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Regular inductive limits of $\mathcal{K}$-spaces

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ABSTRACT

A well-known result for bounded sets in inductive limits of locally convex spaces is the following: If each of the constituent spaces E_n are Féchet spaces and E is the inductive limit of the spaces E_n, then each bounded subset of E is bounded in some E_n iff E is locally complete. Using DeWilde's localization theorem, we show here that the completeness of each E_n and the local completeness of E may be replaced with the conditions that the spaces E_n are all webbed K-spaces and E is locally Baire, respectively.

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1. Introduction

Throughout this paper, "space" refers to a Hausdorff locally convex space. If E is a space and B is a bounded, absolutely convex subset of E (also called a *disk*), we write E_B for the normed space obtained by equipping the linear span of B with the topology generated by the Minkowski functional of B. If E_B is a Banach space, we call B a *Banach disk*. If each closed, bounded disk is a Banach disk, we say that E is *locally complete*. If E_B is a Baire space, we call B a *Baire disk*. If each bounded set is contained in a bounded Baire disk, we say that E is *locally Baire*. Clearly, every locally complete space is locally Baire, but not conversely (see the Remark 1

Presented in honor of the sixtieth birthday of Professor M. Valdivia in Section 3). Also, we will use a number of properties of webs in a locally convex space. The standard references are DeWilde [1], Köthe [2], and Robertson and Robertson [3].

If $\{E_n : n \in \mathbb{N}\}$ is a collection of spaces, with $E_n \subset E_{n+1}$ and $id : E_n \to E_{n+1}$ continuous for each $n \in \mathbb{N}$, we write $E = \text{ind}\ \lim_n E_n$ for the inductive limit of the spaces $\{E_n : n \in \mathbb{N}\}$. If $E = \text{ind}\ \lim_n E_n$, then E is *regular* if each bounded set in E is contained in and bounded in some E_n.

When all the spaces E_n are Fréchet, one can apply Grothendieck's Factorization Theorem to obtain the following well known result: *E is regular iff E is locally complete.* See for example, Pérez-Carreras and Bonet [4, 7.3.3. (i), p. 210; 1.2.20 (i), p. 22]. A natural question to ask is whether we can have a result such as this without requiring so much completeness. The aim of this note is to use locally Baire spaces to show that an inductive limit of webbed locally Baire spaces is regular if and only if it is still locally Baire. We also show that spaces satisfying property $\mathcal{K}$(see below) are locally Baire so that we may apply the above result to webbed $\mathcal{K}$spaces.

We will make use of the following interesting property (for further information, see Antosik and Swartz [5], and the references therein): A space is said to satisfy *property $\mathcal{K}$*if each null sequence has a series convergent subsequence. Thus, property $\mathcal{K}$means that if $x_n \longrightarrow 0$ then there is a subsequence (x_{n_k}) of (x_n) such that

$$\sum_{k=1}^{\infty} x_{n_k}$$

converges in E. In this case, we also call E a *$\mathcal{K}$space.*

The following facts concerning property $\mathcal{K}$are important to us because they indicate that for metrizable locally convex spaces, property $\mathcal{K}$is a more general notion than completeness.

(i) Every Fréchet space is a $\mathcal{K}$space. See the discussion at the beginning of [6].
(ii) A metrizable space which is a $\mathcal{K}$space is a Baire space [6, 2.2, p. 37].
(iii) There exist $\mathcal{K}$spaces which are not complete [7, Theorem 2, p. 94].

2. Main results

Theorem 1

Let $E = \text{ind}\ \lim_n E_n$ be a Hausdorff inductive limit such that each E_n is webbed and locally Baire. Then E is regular if and only if E is locally Baire.

