y\|}}\right)\frac{y}{\|y\|}, & \text{otherwise .} \end{cases}$$

So ∂p is not w-usco.

EXAMPLE 6: Let $C = \{ae_1 + y : a \leq 1 \text{ and } y \in Y\}$. For any $y \in Y$, $e_1 + y$ is on the boundary of C. One can easily see that

$$F(C) = \{ae_1 : a > 0\}$$
$$\partial p(e_1) = \{e_1 + y : y \in Y\}.$$

So ∂p is (n-n) upper semicontinuous, but $\partial p(e_1)$ is not weakly compact. So it is not w-usco.

Remark 4. Let T be a linear operator from X_1 into X_2 such that $T(X_1)$ is dense in X_2. Let C be any closed convex subset of X_1 with property (α) (respectively, weak property (α)). Since T^* is one-to-one, $T^*(F(T(C))) \subset F(C)$. This implies $T(C)$ has property (α) (respectively, weak property (α)).

EXAMPLE 7: Let $\{e_i : i \in \mathbb{N}\}$ be the natural basis of c_o, and let $C = \{ \sum_{i=1}^{\infty} a_i e_i : \sum_{i=1}^{\infty} a_i^2 \leq 1\}$. It is easy to see that

(a) C is a noncompact closed convex set with empty interior, and so C does not have the drop property;

(b) since the unit ball of ℓ_2 has property (α), (by Remark 4) C has property (α);

(c) if $x^* = \sum_{i=1}^{\infty} a_i e_i^* \in F(C)(= \ell_1 \setminus \{0\})$, then

$$\partial p(x^*) = \left\{ \frac{\sum_{i=1}^{\infty} a_i e_i}{\left(\sum_{i=1}^{\infty} a_i^2\right)^{1/2}} \right\}.$$

So ∂p is usco.

Suppose that X is reflexive, and C is a closed convex subset of X with nonempty interior. It is known [6] that if C has the drop property, then C does not contain any asymptote. The following example shows the assumption "$\operatorname{int}(C) \neq \emptyset$"can not be dropped.

EXAMPLE 8: Let $\{e_i : i \in \mathbb{N}\}$ be the natural basis of c_o, and

$$C = \left\{ a e_1 + \sum_{i=2}^{\infty} y_i e_i : a \geq 0, \text{ and } \sum_{i=2}^{\infty} y_i^2 \leq a, \right\}.$$

By Example 4 and Remark 4, C has the property (α). It is easy to see the interior of C is an empty set.

Let $y = \sum_{i=2}^{\infty} e_i/\sqrt{n}$. Then $\{y + \lambda e_1 : \lambda \geq 0\} \cap C = \emptyset$. One can easily verify that $\{y + \lambda e_1 : \lambda \geq 0\}$ is an asymptote of C.

The following example shows there is a closed convex set which satisfies the following condition:

(a) C does not contain any asymptote, but there are x_1^*, $x_2^* \in F(C)$ such that $\partial p(x_1^*) = \emptyset$ and $\partial p(x_2^*)$ is not bounded;

(b) $F(C)$ is nowhere dense subset of X^*.

EXAMPLE 9: Let $\{e_n : n \in \mathbb{N}\}$ be the natural basis of ℓ_2, and let $C = \overline{\text{co}}\{\pm n e_n : n \in \mathbb{N}\}$. It is known that C does not have weak property (α). Indeed, C satisfies the following properties:

(a) C does not contain any asymptote, but there are x_1^*, $x_2^* \in F(C)$ such that $\partial p(x_1^*) = \emptyset$ and $\partial p(x_2^*)$ is not bounded;

(b) $F(C)$ is nowhere dense subset of X^*.

Proof. For any $x^* = \sum_{k=1}^{\infty} a_k e_k^* \in \ell_2$,

$$\sup\{\langle x, x^*\rangle : x \in C\} = \sup\{k|a_k| : k \in \mathbb{N}\}.$$

Hence,

$$F(C) = \left\{ \sum a_k e_k^* : \infty > \sup\{k|a_k| : k \in \mathbb{N}\} > 0 \right\},$$

and so $F(C)$ has empty interior. One easily see that if $x^* = \sum_{k=1}^{\infty} a_k e_k \in F(C)$, then

$$\partial p(x^*) = \overline{\text{co}}\{ \text{ sgn } (a_k) k e_k : k|a_k| = \sup\{j|a_j| : j \in \mathbb{N}\}\}.$$

Hence, if $x_1^* = \sum_{k=1}^{\infty} e_k/(k+1)$ and $x_1^* = \sum_{k=1}^{\infty} e_k/k$, then $\partial p(x_1^*)$ is an empty set and $\partial p(x_2^*) = \overline{\text{co}}\{k e_k : k \in \mathbb{N}\}$ is an unbounded set. □

Remark 5. Let C be a proper closed convex subset of X and x_o^* be any element in $F(C)$. By the well-known Bishop-Phelps Theorem, for any $\epsilon > 0$, there exist $x^* \in X^*$ and $x_o \in C$ such that $\langle x_o, x^*\rangle = \sup\{\langle x, x^*\rangle : x \in C\}$ and $\|x_o^* - x^*\| \leq \epsilon$. Hence, $x_o \in \partial p(x^*)$, and $\partial p(x^*) \neq \emptyset$. (See p. 51 [8] or p. 177 [2].) This implies that if $\partial p(x_o^*) = \emptyset$, then ∂p is not upper semicontinuous at x_o^*.

