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On finite soluble groups verifying an extremal condition on subgroups

FERNANDO TUCCILLO

Dipartimento di Matematica e Applicazioni R Caccioppoli"

Università di Napoli, Via Mezzocannone 8, 80134, Napoli (Italia)

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ABSTRACT

We classify the finite soluble groups satisfying the following condition: if H is a subgroup of G and H is not nilpotent, then the Fitting subgroup of H is the centralizer in H of its derived subgroup H'.

1. Introduction

In this paper we classify all finite soluble groups G such that for every non-nilpotent subgroup H the centralizer of derived subgroup H' in H is the Fitting subgroup of H. We will call such groups F-extremal groups. This on the grounds of the following definition of ρ-extremal property.

Let χ be a group-theoretical class and $\rho_\chi(G)$ the χ-radical of a group G. If f is a function that assigns to each group G a subgroup $f(G)$ of $\rho_\chi(G)$, we will say that a group G is ρ_χ-extremal with respect to f, if the following implication holds:

$$H \leq G \quad \text{and} \quad \rho_\chi(H) \neq H \qquad \Longrightarrow \qquad \rho_\chi(H) = f(H)$$

The F-extremality is an instance of ρ_χ-extremal property. Choose: χ the class N of nilpotent group, f the function that assigns to each group G the centralizer of its derived subgroup G'. Terminology is obviously referred to notation $\rho_N(G) = F(G)$ Fitting subgroup of a group G.

The F-extremality is not inherited by factor groups: a free group is F-extremal. The following proposition assures that the property is inherited by factor groups for finite groups.

1.1 Proposition

Let G be a finite F-extremal group and H a normal subgroup of G. Then G/H is F-extremal.

Proof. Let G be a counterexample of least order and let N be a minimal normal subgroup of G such that G/N is not F-extremal. The minimality of G assures that N is contained in the Frattini subgroup $\Phi(G)$ of G and that $C_{G/N}(NG'/N)$ is a proper subgroup of $F(G/N)$. Since $N \leq \Phi(G)$ we have $F(G/N) = F(G)/N$. On the other hand, as G is F-extremal and is not nilpotent, we get $C_G(G') = F(G)$; it follows the contradiction $F(G/N) = C_{G/N}(NG'/N)$. □

We will often use the following proposition:

1.2 Proposition

Let G be a soluble F-extremal group. If G is not nilpotent then $G' \leq Z(F(G))$ and hence G is metabelian.

Proof. The statement is obvious, if we recall that, if G is a soluble group, then $C_G(F(G)) = Z(F(G))$ (see for instance [2], 5.4.4). □

All the groups considered are finite. The notation is generally standard, but we also use the following: if G is a finite group,

$$\pi(G) = \{p \in \Pi \mid p \mid |G|\}$$
$$\omega(G) = \{p \in \pi(G) \mid \mathrm{cl}(G_p) \leq 2\}$$

where Π is the set of primes and $G_p \in \mathrm{Syl}_p(G)$.

2. Finite soluble F-extremal groups

By means of the following propositions we reduce the problem of the classification of soluble non-nilpotent F-extremal groups to the case of groups whose Sylow subgroups have class at most two.

2.1 Proposition

Let G be a soluble F-extremal group and let G_p be a Sylow p-subgroup of G of class ≥ 3. Then $N_G(G_p)$ is nilpotent.

Proof. The statement easily follows from 1.2. □

2.1.1 Corollary

Let G be a soluble F-extremal group. If all the Sylow subgroups of G have class ≥ 3, then G is nilpotent.

Proof. By 2.1 the normalizer of a Syllow subgroup of G is a Carter subgroup of G. It follows that the normalizers of the Sylow subgroups of G are conjugate and hence G is nilpotent. □

2.2 Proposition

Let G be a soluble F-extremal group. Put $\omega = \omega(G)$ and let G_ω be a Hall ω-subgroup of G. Then:

(i) G_ω is normal in G;

(ii) $G_{\omega'}$ is nilpotent.

Proof.

(i) We argue by induction on $|G|$. If $\omega = \emptyset$ or $\pi(G)$, the statement is obvious. If the order of ω or that of $\omega' \cap \pi(G)$ is at least 2, the statement easily follows from the inductive hypothesis. Suppose that $G = G_p G_q$, with $\omega = \{p\}$ and $\{q\} = \omega' \cap \pi(g)$. If $O_p(G) \neq 1$, the statement immediately follows from the inductive hypothesis. Suppose that $O_p(G) = 1$ and so $F(G) = O_q(G)$. Since G is F-extremal and is not nilpotent, we have $G' \leq Z(O_q(G))$ and therefore $G_q \triangleleft G$. It follows, by 2.1, that G is nilpotent: a contradiction.

