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Fréchet interpolation spaces and Grothendieck operator ideals

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Abstract

Starting with a continuous injection $I : X \longrightarrow Y$ between Banach spaces, we are interested in the Fréchet (non Banach) space obtained as the reduced projective limit of the real interpolation spaces. We study relationships among the pertenence of I to an operator ideal and the pertenence of the given interpolation space to the Grothedieck class generated by that ideal.

0. Introduction

In this paper we construct a Fréchet (non Banach) space F intermediate of a given couple of Banach spaces linked by a continuous injection $I : X \longrightarrow Y$. The construction of the space F resembles that of $\bigcap_{\varepsilon>0} l_{p+\varepsilon}$: F is described as the reduced projective limit of real interpolation spaces $[X,Y]_{\theta,q}$ with respect to the canonical linking maps.

We study the connection between the pertenence of I to a given operator ideal A and the pertenence of F to the class Groth(A) of those locally convex spaces generated by the ideal A. Our choices for A are: W (weakly compact operators), E_p (entropy ideals), N (nuclear operators), K (compact operators), B (completely continuous operators) and U (unconditionally converging operators). We obtain affirmative answers for W, K, U, negative for B, and a necessary gradations of the result for N and E_p.

Perhaps the most interesting analysis occurs in 4.5 for $A = U$, where we show that if Y does not contain c_0 then F does not contain c_0. In the Banach space setting, it was shown by Levy [12] that if X and Y do not contain c_0, then $[X,Y]_{\theta,q}$ does not contain c_0, answering in this way a question of Beauzamy [1].

1. Real Interpolation Spaces

For a more general background on interpolation theory we refer to [2].

Let (X_0, X_1) be a couple of Banach spaces continuously embeded into some Hausdorff topological vector space U; we shall at times refer to this as an interpolation couple. If $\|\cdot\|_0$ and $\|\cdot\|_1$ denote their respective norms, we set $X_\Sigma = X_0 + X_1$ with norm $\|x\| = K(1,x)$, where the K-functional K is defined by the expression

$$K(t,x) = \inf\{\ \|x_0\|_0 + t\|x_1\|_1\ :\ x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\}.$$

Analogously, $X_\Delta = X_0 \cap X_1$ endowed with the norm $\|x\| = J(1,x)$, where the J-functional J is defined by $J(t,x) = \max\{\ \|x_0\|_0, \|x_1\|_1\}$.

Both X_Σ and X_Δ are Banach spaces.

A Banach space X is called an intermediate space of (X_0, X_1) if we have continuous inclusions $X_\Delta \longrightarrow X \longrightarrow X_\Sigma$.

An intermediate space X of (X_0, X_1) is said to be of K-type θ, $0 < \theta < 1$, if $K(t,x) \leq cte\ t^\theta \|x\|_X$ for all $x \in X$. X is said to be of J-type θ, $0 < \theta < 1$, if $\|x\|_X \leq cte\ t^{-\theta} J(t,x)$ for all $x \in X_\Delta$.

It is well-known that if (X_0, X_1) is an interpolation couple, $0 < \theta < 1$, and $1 \leq q \leq \infty$, then the intermediate space $[X_0, X_1]_{\theta,q}$ defined by

$$[X_0, X_1]_{\theta,q} = \left\{x \in X\ :\ \left(\int_0^\infty (t^{-\theta} K(t,x))^q \frac{dt}{t}\right)^{1/q} < +\infty\right\} \qquad \text{if } q < +\infty$$

$$[X_0, X_1]_{\theta,\infty} = \left\{x \in X\ :\ \sup_{t>0}\{t^{-\theta} K(t,x)\} < +\infty\right\} \qquad \text{if } q = +\infty$$

is both of K and J types θ.

2. Entropy Ideals

Let X and Y be Banach spaces with respective unit balls U_X and U_Y and let $T : X \longrightarrow Y$ be a continuous operator. The nth entropy number of T is defined as:

$$e_n(T) = \inf\left\{\sigma > 0\ :\ \exists\ y_1, \ldots, y_q \in Y,\ q \leq 2^{n-1},\ T(U_X) \subseteq \bigcup_1^q \{y_i + \sigma U_Y\}\right\}$$

The following properties of the sequence of entropy numbers come from the definition (see [13]):

1. $\|T\| \geq e_1(T) \geq e_2(T) \geq \ldots \geq 0$.
2. $e_n(T+S) \leq e_n(T) + \|S\|$.
3. $e_n(RTS) \leq \|R\|\, e_n(T)\, \|S\|$ and $e_{n+m-1}(TS) \leq e_n(T)\, e_m(S)$.

