*Collect. Math.* **42**, 2 (1991), 171–176


# On jets of surfaces

FERNANDO ETAYO GORDEJUELA

*Dpto. Geometría y Topología, Facultad de Matemáticas, Universidad Complutense, 28040 Madrid, Spain*

Received March 19, 1991

ABSTRACT

We study the 2-jet bundle of mappings of the real plane into a manifold. We shall prove that there exists an imbedding of this 2-jet bundle into a suitable first order jet bundle, in such a way that its image is the set of fixed points of a canonical automorphism of the biggest jet bundle.

## 1. Introduction

Let $M$ be a paracompact smooth real manifold of dimension $n \geq 2$. Let $J_0^k(\mathbb{R}^p, M)$ be the *tangent bundle of $p^k$-velocities* ([2, 4, 6, 8, 9, 10]), i. e., the $k$-jet bundle of mappings from $\mathbb{R}^p$ to $M$ with source $0 \in R^p$. As is well known, $\pi: J_0^k(\mathbb{R}^p, M) \longrightarrow M$ is a fibre bundle, setting $\pi([\phi]_k) = \phi(0)$, where $[\phi]_k$ stands for the $k$-jet of $\phi$.

In particular, if $p = k = 1$, we have the tangent bundle of $M$, and, if $p = 1$, $k = 2$, the *second tangent bundle*. This latter one satisfies the following property [1]: $J_0^2(\mathbb{R}, M)$ can be immersed in $TTM$ as the invariant set of the *canonical involution* ([1,5,8]). This result has been generalized in [7] to the 1-jet bundle of sections of a fibre bundle.

In this paper, we shall prove that there exists a canonical involution $\alpha$ in $J_0^1\left(\mathbb{R}^2, J_0^1(\mathbb{R}^2, M)\right)$ such that $J_0^2(\mathbb{R}^2, M)$ can be immersed in $J_0^1\left(\mathbb{R}^2, J_0^1(\mathbb{R}^2, M)\right)$ as the invariant set of $\alpha$. The same result is true for $k > 2$, but, for the sake of simplicity, we shall study only the case $k = 2$. The $k$-jet bundle $J_0^k(\mathbb{R}^2, M) \longrightarrow M$ will be called the *$k$-jet bundle of surfaces*, because the image of $\phi$ is a surface in $M$.

Notations in the work are those we have used in [5]. We wish to thank prof. Ignacio Sols for proposing to us this question

## 2. The results

In [8] Morimoto gives the following proposition (that we study in the case $k = 1$, $p = 2$):

**Proposition 1**

*There exists a unique automorphism $\alpha$ in $J_0^1\Big(\mathbb{R}^2, J_0^1(\mathbb{R}^2, M)\Big)$ such that:*

*a) $T_2\pi \circ \alpha = \tilde{\pi}$,*

*b) $\tilde{\pi} \circ \alpha = T_2\pi$,*

*c) $\left(f^{(\mu)}\right)^{(\lambda)} \circ \alpha = \left(f^{(\lambda)}\right)^{(\mu)}$, for all $\lambda, \mu \in N(2,1)$, and all function $f$ of $M$, where $\tilde{\pi}$ is the canonical projection, $T_2\pi$ is the induced map, given by*

$$(T_2\pi)\Big(\Big[[\phi]_1\Big]_1\Big) = \Big[\pi \circ [\phi]_1\Big]_1,$$

*$N(2,1) = \{ (m,n) \in \mathbb{Z} \times \mathbb{Z} \mid m \geq 0, n \geq 0, m+n \leq 1 \}$, $f^{(\alpha)}$ is the function on $J_0^1(\mathbb{R}^2, M)$ given by*

$$f^{(\alpha)}([\phi]_1) = \frac{1}{\alpha!}\Big(\Big(\frac{\partial}{\partial t}\Big)^{\alpha}(f \circ \phi)\Big)_{t=0}$$

*with $t = (t_1, t_2)$ the canonical coordinates in $\mathbb{R}^2$, and $\alpha = (\alpha_1, \alpha_2)$.*

It is easy to see that $\alpha \circ \alpha$ is the identity. We shall call $\alpha$ the *canonical involution of $J_0^1\Big(\mathbb{R}^2, J_0^1(\mathbb{R}^2, M)\Big)$.*

**Proposition 2**

*The map $i$*

$$i \colon J_0^2(\mathbb{R}^2, M) \longrightarrow J_0^1\Big(\mathbb{R}^2, J_0^1(\mathbb{R}^2, M)\Big)$$

*given by $i([\phi]_2) = \Big[[\phi]_1\Big]_1$ is an injective immersion.*

The proof is easy.

*Remark.* This kind of injection is often used in the theory of jet bundles of sections ([7]).

The most difficult problem is the following

**Proposition 3**

*With the above notation, $i\Big(J_0^2(\mathbb{R}^2, M)\Big)$ is the set of fixed points of the canonical involution $\alpha$.*

*Proof.* We have to take suitable local coordinates in all the manifolds, in order to obtain the local expressions of the injection $i$, the involution $\alpha$, the maps $T_2\pi$ and $\tilde{\pi}$, and the lifted functions $\left(f^{(\lambda)}\right)^{(\mu)}$.