Proof. It is clear that a regular inductive limit of locally Baire spaces is again locally Baire. Conversely, suppose E is an inductive limit of webbed locally Baire spaces which is again locally Baire, and let $A \subset E$ be bounded. Let B be a bounded disk containing A for which E_B is a Baire space. $id : E_B \to \text{ind}\lim_n E_n$ is continuous and each E_n is webbed. Hence, we may apply DeWilde's localization theorem, via [1, IV.6.2], and [1, IV.6.5] to conclude that there is some $i \in \mathbb{N}$ such that $id(E_B) = E_B \subset E_i$ and $id : E_B \to E_i$ is continuous, too. Finally, A is bounded in E_B, so A is bounded in E_i. □

Theorem 2

If E is a $\mathcal{K}$space, then E is locally Baire.

Proof. Let $A \subset B$ be a bounded set. Denote by B the closed, absolutely convex hull of A and let (x_n) be a null sequence in E_B. We may choose a subsequence, denoted again by (x_n), such that for each $n \in \mathbb{N}$, $x_n \in 2^{-n}B$. Hence, the partial sums of $\sum_{n=1}^{\infty} x_n$ form a Cauchy sequence in E_B. The topology of E_B is finer than the topology induced by E on the span of B, so, $x_n \longrightarrow 0$ in E. Thus, there is a subsequence (x_{n_k}) of (x_n) and a $y \in E$ such that $\sum_{k=1}^{\infty} x_{n_k} = y$. Define the sequence (y_m) in E by $y_m = \sum_{k=1}^{m} x_{n_k}$. Then (y_m) is a sequence of elements from E_B, converging in E, hence, also with respect to the induced topology on E_B. Moreover, $B \subset E_B$, and B is closed in E, so $\{n^{-1}B : n \in \mathbb{N}\}$ is a base of zero neighborhoods, closed in the induced topology on the span of B. Furthermore, (y_m) is Cauchy in E_B. By [8, 3.2.4, p. 59], (y_m) converges to y with respect to the normed topology of E_B. Because B is closed in E, and (y_m) is a bounded sequence in E_B, we have, for some $\lambda > 0$, $y \in \lambda B \subset E_B$. This means E_B satisfies property $\mathcal{K}$. From [6; 2.2], we know now that E_B is a Baire space, and this finishes the proof. □

We may combine the two theorems above to obtain:

Theorem 3

Let $E = \text{ind}\lim_n E_n$ be a Hausdorff inductive limit such that each E_n is a webbed $\mathcal{K}$space. Then E is regular if and only if E is locally Baire.

3. Remarks and Examples

Below are some simple examples of the relationship between spaces that are locally Baire, webbed, and satisfy property $\mathcal{K}$.

1. Examples of locally Baire spaces that are not locally complete were promised in Section 1. Metrizable, incomplete $\mathcal{K}$spaces satisfy this condition because they are locally Baire by Theorem 2, but cannot be locally complete because a metrizable space is locally complete if and only if it is complete [9, Th. II.2.2].
2. Theorem 3 generalizes the result in Section 1 for Fréchet as follows. Consider the space ℓ_1, equipped with its weak topology $\sigma(\ell_1, \ell_\infty)$. [10, Example 15, p. 480] shows that this space is webbed and locally complete, hence, locally Baire. Of course, $(\ell_1, \sigma(\ell_1, \ell_\infty))$ is not metrizable so it is not a Fréchet space. However, an application of Shur's Theorem and straightforward calculations show that this space satisfies property $\mathcal{K}$. The author believes that there exists a webbed $\mathcal{K}$space which is not even locally complete.
3. There are locally Baire spaces that do not satisfy property $\mathcal{K}$, obtained as follows: Let E be any infinite dimensional Banach space which has weakly convergent sequences that are not convergent in norm. Equip E with is weak topology, $\sigma(E, E')$ and suppose that this space is a $\mathcal{K}$space. Then by [5, 3.7, p. 17], each null sequence would converge in norm. This is impossible.
4. Kucera and McKennon exhibit in [11] an inductive limit of nuclear Fréchet spaces that is not regular. Because each Fréchet space is a webbed $\mathcal{K}$space, Theorem 3 applies: This inductive limit is not locally Baire.

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