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The Product Formula

Genaro López Acedo

Facultad de Matemáticas, Universidad de Sevilla, Tarfia sn, 41012 Sevilla (Spain)

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Abstract

A useful property of the Brouwer degree relates the degree of a composition of maps to the degree of each map.This property, which can be generalized for the Leray Schauder degree and in some cases for the A-proper maps [see 4], is called the Product Formula.

In [3] we develop a generalized degree theory for a class of mappings, this class contains the class of A-proper mappings and compact mappings. In this paper we prove a generalization of the Product Formula when one factor is of the Identity+Compact type and the other is an A-compact mapping.

1. Introduction

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In [3] we develop a generalized degree theory for a class of mappings, this class contains the class of A-proper mappings and compact mappings. In this paper we prove a generalization of the Product Formula when one factor is of the Identity+Compact type and the other is an A-compact mapping.

2. Notation and Previous Results

Throughout this paper $\Gamma_n = \{X, P_n, X_n\}$ will be a projectionally complete Banach space (see definition in [2]). For any subset G of X, $Cl(G)$ will denote the closure of G, bdry(G) the boundary of G and $G_n = G \cap X_n$. We consider the following classes of mappings:

1. $A(G) = \{F : Cl(G) \longmapsto X \mid F \quad is \quad A - proper \quad and \quad continuous\}$ (see definition in [1])
2. $K(G) = \{C : Cl(G) \longmapsto X \mid C \quad is \quad Compact\}$
3. $\Delta(G) = \{T : Cl(G) \longmapsto X \mid \lim_{n\to\infty} \sup_{x\in G_n} d(T(x), X_n) = 0\}$

By $\mathbb{Z}$ we denote the set of integer numbers and $\mathbb{Z}^* = \mathbb{Z} \cup \{+\infty\} \cup \{-\infty\}$.

If $A, B \subset \mathbb{Z}^*$, $A + B = \{a \in \mathbb{Z}^* \mid a = a_1 + a_2, a_1 \in A \quad a_2 \in B\}$. By definition $\{+\infty\} + \{-\infty\} = \mathbb{Z}^*$

DEFINITION 1. Let $\Gamma_n = \{X, P_n, X_n\}$ be a projectionally complete Banach space, G an open bounded subset of X, T a mapping from $Cl(G)$ into X such that P_nT is continuous. We say that T is A-compact on $Cl(G)$ with respect to the approximation scheme Γ_n if for any $y \in X$, for any sequence $\{n_j\}$ of positive integers with $n_j \longmapsto \infty$ and for any sequence $\{x_{n_j}\}$, $x_{nj} \in G \cap X_{n_j}$, if:

$$\lim_{n_j\to\infty} P_{n_j} T x_{n_j} = y$$

then there exists a subsequence $\{x_{n_{j_k}}\}$ such that:

$$\lim_{n_{j_k}\to\infty} T x_{n_{j_k}} = y$$

We will denote by $A - K(G)$ the class of A-compact mappings.

Lemma 1. ([3])

Let T be an A-compact mapping from $Cl(G)$ into X with respect to Γ_n. Suppose $y \notin Cl(T(\text{bdry}(G)))$, then there exist $\delta > 0$ and $n_o \in \mathbb{N}$ such that for every $n > n_o$ we obtain:

$$\|P_nT(x) - P_n(y)\| \geq \delta$$

for every $x \in \text{bdry}(G) \cap X_n$

DEFINITION 2. Let Γ_n be a projectionally complete Banach space, G an open bounded subset of X, T a mapping in $A - K(G)$ and let us assume that $y \notin Cl(T\text{bdry}(G))$, We define:

$$D(T,G,y) = \Big\{ z \in Z : z = \lim_{n_j \to \infty} \{d(P_{n_j}T, G_{n_j}, P_{n_j}y)\} \quad \textit{and}$$
$$\{n_j\} \quad \textit{an increasing sequence of positive integers} \Big\}$$

where $G_{n_j} = G \cap X_{n_j}$ and $d(P_{n_j}T, G_{n_j}, P_{n_j}y)$ is the classical Brouwer degree.