For a given couple of Banach spaces X, Y, $E_{p,q}(X,Y)$ denotes the space of all operators of $L(X,Y)$ whose sequence of entropy numbers belongs to the Lorentz sequence space $l_{p,q}$. When $p = q$ we simply write E_p.

3. Locally Convex Spaces and Grothendieck Ideals

We refer to [11] for general background on locally convex spaces, and to [13] for operator ideals.

If E is a Hausdorff locally convex spaces (in short lcs), for any given continuous seminorm p_k on E we note $\hat{E}_k$ to the completion of the normed space

$$\left({}^{\mathrm{E}}/_{\mathrm{Ker}\; p_k}, \|\cdot\|_k\right)$$

where $\|\phi_k x\|_k = p_k(x)$ and ϕ_k is the quotient map. The spaces $\hat{E}_k$ will be referred to as the associated Banach spaces. If $p_k \leq p_j$ are continuous seminorms, the canonical linking mapping $\hat{T}_{jk}$ is the extension to the completions of the operator $T_{jk}\phi_j x = \phi_k x$ of $L(E_j, E_k)$.

A Hausdorff lcs is said to be generated by an ideal A of operators when for each continuous seminorm p_k there is a continuous seminorm p_j such that the canonical linking map $\hat{T}_{jk}$ belongs to A. E is also called an A-space. The class formed by all A-spaces is called the Grothendieck space ideal generated by A, and noted Groth(A). A study of the classes Groth(A) occurs in [3].

4. The space $[X,Y]_{(\theta),q}$

Let X and Y be Banach spaces such that there is a continuous injection $I : X \longrightarrow Y$ with dense range. Then (X,Y) can be considered as an interpolation couple. For $0 < \theta < 1$, and $1 \leq q \leq +\infty$ the real interpolation method gives an intermediate Banach space $[X,Y]_{\theta,q}$. It is well known that for $\theta' < \theta$ there is an induced continuous injection $I_{\theta'\theta} : [X,Y]_{\theta',q} \longrightarrow [X,Y]_{\theta,q}$.

In this way, for a fixed q we can construct a Fréchet space

$$[X,Y]_{(\theta),q} := \varprojlim I_{\theta'\theta}\left([X,Y]_{\theta,q}\right)$$

taking the projective limit of the Banach spaces $[X,Y]_{\theta,q}$ with respect to the maps $I_{\theta'\theta}$.

The space $[X,Y]_{(\theta),q}$ is, to an extent, a generalization of some interesting spaces. If, for instance, we take $X = l_p$, $Y = l_\infty$ with the natural inclusion $l_p \longrightarrow l_\infty$, then

$$[l_p, l_\infty]_{(\theta),q} = \bigcap_{\varepsilon>0} l_{p+\varepsilon}.$$

We are interested in the locally convex structure of $[X,Y]_{(\theta),q}$ from the point of view of Grothendieck space ideals. To be precise, we study the relationships between "$I \in A$" and "$[X,Y]_{(\theta),q} \in \text{Groth}(A)$" for several operator ideals A.

Since $[X,Y]_{(\theta),q}$ has $[X,Y]_{\theta,q}$ as associated Banach spaces it is clear that if $[X,Y]_{(\theta),q}$ is an A-space then $I \in A^n$ for all $n \in \mathbb{N}$. It also suggests that we should focus our attention on idempotent operator ideals.

4.1. Weakly compact operators

The following result is due to Beauzamy ([1], p.32): *$I : X \longrightarrow Y$ is weakly compact if and only if $[X,Y]_{\theta,q}$ $(0 < \theta < 1,\ 1 < q < +\infty)$ is reflexive.*

Therefore:

Proposition W.

$I : X \longrightarrow Y$ is weakly compact if and only if $[X,Y]_{(\theta),q} \in Groth(W)$ for $1 < q < +\infty$.

4.2. Compact operators

Proposition K.

$I : X \longrightarrow Y$ is compact if and only if $[X,Y]_{(\theta),q}$ is a Schwartz space.

Proof. This is an easy consequence of the reiteration theorem ([2], 3.5.3) and the *compactness theorem* ([3], 3.8.1): if $I : X \longrightarrow Y$ is compact then both $[X,Y]_{\theta,q} \longrightarrow Y$ and $X \longrightarrow [X,Y]_{\theta,q}$ are compact. Given $\theta' < 0$, since $[X,Y]_{\theta',q}$ is an interpolation space intermediate between X and $[X,Y]_{\theta,q}$ we see that $I_{\theta',\theta}$: $[X,Y]_{\theta',q} \longrightarrow [X,Y]_{\theta,q}$ is again compact. □

4.3. Entropy ideals

When the ideal is not idempotent we can only get a gradation of the results for $I_{\theta',\theta}$. This is the case of entropy ideals due to the composition formula

$$E_r = E_p \circ E_q,\ r^{-1} = p^{-1} + q^{-1}.$$

Applications to the locally convex structure of the space $[X,Y]_{(\theta),q}$ will take place in 4.4.

