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A Kratzel's integral transformation of distributions

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Abstract

In this paper we study an integral transformation introduced by E. Kratzel in spaces of distributions. This transformation is a generalization of the Laplace transform. We employ the usually called kernel method. Analyticity, boundedness, and inversion theorems are established for the generalized transformation.

1. Introduction

E. Kratzel [6] introduced a generalization of the well known Laplace transformation by the integral

$$F(x) = \mathcal{L}_\nu^{(n)}\{f\}(x) = \int_0^\infty \lambda_\nu^{(n)}(xt)f(t)\,dt, \qquad \text{for } x > 0, \tag{1}$$

where

$$\lambda_\nu^{(n)}(x) = (2\pi)^{(n-1)/2}(x/n)^{n\nu}\frac{\sqrt{n}}{\Gamma(\nu+1-(1/n))}\int_1^\infty (t^n-1)^{\nu-(1/n)}e^{-xt}\,dt,$$

for $x > 0$, $\nu > -1 + 1/n$ and $n = 1, 2, \ldots$.

The $\mathcal{L}_\nu^{(n)}$ transformation reduces to the Laplace transformation when $n = 1$. Moreover the K_ν transformation [16] is obtained from $\mathcal{L}_\nu^{(n)}$ by taking $n = 2$.

E. Kratzel studied the main propierties of the function $\lambda_\nu^{(n)}(x)$ and the $\mathcal{L}_\nu^{(n)}$ transformation in a series of papers [6–10]. Later G. L. N. Rao and L. Debnath [12] investigated the integral transformation (1) in a certain space of distributions. More recently, the authors have completed in some aspects the study of E. Kratzel and established real inversion formulas [1] and integrability theorems [2] for the $\mathcal{L}_\nu^{(n)}$ transformation.

In this paper we define the Kratzel integral transforms of distributions by using the known kernel method. We firstly introduce a Fréchet space of functions denoted by $\mathcal{A}$ and constituted by the infinitely differentiable functions $\phi(t)$, $0 < t < \infty$, such that

$$\sup_{1/m<t<\infty} \left| \frac{d^k}{dx^k}\phi(t) \right| < \infty, \qquad \text{for every } m, k \in \mathbb{N}.$$

The generalized $\mathcal{L}_\nu^{(n)}$ transform $\mathcal{L}_\nu^{(n)} f$ of $f \in \mathcal{A}'$ is defined in section 3 by

$$(\mathcal{L}_\nu^{(n)} f)(x) = \left\langle f(t), \lambda_\nu^{(n)}(xt) \right\rangle, \qquad \text{for } x > 0.$$

We establish analyticity, boundedness and inversion theorems for the generalized $\mathcal{L}_\nu^{(n)}$ transformations.

In Section 4 we define the generalized $\mathcal{L}_\nu^{(n)}$ transformation of distributions with compact support. A new inversion formula for $\mathcal{L}_\nu^{(n)}$ is obtained.

We now list some properties of the function $\lambda_\nu^{(n)}(x)$ due to E. Kratzel [6] that will be useful in the sequel.

The behaviour of $(d^k/dx^k)(\lambda_\nu^{(n)}(x))$, $k \in \mathbb{N}$, near the origin and the infinity is as follows:

$$\frac{d^k}{dx^k}\left(\lambda_\nu^{(n)}(x)\right) = O\left(x^{-k}\right), \qquad \text{as } x \longrightarrow 0^+, \quad \text{for } \nu > 0, \tag{2}$$

$$\frac{d^k}{dx^k}\left(\lambda_\nu^{(n)}(x)\right) = O\left(x^{(n-1)\nu+(1/n)-1} e^{-x}\right), \qquad \text{as } x \longrightarrow \infty, \quad \text{for } \nu > -1 + 1/n. \tag{3}$$

From (2) and (3) it deduces that there exists $M > 0$ for which

$$\left| \frac{d^k}{dx^k}\left(\lambda_\nu^{(n)}(x)\right) \right| \leq M\left(1 + x^{-k}\right) x^{n-1)\nu+(1/n)-1} e^{-x}, \tag{4}$$

for $x > 0$ and with $0 < \nu < 1/n$.

Moreover, $\lambda_\nu^{(n)}(x)$ satisfies the differential equation

$$\big(B_{\nu,n} - (-1)^n\big)y = 0 \tag{5}$$

where

$$B_{\nu,n} = x^{n\nu-1}\,\frac{d^{n-1}}{dx^{n-1}}\,x^{1-n\nu}\,\frac{d}{dx}\,.$$

If the Mellin integral transformation is defined by

$$M\{f\}(s) = \int_0^\infty x^{s-1} f(x)\,dx$$

then

$$M\{\lambda_\nu^{(n)}(x)\}(s) = (2\pi)^{(n-1)/2} n^{-n\nu-(1/2)}\,\frac{\Gamma(s+n\nu)\,\Gamma(s/n)}{\Gamma((s/n)+\nu+1-(1/n))}, \tag{6}$$

for $\Re s > \max\{0, -n\nu\}$.