(ii) It immediately follows from 2.1.1. □

2.3 Proposition

Let G be a soluble F-extremal group. If $cl(G_p) \leq 2$ for every $p \in \pi(G)$, then

$$\langle (G_p)' \mid p \in \pi(G) \rangle \leq Z(G).$$

Thus $G/Z(G)$ is an A-group (that is, the Sylow subgroups of $G/Z(G)$ are abelian).

Proof. We argue by induction on $|G|$. The statement is obvious if G is nilpotent or if G is an A-group. Suppose that G is neither an A-group nor nilpotent and let G_p be a non-abelian Sylow p-subgroup of G. If the order of $p' \cap \pi(G)$ is at least 2, the statement easily follows from the inductive hypothesis. Suppose that $G = G_pG_q$ (q prime $\neq p$). If N is a minimal normal subgroup of G, the inductive hypothesis provides

$$\left[G, (G_p)'\right] \leq N$$

and therefore we get the statement, if G has at least two minimal normal subgroups. Suppose that N is the only minimal normal subgroup of G. Since G is metabelian, we have

$$F(G) = G_p \qquad \text{and} \qquad N \leq \Omega_1((G_p)') = \left\{x \in (G_p)' \mid x^p = 1\right\} \leq Z(G_p).$$

It follows (Maschke's theorem) that $N = \Omega_1((G_p)')$.

On the other hand we have (Maschke's theorem) that

$$G_p \big/ \Phi(G_p) = G'\Phi(G_p) \big/ \Phi(G_p) \times L \big/ \Phi(G_p),$$

where L is normal in G. It follows $G_p = G'L$. Since G is not nilpotent and hence $G' \not\leq \Phi(G_p)$, L is a proper subgroup of G_p; moreover, as $G' \leq Z(G_p)$ and G_p is not abelian, L is not abelian. Then the inductive hypothesis assures that $L' \leq Z(G)$; it follows

$$N = \Omega_1((G_p)') \leq Z(G)$$

and so, as N is the only minimal normal subgroup of G, $(G_p)'$ is cyclic. We have then that the automorphism group induced on $(G_p)'$ by G_q is trivial, since it acts trivially on $\Omega_1((G_p)')$, and so $(G_p)' \leq Z(G)$. □

2.4 Proposition

Let G be a central extension by an A-group. Then G is F-extremal if and only if G is metabelian.

Proof. The necessity of the condition is obvious. Conversely, let G be metabelian and let A be a subgroup of $Z(G)$ such that G/A is an A-group. By induction on the group order, proper subgroups of G are F-extremal and therefore it is sufficient to show that $G' \leq Z(F(G))$, whenever G is not nilpotent. If $A = 1$, G is a metabelian A-group and so $F(G) = G' \times Z(G)$ (see for instance [1], VI, 14.7 Satz). Suppose that $A \neq 1$. If N is a minimal normal subgroup of G, the inductive hypothesis implies that $[G', F(G)] \leq N$ and therefore we get the statement, if G has at least two minimal normal subgroups. Suppose that N is the only minimal subgroup of G. We have obviously that the order of N is a prime p, A is a cyclic p-group and $F(G) = G_p$. Let $M = G_{p'}G'$. M is an A-group and so $M' \cap Z(M) = 1$; it follows, as $N \leq Z(M)$, that M is abelian and therefore $M = G'$. We have then $G = G_p$ and so, obviously, the statement. □

2.4.1 Corollary

The soluble F-extremal groups, whose Sylow subgroups have class at most two, are all the metabelian central extensions by an A-group.

2.5 Classification of the soluble F-extremal groups

2.5.1 Proposition

Let G be a soluble F-extremal group. Let H be a Hall ω-subgroup of G where $\omega = \omega(G)$. Then $H' \cap Z(H) \leq Z(G)$.

Proof. We argue by induction on $|G|$. If G is nilpotent or $H = G$, the statement is obvious. Suppose that G is not nilpotent, $H < G$ and $H' \cap Z(H) \neq 1$. Denote by N a minimal normal subgroup of G contained in H. The induction hypothesis may be applied to G/N to give

$$\Big[G, H' \cap Z(H)\Big] \leq N.$$

Thus we can assume that N is the unique minimal normal subgroup of G contained in H. As $G' \leq Z(F(G))$, we get

$$H' \leq G_p \in \mathrm{Syl}_p(H),$$

for some $p \in \omega$. On the other hand we can assume, by inductive hypothesis, that