Proposition E.

Let (X,Y) be an interpolation couple with linking map $I : X \longrightarrow Y$. $I \in E_p$ implies that for $\theta' < \theta$, the canonical induced operator $I_{\theta',\theta}$ belongs to $E_{p/\theta-\theta'}$.

Proof. We use the same idea as in Proposition K above. Recall that if Z is an intermediate space to the couple (X,Y) of K-type θ then $I \in E_p$ implies that the induced operator $Z \longrightarrow Y$ belongs to $E_{p/1-\theta}$. If Z is of J-type θ then $X \longrightarrow Z$ belongs to $E_{p/\theta}$.

Since $[X,Y]_{\theta',q}$ is of K-type θ'/θ with respect to the interpolation couple $(X,\ [X,Y]_{\theta,q})$ we get that $I_{\theta',\theta}$ belongs to E_r, with

$$r = \frac{p/\theta}{1-(\theta'/\theta)} = \frac{p}{\theta-\theta'}. \qquad \square$$

Remark. The behaviour of entropy numbers under real interpolation with a functional parameter was calculated in [4].

4.4. Nuclear operators

Since N is not an idempotent ideal we pass to $\bigcap_{n\in\mathbb{N}} N^n = N_0$, the ideal of strongly nuclear or (s)-nuclear operators. We have:

Proposition N.

$I : X \longrightarrow Y$ is strongly nuclear if and only if $[X,Y]_{(\theta),q}$ is a nuclear space.

Proof. We prove only the "only if" part. Recall that the class Groth(N) coincides ([13], 29.7) with the classes Groth(E_p) for all $p > 0$. Also recall that $T \in N_0$ if and only if

$$T \in E_{(s)} = \bigcap_{p>0} E_p.$$

In this way, for θ given we can choose r and $\theta' < \theta$ such that $p = r/(\theta - \theta')$. Next apply Proposition E. □

4.5. Unconditional converging operators

Recall that an operator $T : X \longrightarrow Y$ is said to be unconditionally converging if it sends weakly -1-summable sequences (in short wuC), i.e., sequences (x_n) such that, for each $x^* \in X^*$, $\left(x^*(x_n)\right)_n \in l_1$, into summable sequences. The ideal of unconditionally convergent operators will be denoted by U.

In ([1], p.36) is stated the problem whether X *and* Y *do not contain* c_0 implies $[X,Y]_{\theta,q}$ *does not contain* c_0. This is in connection with the problem of whether the ideal U is idempotent ([13], 3.1.9) and ([8], p.260). The first problem was solved affirmatively by Levy in [12]. The second one was solved negatively by Ghoussoub et alt [9].

A simpler proof for $[X,Y]_{(\theta),q}$, which does not implies, nor is implied by, that of Levy, follows:

Firstly, it is easy to see that:

Lemma 4.5.1

Let (X,Y) be an interpolation couple of Banach spaces. Let F be an intermediate space of J-type θ, $0 < \theta < 1$. If $T : E \longrightarrow (X,Y)_\Delta$ is a continuous operator such that $T : E \longrightarrow Y$ is unconditionally converging then $T : E \longrightarrow F$ is unconditionally converging.

Proof. By ([13], c.5.7), having J-type θ means that $\|x\|_F \leq \|x\|_X^{1-\theta}\,\|x\|_Y^{\theta}$ for all $x \in (X,Y)_\Delta$. Therefore, if (x_n) is a wuC sequence of E then $\|Tx_n\| \longrightarrow 0$ in Y. Since $Tx_n \in (X.Y)_\Delta$ and is bounded in X we also have $\|Tx_n\| \longrightarrow 0$ in F. That is all we need. □

It is well known that the identity of a Banach space is unconditionally converging if and only if the space does not contain a copy of c_0 (see [7], p. 45). We extend this result to Fréchet spaces.

Proposition 4.5.2

Let E be a Fréchet space. Then $Id_E \in U$ if and only if E does not contain a copy of c_0.