Theorem 1. ([3])

Let T be an A-compact mapping from $Cl(G)$ into X with respect to Γ_n. Suppose that $y \notin Cl(T(\text{bdry}(G)))$. Then:

1. $D(G,T,y) \neq \emptyset$

2. $D(I,G,y) = \{1\}$

3. If G_1 and G_2 are open subsets of G, $G_1 \cap G_2 = \emptyset$ and $y \notin Cl(T(Cl(G - (G_1 \cup G_2)))$ we obtain:

$$D(T,G,y) \subseteq D(T,G_1,y) + D(T,G_2,y)$$

where equality holds if either $D(T,G_1,y)$ or $D(T,G_2,y)$ is a singleton.

4. Let H be a mapping from $[0,1] \times Cl(G)$ into X. Suppose that the following hypotheses are all satisfied:

$H(t,.) \in A - K(G)$ for every $t \in [0,1]$

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $\|H(t_1,.) - H(t_2,.)\| < \epsilon$ for every $x \in Cl(G)$. Then $D(H(t,.),G,y(t))$ is independent of t, where $y : [0,1] \longmapsto X$ is continuous and $y(t) \notin Cl(H(t,\text{bdry}(G))$

5. If $T(Cl(G))$ is closed and $D(T,G,y) \neq \{0\}$ there exists a point $x \in G$ such that:

$$T(x) = y$$

6. $K(G) \bigcup \Delta(G) \bigcup A(G) \subset A - K(G)$

7. If $T \in A - K(G)$ and $C \in K(G)$ we obtain $T + C \in A - K(G)$

3. Results

Lemma 2.

Let $\Gamma_n = \{X, P_n, X_n\}$ be a projectionally complete Banach space, G and U open bounded subsets of X, $C \in K(G)$, $I + C(Fr(G) \subset Cl(U)$ and T continuous and A-compact, if:

(1) There exists n_0 such that if $n > n_0$ $P_n(I + C)(Cl(G)) \subset Cl(U)$.

(2) $y \notin Cl(T(I + C)\text{bdry}(G))$.

Then there exists n_1 such that if $n > n_1$

$$d(P_nTP_n(I + C), G_n, P_ny) = d(P_nT, G_n, P_ny)$$

Proof. By the homotopy property for the Brouwer degree it is sufficient to prove that there exists n_1 such that for all $n > n_1$:

$P_ny \notin (tP_nT(I + C) + (1 - t)P_nTP_n(I + C))(\text{bdry}(G_n)$ for all $t \in [0, 1]$.

If we suppose this assertion is false there exist sequences $\{x_{n_i}\}$, $\{t_{n_i}\}$ with $x_{n_i} \in X_{n_i}$ and $t_{n_i} \in [0, 1]$ such that :

$$t_{n_i}P_{n_i}T(I + C)x_{n_i} + (1 - t_{n_i})P_{n_i}TP_{n_i}(I + C))x_{n_i} = P_{n_i}y$$

We can suppose $t_{n_i} \longmapsto t_0 \in [0, 1]$ and $P_{n_i}y \longmapsto y$, thus:

$$\lim_{n_i \to \infty} t_0P_{n_i}T(I + C)x_{n_i} + (1 - t_0)P_{n_i}TP_{n_i}(I + C))x_{n_i} = y$$

As C is a compact mapping there exists a convergent subsequence of $\{Cx_{n_i}\}$ and so there exists a subsequence of $\{P_{n_i}T(I + C)x_{n_i}\}$ convergent to y which is not possible by lemma 1 and hypothesis (2). □

Lemma 3.

If in addition to the hypothesis of lemma 2 we have:

(1) T is proper

(2) $U - (I + C)\text{bdry}(G)) = V$ is connected.