$$G_{\omega'} = G_q \in \mathrm{Syl}_q(G) \qquad (q \text{ prime}).$$

Since $\mathrm{cl}(G_q) \geq 3$ and $G' \leq Z(F(G))$, there exists a minimal normal subgroup L of G contained in G_q. If $\mathrm{cl}(G_q/L) \geq 3$, the statement easily follows from the inductive hypothesis. Suppose that $\mathrm{cl}(G_q/L) \leq 2$. By 2.3 we obtain $[G, (G_p)'] \leq L$ and obviously the statement, if $H = G_p$, so we can assume that $G_p < H$. The inductive hypothesis, applied to G_pG_q, implies that $G_q \leq C_G((G_p)')$ and therefore, as $N \leq H' \cap Z(H)$, we have $N \leq Z(G)$. It follows, as

$$\Big[G_q, H' \cap Z(H)\Big] \leq N,$$

that the automorphism group induced on $H' \cap Z(H)$ by G_q is a p-group; it follows that G_q centralizes $H' \cap Z(H)$ and so we get the statement. □

2.5.2 Proposition

Let G be a soluble F-extremal group. Write $H = G_\omega$, $K = G_{\omega'}$ and $C = C_G(H)$, where $\omega = \omega(G)$. If T/C is a Carter subgroup of G/C, then:

(i) T is nilpotent;

(ii) $(G/C)' = C \times (H \cap G')/C$;

(iii) $H \cap G' \cap T = H' \cap Z(H) \simeq (G/C)' \cap Z(G/C)$;

(iv) $G = (H \cap G')\,T$;

(v) If G is not nilpotent, $K' \leq Z(C)$.

Proof. If G is nilpotent, the statement is obvious. Suppose that G is not nilpotent. Since $C = K \cap F(G)$ and $G' \leq Z(F(G))$, we have that $K' \leq Z(C)$ and hence K/C is an A-group. Thus the Sylow subgroups of G/C have class at most two and therefore G/C is a metabelian central extension by an A-group (see 2.4.1). We have then

$$G/C = (G/C)'\,T/C.$$

It follows, since $G' = (H \cap G') \times K'$ and hence $(G/C)' = C \times (H \cap G')/C$, that $G = (H \cap G')\,T$.

On the other hand, as G/H is nilpotent, so is also $G/H \cap G'$ and so, obviously, T. Finally, since $H' \cap Z(H) \leq Z(G)$ (see 2.5.1), we have that $H' \cap Z(H) \leq T$ and that $T/C \times (H' \cap Z(H))$ is a Carter subgroup of $G/C \times (H' \cap Z(H))$.

On the other hand, $H/H' \cap Z(H)$ and K/C are A-groups and hence so is also

$$G/C \times (H' \cap Z(H)).$$

It follows that

$$\left(G/C \times (H' \cap Z(H))\right)' \cap \left(T/C \times (H' \cap Z(H))\right) = 1,$$

from which, obviously, we get

$$H \cap G' \cap T = H' \cap Z(H) \simeq (G/C)' \cap T/C = (G/C)' \cap Z(G/C). \qquad \square$$

2.5.3 A class of soluble F-extremal groups

Let E be a metabelian central extension by an A-group and E be not nilpotent. If D is a Carter subgroup of E, we have $E = D\,E'$. Let now T be a nilpotent group satisfying the following condition. With $T = T_\omega \times T_{\omega'}$, where $\omega = \omega(T)$, there exists a subgroup C of $T_{\omega'}$ such that $(T_{\omega'})' \leq Z(C)$ and $T/C \simeq D$. If $\sigma: D \longrightarrow T/C$ is an isomorphism, with $Z = (D \cap E')^{\sigma}$, we consider the semidirect product

$$E^* = T \dot{\times}_{\phi} E'$$

with the subgroups $D \cap E'$ and Z amalgamated under σ, where $\phi = \pi \sigma^{-1} \psi$, being $\pi: T \longrightarrow T/C$ the canonical homomorphism and $\psi: D \longrightarrow \mathrm{Aut}E'$ the map of D in the automorphism group induced by D itself on E' in E. It is easy to verify that E^* is a soluble, non-nilpotent, F-extremal group. We will call such a group an F-extremal E-extension. Notice that E itself is an F-extremal E-extension (choose $T = D$).

2.5.4 Theorem

Let G be a soluble group. Then G is F-extremal if and only if one of the following conditions hold:

(i) G is nilpotent;

(ii) G is an F-extremal E-extension, where E is a metabelian, non-nilpotent, central extension by an A-group.

Proof. It immediately follows from 2.4.1, 2.5.2 and 2.5.3. □

References

1. B. Huppert, *Endliche Gruppen I*, Springer–Verlag, Berlin, 1967.
2. D. J. S. Robinson, *A course in the Theory of Groups*, Springer–Verlag, Berlin, 1982.