Proof. One implication is trivially true. Let us go for the other. Let E be a Fréchet space such that $Id_E \notin U$. There is a sequence (x_n) in E weakly-1-summable but not strongly summable. Assume a sequence of seminorms $\|\cdot\|_1 < \|\cdot\|_2 < \dots$ defining the topology of E. There must be a sequence $p_1 < q_1 < p_2 < q_2 < \dots$ of N, a $d > 0$ and a $k_0 \in N$ such that

$$\left\|\sum_{p_i}^{q_i} x_n\right\|_{k_0} > d$$

Let us call $y_k = \sum_{p_k}^{q_k} x_n$. (y_k) is a weakly-1-summable sequence of E satisfaying $\|y_k\|_{k_0} > d$ for all k. We may easily suppose $k_0 = 1$. Clearly $(\phi_m(y_k))_k$ is a wuC sequence in E_m and $\|\phi_m(y_k)\|_m = \|y_k\|_m > d$. By the Bessaga-Pelczynski selection principle we can pick a basic subsequence $(\phi_1(y_k^1))$ equivalent to the unit vector basis of c_0 (see [7], p.42-46). We proceed inductively to obtain a subsequence (y_k^{m+1}) of (y_k^m) such that $(\phi_{m+1}(y_k^{m+1}))$ is a basic sequence in E_{m+1} equivalent to the unit vector basis of c_0.

If we diagonalize to obtain (y_n^n) we get a basic sequence in E which is equivalent to the unit vector basis of c_0 due to the following straightforward extension of a classical result (see [7], Th. 6, p. 44):

Lemma 4.5.3

Let $\sum_n x_n$ be a formal series in a Fréchet space. They are equivalent:
1. $\sum_n x_n$ is weakly-1-summable.
2. For any $(t_n) \in c_0$, $\sum_n t_n x_n$ converges.

Proof. If $\sum_n x_n$ is weakly-1-summable in a Fréchet space E, then for each continuous seminorm $\|\cdot\|_k$ in E, $\sum_n \phi_k(x_n)$ is wuC in E_k; therefore for each $(t_n) \in c_0$, $\sum_n t_n \phi_k(x_n)$ converges in $\widehat{E_k}$. This proves $1 \Rightarrow 2$. To prove $2 \Rightarrow 1$, consider a sequence (x_n) in E such that for each $(t_n) \in c_0$ $\sum_n t_n x_n$ converges. This implies that for each continuous seminorm $\|\cdot\|_k$ in E, $\sum_n t_n \phi_k(x_n)$ converges in $\hat{E}_k$. For each $x' \in E'$ there exists a k such that $x' \in E_k{}'$.

Therefore $\sum_n |x'(x_n)| = \sum_n |x'\phi_k(x_n)| < +\infty$. □

Now the proof of 4.5.2 can be rounded off with a standard argument:

If $(t_n) \in c_0$, $\sum_n t_n y_n^n$ converges in E; if, on the other hand, $\sum_n t_n y_n^n$ converges in E, then $(t_n) \in c_0$, since $|t_n| \left\|y_n^n\right\|_{k_0}$ tends to 0 but $\left\|y_n^n\right\|_{k_0} \geq d > 0$ for all $n \in \mathbb{N}$. □

Corollary

A sequence (x_n) in a Fréchet space E with defining sequence of seminorms $(\|\cdot\|_n)$ is weakly-1-summable if and only if for any $k \in \mathbb{N}$ the sequence $(\phi_k x_n)$ is weakly-1-summable in $\hat{E}_k$; and if and only if for any continuous seminorm p in E exists a constant K_p such that

$$p\Big(\sum_{n\in\Delta} \pm x_n\Big) \leq K_p$$

for any choice of signs, and any finite subset $\Delta \subseteq \mathbb{N}$.

The following proposition is very simple but necessary

Proposition 4.5.4

Let E be any lcs. If $E \in Groth(U)$ then $Id_E \in U$.

Proposition 4.5.5

Let (X, Y) be an interpolation couple with linking map $I : X \longrightarrow Y$. Assume that Y does not contain c_0. Then $[X, Y]_{(\theta),q}$ does not contain c_0.

Proof. Let $0 < \theta' < \theta < 1$. Looking at the diagram

$$X \longrightarrow [X, Y]_{\theta',q} \longrightarrow [X, Y]_{\theta,q} \longrightarrow Y$$

we see that $[X, Y]_{\theta',q} \longrightarrow Y$ belongs to U by the hypothesis. Therefore

$$[X, Y]_{\theta',q} \longrightarrow [X, Y]_{\theta,q}$$

also belongs to U, by 4.5.1 and the reiteration theorem.