Thoughout this paper, I denotes the real interval $0 < x < \infty$. $D(I)$, $E(I)$, $D'(I)$ and $E'(I)$ denote well known spaces of functions and distributions encountered in [13] and [18].

2. The function space $\mathcal{A}$ and its dual

We introduce the function space $\mathcal{A}$ consisting of all infinitely differentiable functions $\phi(t)$, $0 < t < \infty$, such that

$$\gamma_{m,k}(\phi) = \sup_{1/m<t<\infty} \left|\frac{d^k}{dx^k}\,\phi(t)\right| < \infty, \qquad \text{for every } m,k \in \mathbb{N}.$$

$\mathcal{A}$ is endowed with the weak topology generated by the family of seminorms $\{\gamma_{m,k}\}_{m,k\in\mathbb{N}}$. Thus $\mathcal{A}$ is a Fréchet space. Furthermore, as it is easy to see, $D(I) \subset \mathcal{A} \subset E(I)$ and the inclusions are continuous.

Proposition 1

The operator $B_{n,\nu}$ defines a continuous linear mapping from $\mathcal{A}$ into itself.

Proof. It is sufficient to note that for every $m, k \in \mathbb{N}$ there exists $M > 0$ such that

$$\gamma_{m,k}(B_{\nu,n}\phi) \leq M \sum_{j=1}^{n} \gamma_{m,k+j}(\phi), \qquad \text{for each } \phi \in \mathcal{A}. \ \square$$

The dual space of $\mathcal{A}$ is denoted as usual by $\mathcal{A}'$. $\mathcal{A}$ is equipped with the weak topology.

We define the generalized operator $B^*_{\nu,n}$ on $\mathcal{A}'$ as the adjoint of the classical operator $B_{\nu,n}$. More specifically,

$$\left\langle B^*_{\nu,n} f, \phi \right\rangle = \langle f, B_{\nu,n}\phi \rangle, \qquad \text{for } f \in \mathcal{A}' \text{ and } \phi \in \mathcal{A}.$$

Hence, by virtue of Proposition 1, $B^*_{\nu,n}$ is a continuous linear mapping from $\mathcal{A}'$ into itself.

The following proposition allows to define some members of $\mathcal{A}'$.

Proposition 2

Let $f(t)$ be a locally integrable function on $(0, \infty)$. If there exists $a > 0$ such that $f(t) = 0$, for $t < a$, and $\int_a^\infty |f(t)| dt < \infty$, then F defines a regular generalized function in $\mathcal{A}'$ through

$$\langle f, \phi \rangle = \int_0^\infty f(t)\phi(t)\, dt, \qquad \textit{for every } \phi \in \mathcal{A}.$$

Proof. The linearity of the mapping is clear. Moreover F is a continuous mapping because

$$|\langle f, \phi \rangle| \leq \int_a^\infty |f(t)| dt \sup_{1/m < t < \infty} |\phi(t)|, \qquad \text{for } \phi \in \mathcal{A},$$

where $M \in \mathbb{N}$ and $1/m < a$. $\square$

3. The generalized $\mathcal{L}_\nu^{(n)}$ transformation in $\mathcal{A}'$.

Let $\nu > -1 + 1/n$ and $n \in \mathbb{N}$. For $f \in \mathcal{A}'$, define its generalized $\mathcal{L}_\nu^{(n)}$ transform by the relation

$$F(x) = (\mathcal{L}_\nu^{(n)} f)(x) = \left\langle f(t), \lambda_\nu^{(n)}(xt) \right\rangle, \qquad x > 0. \tag{7}$$

Note that (7) is well defined and if follows from the asymptotic behaviour easily that $\lambda_\nu^{(n)}(xt)$ is in $\mathcal{A}$ for fixed $x > 0$. Moreover if f is a function satisfying the requirements in Proposition 2, the generalized $\mathcal{L}_\nu^{(n)}$ transform of F reduces to the classical $\mathcal{L}_\nu^{(n)}$ transform of f.

We now establish several properties of the generalized $\mathcal{L}_\nu^{(n)}$transformation.