Let $(x^1, \dots, x^n) = (x^i)$ be local coordinates in $M$, $n$ being the dimension of $M$. Then we obtain an induced local chart in $J_0^1(\mathbb{R}^2, M)$ given by $(x^i, y^i, z^i)$, where

$$x^i([\phi]_1) = (x^i \circ \phi)(0) = \phi^i(0).$$
$$y^i([\phi]_1) = \frac{\partial \phi^i}{\partial t_1}(0)$$
$$z^i([\phi]_1) = \frac{\partial \phi^i}{\partial t_2}(0)$$

Using this idea, we obtain coordinates

$$(x^i, y^i, z^i, x^{i+n}, x^{i+2n}, y^{i+n}, y^{i+2n}, z^{i+n}, z^{i+2n})$$

in $J_0^1\left(\mathbb{R}^2, J_0^1(\mathbb{R}^2, M)\right)$, where

$$x^{i+n}([\phi]_1) = \frac{\partial \phi^i}{\partial t_1}(0)$$
$$x^{i+2n}([\phi]_1) = \frac{\partial \phi^i}{\partial t_2}(0)$$
$$y^{i+n}([\phi]_1) = \frac{\partial \phi^{i+n}}{\partial t_1}(0)$$
$$y^{i+2n}([\phi]_1) = \frac{\partial \phi^{i+n}}{\partial t_2}(0)$$
$$z^{i+n}([\phi]_1) = \frac{\partial \phi^{i+2n}}{\partial t_1}(0)$$
$$z^{i+2n}([\phi]_1) = \frac{\partial \phi^{i+2n}}{\partial t_2}(0)$$

where $x^i \circ \phi = \phi^i$, $y^i \circ \phi = \phi^{i+n}$, $z^i \circ \phi = \phi^{i+2n}$.

And we obtain for $J_0^2(\mathbb{R}^2, M)$, induced local coordinates

$$(x^i, y^i, z^i, a^i, b^i, c^i)$$

where the three last ones are the second partial derivatives

$$a^i([\phi]_1) = \frac{\partial^2 \phi^i}{(\partial t_1)^2}(0), \qquad b^i([\phi]_1) = \frac{\partial^2 \phi^i}{\partial t_1\, \partial t_2}(0), \qquad c^i([\phi]_1) = \frac{\partial^2 \phi^i}{(\partial t_2)^2}(0).$$

We introduce the following notation

$$\xi = (x^i, y^i, z^i, x^{i+n}, x^{i+2n}, y^{i+n}, y^{i+2n}, z^{i+n}, z^{i+2n}).$$

Then we obtain

$$\tilde{\pi}(\xi) = (x^i, y^i, z^i),$$

$$(T_2\pi)(\xi) = (x^i, x^{i+n}, x^{i+2n}),$$

$$i([\phi]_2) = \left(\phi^i, \frac{\partial \phi^i}{\partial t_1}, \frac{\partial \phi^i}{\partial t_2}, \frac{\partial \phi^i}{\partial t_1}, \frac{\partial \phi^i}{\partial t_2}, \frac{\partial^2 \phi^i}{(\partial t_1)^2}, \frac{\partial^2 \phi^i}{\partial t_1\, \partial t_2}, \frac{\partial^2 \phi^i}{\partial t_1\, \partial t_2}, \frac{\partial^2 \phi^i}{(\partial t_2)^2}\right)$$

and then,

$$i\Big(J_0^2(\mathbb{R}^2, M)\Big) = \{(x^i, y^i, z^i, y^i, z^i, y^{i+n}, y^{i+2n}, y^{i+2n}, z^{i+2n})\},$$

and the nine types of lifted functions are:

$$\left(f^{(0,0)}\right)^{(0,0)}(\xi) = f(x^i)$$

$$\left(f^{(0,0)}\right)^{(1,0)}(\xi) = \frac{\partial f}{\partial x^i} \cdot x^{i+n}$$

$$\left(f^{(0,0)}\right)^{(0,1)}(\xi) = \frac{\partial f}{\partial x^i} \cdot x^{i+2n}$$

$$\left(f^{(1,0)}\right)^{(0,0)}(\xi) = \frac{\partial f}{\partial x^i} \cdot y^i$$

$$\left(f^{(1,0)}\right)^{(1,0)}(\xi) = \frac{\partial^2 f}{(\partial x^i)^2} \cdot y^i x^{i+n} + \frac{\partial f}{\partial x^i} \cdot y^{i+n}$$

$$\left(f^{(1,0)}\right)^{(0,1)}(\xi) = \frac{\partial^2 f}{(\partial x^i)^2} \cdot y^i x^{i+2n} + \frac{\partial f}{\partial x^i} \cdot y^{i+2n}$$

$$\left(f^{(0,1)}\right)^{(0,0)}(\xi) = \frac{\partial f}{\partial x^i} \cdot z^i$$

$$\left(f^{(0,1)}\right)^{(1,0)}(\xi) = \frac{\partial^2 f}{(\partial x^i)^2} \cdot z^i x^{i+n} + \frac{\partial f}{\partial x^i} \cdot z^{i+n}$$

$$\left(f^{(0,1)}\right)^{(0,1)}(\xi) = \frac{\partial^2 f}{(\partial x^i)^2} \cdot z^i x^{i+2n} + \frac{\partial f}{\partial x^i} \cdot z^{i+2n}$$