This implies that $[X, Y]_{(\theta),q}$ belongs to Groth(U), and $\mathrm{Id}_{[X,Y]_{(\theta),q}} \in U$. By 4.5.2 it cannot contain c_0. □

And as a consequence:

Proposition U

$I : X \longrightarrow Y$ belongs to U if and only if $[X, Y]_{(\theta),q} \in Groth(U)$.

Remark. That 4.5.5 is not a rewarding of Levy's result is a consequence of the fact that a Fréchet space F having a fundamental sequence of associated Banach spaces (F_k) not containing c_0 can contain c_0 or, in other words, that proposition 4.5.4 has no counterpart; that is, $\mathrm{Id}_E \in U$ does not imply that $E \in \mathrm{Groth}(U)$, even in the Fréchet space setting: just consider F a Köthe echelon space of order 0 which is Fréchet-Montel but not a Schwartz space (see [11] or [10]). F is a reduced projective limit of copies of c_0 with diagonal maps. Since F is Montel it cannot contain c_0. Since it is not Schwartz, the diagonal maps cannot be in U, since in that case D_σ would be compact.

A simpler example in the general locally convex setting is as follows: consider I an uncountable set of cardinality d, and consider the sum space $\varphi_d := \oplus_I K$ endowed with the so-called box-topology, for which a fundamental system of neighborhoods of 0 is given by the sets: $\varphi_d \cap \prod_I r_i B$, where B is the unit ball of K and

$$(r_i) \in l_\infty^+(I) = \{(r_i) \in K^I : r_i > 0\}.$$

It is easy to see that

$$[\varphi_d, t_{\mathrm{box}}] = \varprojlim D_\sigma(c_0(I)), \qquad \sigma \in l_\infty^+(I),$$

where $D_\sigma : c_0(I) \longrightarrow c_0(I)$ is the diagonal operator defined by σ. $|\varphi_d, t_{\mathrm{box}}|$ is therefore complete.

$[\varphi_d, t_{\mathrm{box}}]$ does not belong to $\mathrm{Groth}(U)$ since the diagonal operators D_σ acting from $c_0(I)$ into $c_0(I)$ are not unconditionally converging: there must be an $\varepsilon > 0$ such that $\sigma_i > \varepsilon$ for an uncountable set of i. Let us call this set J. Choose now a sequence (e_n) with all $n \in J$. This sequence is wuC, while $\|D_\sigma e_n\| > \varepsilon$, which prevents $(D_\sigma e_n)$ from being summable.

On the other hand, the identity of $[\varphi_d, t_{\mathrm{box}}]$ belongs to U since given any sequence of this space it lies inside a copy of $\varphi := \oplus_{\mathbb{N}} K$.

4.6 Completely continuous operators

In this case $[X, Y]_{(\theta),q} \in \mathrm{Groth}(B)$ is not implied by $I \in B$ as the example $l_1 \to l_\infty$ shows.

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References

1. B. Beauzamy, *Espaces d'Interpolation Réels: Topologie et Géometrie*, Lecture Notes in Mathematics 666, Springer, 1978.
2. J. Bergh and J. Löfstrom, *Interpolation spaces. An introduction*, Springer, 1976.
3. Jesús M. F. Castillo, On Grothendieck space ideals, *Collectanea Math.* **39, 1** (1988), 67–82.
4. Jesús M. F. Castillo, Factorization of Entropy ideals: Proportional Case, *Portugaliae Math.* **48** (1991), 11–23.
5. J. C. Díaz, Montel subspaces in the countable projective limits of $L^p(m)$-spaces, *Canadian Math. Bull.* **32, 2** (1989), 169–176.
6. J. C. Díaz and J. A. López Molina, Projective tensor products of Fréchet spaces, *Proc. Edinburgh Math. Soc.* **34** (1991), 169–178.
7. J. Diestel, *Sequences and Series in Banach spaces*, GTM **92**, Springer, 1984.
8. J. Diestel and J. J. Uhl, Jr. *Vector Measures*, AMS Math Surveys **15**, 1977.
9. N. Ghoussoub and W.B. Johnson, Counterexamples to several problems on the factorization of bounded linear operator, *Proc. AMS* **92** (1984), 233–238.
10. H. Junek,*Locally convex spaces and operator ideals*, Teubner-Texte **56**.
11. G. Köthe, *Topological Vector Spaces I*, Springer 1969.
12. M. Levy, L'espace d'interpolation réel $(A_0, A_1)_{\theta,q}$ contient l_p, *CRAS* **289** (1979), 675–677.
13. A. Pietsch, *Operator Ideals*, North Holland 1980.