Proposition 3

Let $f \in \mathcal{A}'$. If $F(x)$ denotes the generalized $\mathcal{L}_\nu^{(n)}$ transform of f then $F(x)$ is infinitely differentiable on $(0, \infty)$ and

$$\frac{d^r}{dx^r} F(x) = \left\langle f(t), \frac{\partial^r}{\partial x^r} \lambda_\nu^{(n)}(xt) \right\rangle, \qquad \text{for } x > 0 \text{ and } r \in \mathbb{N}.$$

Proof. Let h be an arbitrary increment in $x > 0$. Without any loss of generality assume $0 < |h| < x/2$.

As it is easy to see

$$\frac{F(x+h) - F(h)}{h} = \left\langle f(t) m \frac{1}{h} (\lambda_\nu^{(n)}(t(x+h)) - \lambda_\nu^{(n)}(tx)) \right\rangle. \tag{8}$$

We will prove that

$$\varphi_h(x,t) = \frac{1}{h} (\lambda_\nu^{(n)}(t(x+h)) - \lambda_\nu^{(n)}(tx)) - \frac{\partial}{\partial x} (\lambda_\nu^{(n)}(tx)) \longrightarrow 0, \qquad \text{as } h \longrightarrow 0,$$

in the sense of convergence in $\mathcal{A}$. Our result for $k = 1$ will then follow from (8) and the continuity of $f(t)$.

For every $r \in \mathbb{N}$, we can write

$$\begin{aligned}
\frac{\partial^r}{\partial t^r} \phi_h(x,t) &= \frac{1}{h} \int_x^{x+h} \int_x^u \frac{\partial^2}{\partial y^2} \left(\frac{\partial^r}{\partial t^r} \left(\lambda_\nu^{(n)}(ty) \right) \right) dy\, du \\
&= \frac{1}{h} \int_x^{x+h} \int_x^u \frac{\partial^2}{\partial y^2} \left(y^r \frac{d^r}{d(ty)^r} \left(\lambda_\nu^{(n)}(ty) \right) \right) dy\, du \\
&= \frac{1}{h} \int_x^{x+h} \int_x^u \Big(r(r-1) y^{r-2} \frac{d^r}{d(ty)^r} \left(\lambda_\nu^{(n)}(ty) \right) \\
&\quad + 2ry^{r-1} t \frac{d^{r+1}}{d(ty)^{r+1}} \left(\lambda_\nu^{(n)}(ty) \right) + y^r t^2 \frac{d^{r+2}}{d(ty)^{r+2}} \left(\lambda_\nu^{(n)}(ty) \right) \Big) dy\, du.
\end{aligned}$$

By virtue of (3) for each $m \in \mathbb{N}$ there exists a positive constant M such that

$$\left|\frac{d^{r+j}}{d(ty)^{r+j}}\left(\lambda_\nu^{(n)}(ty)\right)\right| \le M(yt)^{(n-1)\nu+(1/n)-1}e^{-tx/2},$$

for $t > 1/m$, $y > x/2$ and $j = 0, 1, 2$.

Hence

$$\begin{aligned}\left|\frac{\partial^r}{\partial t^r}\phi_h(x,t)\right| &\le M_1 t^{(n-1)\nu+(1/n)-1}e^{-tx/2}\frac{1}{h}\int_x^{x+h}\int_x^u \left(y^{r-2} + y^{r-1}t + y^r t^2\right) dy\,du \\ &\le M_2 e^{-tx/4}\end{aligned}$$

for every $t > 1/m$ and $0 < |h| < x/2$. Here $M_1 < 0$, for $i = 1, 2$.

Therefore, if $\varepsilon > 0$ then there exists $t_0 > 1/m$ such that

$$\left|\frac{\partial^r}{\partial t^r}\phi_h(x,t)\right| < \varepsilon, \qquad \text{for } t > t_0 \text{ and } 0 < |h| < \frac{x}{2}.$$

Moreover, for every $t \in (1/m, t_0)$ one has

$$\left|\frac{\partial^r}{\partial t^r}\phi_h(x,t)\right| \le M_3 \left|\frac{1}{h}\int_x^{x+h}\int_x^u \left(y^{r-2} + y^{r-1} + y^r\right) dy\,du\right|,$$

M_3 being a positive constant. Hence $(\partial^r/\partial t^r)\phi_h(x,h) \longrightarrow 0$, as $h \longrightarrow 0$, uniformly in $t \in (1/m, t_0)$.

Therefore we conclude that $\gamma_{r,m}(\phi_h(x,t)) \longrightarrow 0$, as $h \longrightarrow 0$.