Using proposition 1, a straightforward calculation shows that

$$\alpha(x^i, y^i, z^i, x^{i+n}, x^{i+2n}, y^{i+n}, y^{i+2n}, z^{i+n}, z^{i+2n})$$
$$=(x^i, x^{i+n}, x^{i+2n}, y^i, z^i, y^{i+n}, z^{i+n}, y^{i+2n}, z^{i+2n}).$$

Then the set of fixed points is $i\Big(J_0^2(\mathbb{R}^2, M)\Big)$, as we wanted. □

*Remark.* Condition c) of proposition 1 is sufficient to obtain $\alpha$.

We can obtain more information on these jets of surfaces.

**Proposition 4**

*The canonical coordinates in $\mathbb{R}^2$ define a diffeomorphism*

$$\beta: J_0^1(\mathbb{R}^2, M) \longrightarrow TM \oplus TM.$$

*Proof.* We define $\beta([\phi]_1) = \left(\phi_*(\frac{\partial}{\partial t_1}), \phi_*(\frac{\partial}{\partial t_2})\right)$. It is easy to see that $\beta$ is well defined and injective.

Let $v$ and $w$ be two vectors in $T_xM$, for some $x \in M$. Let $\Gamma$ be a linear connection on $M$ (it exists, because $M$ is paracompact) and $V$ a normal neighbourhood of $x \in M$. Let $U$ be a small neighbourhood of $0 \in \mathbb{R}^2$, such that for all $(t_1, t_2) \in U$, $t_1 v + t_2 w \in V$. Then, we define $\phi: U \subset \mathbb{R}^2 \longrightarrow M$, by setting $\phi(t_1, t_2) = \exp(t_1 v + t_2 w)$. It is obvious that $\beta([\phi]_1) = (v, w)$, and that the construction is independent of the chosen connection. □

*Remark.* The 1-jet of the surface $[\phi]_1$ is given by $\phi(0)$ and the velocities of the curves images of the axis of $\mathbb{R}^2$. These vectors generate the tangent plane to the surface $\mathrm{im}(\phi)$, that can be degenerate, but their information is strong: it is possible that different 1-jets of surfaces define the same tangent plane.

**Corollary 5**

*Given a linear connection on $M$, there exists a canonical diffeomorphism between $J_0^1(\mathbb{R}^2, M)$ and $J_0^2(\mathbb{R}, M)$.*

*Proof.* It is a consequence of the above proposition and the fact that a linear connection defines a diffeomorphism [3]

$$\pi_{TM} \oplus \kappa: J_0^2(\mathbb{R}, M) \longrightarrow TM \oplus TM$$

where $\kappa$ is the *connection map* [5] and $\pi_{TM}$ the tangent projection $T(TM) \longrightarrow TM$ when $J_0^2(\mathbb{R}, M)$ is included in $TTM$. □

**Corollary 6**

*There exists a canonical diffeomorphism*

$$J_0^1\left(\mathbb{R}^2, J_0^1(\mathbb{R}^2, M)\right) \xrightarrow{\simeq} T\left(TM \oplus TM\right) \oplus T\left(TM \oplus TM\right).$$

## References

1. A. Besse, *Manifolds all of whose geodesic are closed*, Springer, Berlin-Heidelberg, 1978.
2. L. A. Cordero, C. T. J. Dodson and M. de León, *Differential Geometry of Frame Bundles*, Kluwer, Dordretch, 1989.
3. C. T. J. Dodson and M. S. Radivoiovici, Second Order Tangent Structures, *Int. Jour. Theor. Physics* **21** (1982), 151–161.
4. C. Ehresmann, Les prolongements d'un space fibré différentiable, *C. R. Acad. Sci. Paris* **240** (1955), 1755-1757.
5. F. Etayo, Lifts and isomorphisms of commutation in bundles of jets, *Coll. Math.* **39** (1988), 231–247.
6. M. de León and P. R. Rodriques, *Generalized Classical Mechanics and Field Theory*, North-Holland, Amstrdam, 1985.
7. M. Modugno, Jet involution and prolongations of connections, *Casopis Pest. Mat.* **114** (1989), 356–365.
8. A. Morimoto, *Prolongation of Geometric Structures*, Math. Inst. Nagoya, Nagiya Univ., 1969.
9. W. Tulczyjew, Les jets généralisés, *C. R. Acad. Sci. Paris* **281** A (1975), 349–352.
10. W. Tulczyjew, The Lagrange complex, *Bull. Soc. Math. France* **105** (1977), 419–431.