There exists n_0 such that for all $n > n_0$:

$$d(P_nTP_n(I + C), G_n, P_ny) = D(I + C, G, V)d(P_nT, V_n, P_ny)$$

Where $D(I + C, G, V)$ is the Leray-Schauder degree for any $x \in V$

Proof. By the Product Formula for the Brouwer degree whenever n is sufficiently large

$$d(P_nTP_n(I+C),G_n,P_ny)=\sum_{i_n=1}^{k} d(P_n(I+C),G_n,V_n^{i_n})d(P_nT,V_n^{i_n},P_ny)$$

where $V_n^{i_n}$ are the finite connected components of $V_n - P_n(I+C)(\mathrm{bdry}(G_n))$ which satisfy $d(P_nT,V_n^{i_n},P_ny)\neq 0$

The lemma will be proved if when $h\neq k$ whatever the subsequences, whenever $d(P_{n_j}T,V_{n_j}^k,P_{n_j}y)$ and $d(P_{n_j}T,V_{n_j}^h,P_{n_j}y)$ are not zero:

$$d(P_{n_j}(I+C),G_{n_j},V_{n_j}^h)=d(P_{n_j}(I+C),G_{n_j},V_{n_j}^k)=D(I+C,G,V)$$

is satisfied.

In order to prove the equality it suffices to show that if $\{x_n\}$ is a sequence of points with $x_n\in V_n - P_n(I+C)\mathrm{bdry}(G_n))$ and $P_nTx_n=P_ny$, there exists n_1 such that for all $n>n_1$:

$$d(P_n(I+C),G_n,x_n)=D((I+C),G,V) \tag{1}$$

At first we prove that there exists $\delta>0$ satisfying

$$d(x_n,(I+C)(\mathrm{bdry}(G)))>\delta>0$$

Suppose that this assertion is false. If we consider the sets

$$V^j=\{x\in V \mid d(x,(I+C)\mathrm{bdry}(G))<1/j\}$$

for all j there would exist a subsequence $\{x_{n_j}\}\subset V^j$. As T is A-compact and proper it must be closed, so there exists $\{x_j\}$ with $x_j\in V^j$ such that $T(x_j)=y$. As T is proper and $I+C$ is closed then $T(x)=y$ with $x\in I+C(\mathrm{bdry}(G))$, which is not possible from the hypothesis.

Let $x\in V$ and $d(x,(I+C)(\mathrm{bdry}(G)))>\delta>0$. If we prove that there exits n_0 such that for $n>n_0$

$$d(P_n(I+C),G_n,x_n)=d(P_n(I+C),G_n,P_nx)$$

is verified, the equality (1) would be proved since $d(P_n(I+C),G_n,P_nx)=D((I+C),G,x)=D((I+C),G,V)$ whenever n is sufficiently large.

By the homotopy property we only need to show that:

$$tP_nx + (1-t)x_n \notin P_n(I+C)(\text{bdry}(G))$$

If this assertion were false there would exist sequences $\{z_{n_i}\}$, $\{t_{n_i}\}$ with $z_{n_i} \in \text{bdry}(G_{n_i})$ and $t_{n_i} \in [0,1]$ such that :

$$t_{n_i}P_{n_i}x + (1-t_{n_i})x_{n_i} = P_{n_i}(I+C)z_{n_i}$$

Hence there would exist $t_0 \in [0,1]$ and a subsequence of $\{n_i\}$, which we denote $\{n_i\}$ again, such that there exists n_0 satisfying for all $n > n_1$

$$\|t_0x + (1-t_0)x_{n_i} - P_{n_i}(I+C)z_{n_i}\| < \frac{1}{j}$$

which is not possible because $d(x_n, (I+C)(\text{bdry}(G))) > \delta > 0$ for all n and $d(x, (I+C)(\text{bdry}(G))) > \delta > 0$. □

Theorem 2.

With the hypotheses of lemmas 2 and 3

$$D(T(I+C), G, y) = D(I+C, G, V)D(T, V, y)$$

The proof is deduced from lemmas 1 and 2.

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