By proceeding inductively the proof can be completed. □

Proposition 4

Let $0 < \nu < 1/n$. If $F(x)$ denotes the generalized $\mathcal{L}_\nu^{(n)}$ transform of $f \in \mathcal{A}'$ then

$$|F(x)| \le P(x)\,x^{(n-1)\nu+(1/n)-1}\,e^{-x/r}, \qquad x > 0,$$

for a certain polynomial P and some $r \in \mathbb{N}$.

Proof. According to [18, Theorem 1.8-1] there exists $r \in \mathbb{N}$ and $m > 0$ for which

$$|F(x)| \le M \max_{\substack{0\le k\le r\\ 1\le l\le r}} \gamma_{l,k}\left(\lambda_\nu^{(n)}(xt)\right), \qquad \text{for every } x > 0.$$

Therefore, from (4) one deduces

$$\begin{aligned}|F(x)| &\le M \max_{p\le k\le r}\ \sup_{1/r<t<\infty}\left|x^k\frac{d^k}{d(xt)^k}\left(\lambda_\nu^{(n)}(xt)\right)\right| \\ &\le M_1 \max_{0\le k\le r}\ \sup_{1/r<t<\infty}\left|x^k\left(1+(xt)^{-k}\right)(xt)^{(n-1)\nu+(1/n)-1}\,e^{-xt}\right| \\ &\le P(x)x^{(n-1)\nu+(1/n)-1}e^{-x/r}, \qquad \text{for } x > 0,\end{aligned}$$

where $M_1 > 0$ and $P(x)$ is a suitable polynomial. □

In the following proposition we show an operational formula for the generalized $\mathcal{L}_\nu^{(n)}$ transformation involving the operator $B_{\nu,n}^*$.

Proposition 5

Let P be a polynomial. If $f \in \mathcal{A}'$ then

$$\left(\mathcal{L}_\nu^{(n)} P\left(B_{\nu,n}^*\right) f\right)(x) = P\big((-x)^n\big)\left(\mathcal{L}_\nu^{(n)} f\right)(x), \qquad \text{for } x > 0.$$

Proof. By virtue of (5) and according to Proposition 1, we can write

$$\begin{aligned}\left(\mathcal{L}_\nu^{(n)} P\left(B_{\nu,n}^*\right) f\right)(x) &= \left\langle P\left(B_{\nu,n}^*\right) f(t), \lambda_\nu^{(n)}(xt)\right\rangle \\ &= \left\langle f(t), P\left(B_{\nu,n}^*\right) \lambda_\nu^{(n)}(xt)\right\rangle \\ &= \left\langle f(t), P\left((-X)^n\right) \lambda_\nu^{(n)}(xt)\right\rangle \\ &= P\big((-x)^n\big)\left(\mathcal{L}_\nu^{(n)} f\right)(x), \qquad \text{for } x > 0. \ \Box\end{aligned}$$

We now establish an inversion theorem for the generalized $\mathcal{L}_\nu^{(n)}$ transformation.

To prove the inversion formula we use a procedure similar to the one employed by S. P. Malgonde and R. K. Saxena [11]. We need previously to show some results.

Lemma 1

Let $f \in \mathcal{A}'$ and $0 < \nu < 1/n$. Then

$$\int_0^\infty x^{-s}\left\langle f(t), \lambda_\nu^{(n)}(xt)\right\rangle dx = \left\langle f(t), \int_0^N x^{-s}\lambda_\nu^{(n)}(xt)\, dx\right\rangle$$

provided that $\Re s \leq (n-1)\nu + 1/n - 1$.

Proof. Let $N \in \mathbb{N}$. We will see firstly that

$$\int_0^N x^{-s}\left\langle f(t), \lambda_\nu^{(n)} xt\right\rangle dx = \left\langle f(t), \int_0^N x^{-s}\lambda_\nu^{(n)}(xt)\, dx\right\rangle. \tag{9}$$

If $\{x_{r,l}\}_{r=0}^{l}$ is a partition of the interval $[0, N]$ being $d_l = x_{r,l} - x_{r-1,l}$ for every $r = 1, 2, \ldots, l$, we can write

$$\int_0^N x^{-s}\left\langle f(t), \lambda_\nu^{(n)}(xt)\right\rangle dx = \lim_{l\to\infty} d_l \sum_{r=0}^{l} x_{r,l}^{-s}\left\langle f(t), \lambda_\nu^{(n)}(x_{r,l}t)\right\rangle$$

and

$$d_l \sum_{r=0}^{l} x_{r,l}^{-s} \langle f(t), \lambda_\nu^{(n)}(x_{r,l}t) \rangle = \left\langle f(t), d_l \sum_{r=0}^{l} x_{r,l}^{-s} \lambda_\nu^{(n)}(x_{r,l}t) \right\rangle .$$

Hence (9) will be proved when we derive

$$\lim_{l\to\infty} d_l \sum_{r=0}^{l} x_{r,l}^{-s} \lambda_\nu^{(n)}(x_{r,l}t) = \int_0^N x^{-s} \lambda_\nu^{(n)}(xt)\, dx \tag{10}$$

in the sense of convergence in $\mathcal{A}$.