Proof of Theorem 3. Suppose C has (weak) property (α). By remark 1.c, $\partial p(x^*)$ is nonempty norm (respectively, weakly) compact for every $x^* \in F(C)$. Hence, we only need to show ∂p is (n-n) (respectively, (n-w)) upper semicontinuous on $F(C)$.

Let x^* be any element of $F(C)$ such that $x \in C$ and $\langle x, x^*\rangle = M = \sup_{y \in C} \langle y, x^*\rangle$. It is known that $S(x^*, C, 1)$ is a bounded set. Hence, there exists $N > 1$ such that $\|y\| < N/2$ whenever $y \in S(x^*, C, 1)$. If $\|y^* - x^*\| \leq 1/(nN)$ $(n \geq 4)$, then

(a) $\langle x, y^*\rangle \geq \langle x, x^*\rangle + \langle x, y^* - x^*\rangle \geq M - 1/(2n)$;

(b) for any $y \in S(x^*, C, 1) \setminus S(x^*, C, 3/n)$,

$$\langle y, y^*\rangle = \langle y, x^*\rangle + \langle y, y^* - x^*\rangle \leq M - \frac{3}{n} + \frac{N}{nN} = M - \frac{2}{n}.$$

If $y \notin S(x^*, C, 1)$, then there exists β, $0 < \beta < 1$, such that $\beta x + (1-\beta)y \in S(x^*, C, 1) \setminus S(x^*, C, 3/n)$. This implies

(c) if $y \notin S(x^*, C, 3/n)$, then $\langle y, y^* \rangle \leq M - 2/n$;

(d) $y^* \in F(C)$ (so $F(C)$ is an open subset of X^*);

(e) $\partial p(y^*) \subseteq S(x^*, C, 3/n)$.

But C has (weak) property (α). Hence, if $y_n^* \in F(C)$, $\|y_n^* - x^*\| < 1/n$, and $y_n \in \partial p(y_n^*)$, then $\{y_n\}$ contains a subsequence which converges (weakly) to some element in $\partial p(x^*)$. So ∂p is usco (respectively, w-usco) on $F(C)$.

Assume that the subdifferential mapping $x^* \to \partial p(x^*)$ from $F(C)$ into subsets of X^{**} is usco (w-usco). For $x^* \in F(C)$, let $x_n \in S(x^*, C, 1/n)$ be any sequence. By the Bishop-Phelps Theorem [2, p.177], there exist $y_n \in C$ and $y_n^* \in F(C)$ such that

(g) $y_n \in \partial p(y_n^*)$;

(h) $\lim_{n\to\infty} \|y_n - x_n\| = 0$ and $\lim_{n\to\infty} \|y_n^* - x^*\| = 0$.

We only need to show $\{y_n\}$ contains a (weakly) convergent subsequence. First, we claim that $A = \{y_n : n \in \mathbb{N}\} \cup \partial p(x^*)$ is (weak) compact.

Let $\{O_i : i \in I\}$ be any (weak) open covering of A. Since $\partial p(x^*)$ is (weak) compact, there exist a finite subcovering $\{O_i : 1 \leq i \leq n\}$ of $\partial p(x^*)$ and an open (respectively, a weak open) neighborhood W of 0 such that

$$\partial p(x^*) + W \subseteq \cup_{1 \leq i \leq n} O_i.$$

But ∂p is usco (respectively, w-usco). There exists N such that if $n > N$, then $y_n \in \partial p(x^*) + W$. This implies A is a (weakly) compact set.

X is (weakly) closed subset of X^{**}. So $A \cap X$ is (weak) compact. Hence, $\{y_n\}$ contains a (weak) convergent sequence, and C has (weak) property (α). □

Combining Theorem A, Theorem B, Theorem 1, and Theorem 3, we have the following theorem.

Theorem 4.

Let C be a closed convex set such that $0 \in \text{int}(C)$ (the set of all interior points of C). Then the following are equivalent.

(1) C has the drop property (respectively, weak drop property);

(2) the subdifferential mapping $x^* \to \partial p(x^*)$ (from $F(C)$ into subsets of X^{**}) is usco (respectively, w-usco);

(3) C has property (α) (respectively, weak property (α) or property (ω)).

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