For every $k \in \mathbb{N}$, according to (4), we get

$$\begin{aligned}
&\left| \frac{\partial^k}{\partial t^k} \left\{ d_l \sum_{r=0}^{l} x_{r,l}^{-s} \lambda_\nu^{(n)}(x_{r,l}t) - \int_0^N x^{-s} \lambda_\nu^{(n)}(xt) dx \right\} \right| \\
&\quad \leq d_l \sum_{r=0}^{l} x_{r,l}^{-\Re s + k} \left| \frac{d^k}{d(x_{r,l}t)} \lambda_\nu^{(n)}(x_{r,l}t) \right| + \int_0^N x^{-\Re s + k} \left| \frac{d^k}{d(xt)^k} \lambda_\nu^{(n)}(xt) \right| dx \\
&\quad \leq M\left(t^{-k} + 1\right) t^{(n-1)\nu + (1/n) - 1} \left(d_l \sum_{r=0}^{l} x_{r,l}^{-\Re s + (n-1)\nu + (1/n) - 1} (x_{r,l}^k + 1) \right. \\
&\qquad \left. + \int_0^N x^{-\Re s + (n-1)\nu + (1/n) - 1} (x^k + 1)\, dx \right)
\end{aligned}$$

Therefore

$$\lim_{t\to\infty} \frac{\partial^k}{\partial t^k} \left\{ d_l \sum_{r=0}^{l} x_{r,l}^{-s} \lambda_\nu^{(n)}(x_{r,l}t) - \int_0^N x^{-s} \lambda_\nu^{(n)}(xt)\, dx \right\} = 0.$$

Then, since the function

$$g(t,x) = \begin{cases} x^{-s+k} \dfrac{d^k}{d(xt)^k} \left(\lambda_\nu^{(n)}(xt) \right), & \text{if } t \in [a,b] \text{ and } x \in (0,N] \\ 0, & \text{if } t \in [a,b] \text{ and } x = 0, \end{cases}$$

with $0 < a < b < \infty$, is uniformly continuous on $(t,x) \in [a,b] \times [0,N]$, if $\varepsilon > 0$ and $m \in \mathbb{N}$ there exists $l_0 \in \mathbb{N}$ such that

$$\sup_{1/m < t < \infty} \left| \frac{\partial^k}{\partial t^k} \left\{ d_l \sum_{r=0}^{l} x_{r,l}^{-s} \lambda_\nu^{(n)}(x_{r,l}t) - \int_0^N x^{-s} \lambda_\nu^{(n)}(xt) dx \right\} \right| < \varepsilon$$

for $l > l_0$. Thus (10) is shown.

On the other hand, by invoking again (4), for every $m, k \in \mathbb{N}$,

$$\left| \frac{d^k}{dt^k} \int_N^\infty x^{-s} \lambda_\nu^{(n)}(xt) dx \right| \leq M t^{(n-1)\nu+(1/n)-1} \left(1 + t^{-k}\right) \int_N^\infty x^{-\Re s+(n-1)\nu+(1/n)-1} \left(x^k + 1\right) e^{-xt}\, dx$$

and hence

$$\frac{d^k}{dt^k} \int_N^\infty x^{-s} \lambda_\nu^{(n)}(xt) dx \longrightarrow 0, \qquad \text{as } N \longrightarrow \infty, \tag{11}$$

uniformly in $t \in (1/m, \infty)$.

Moreover, by virtue of Proposition 4, for certain P polynomial and $r \in \mathbb{N}$

$$\left| \int_N^\infty x^{-s} \langle f(t), \lambda_\nu^{(n)}(xt) \rangle \, dx \right| \leq \int_N^\infty P(x) x^{-\Re s+(n-1)\nu+(1/n)-1} e^{-x/r}\, dx.$$

Hence

$$\lim_{N \to \infty} \int_N^\infty x^{-s} \langle f(t), \lambda_\nu^{(n)}(xt) \rangle \, dx = 0. \tag{12}$$

From (9), (11) and (12) we can conclude that

$$\int_0^\infty x^{-s} \langle f(t), \lambda_\nu^{(n)}(xt) \rangle \, dx = \left\langle f(t), \int_0^\infty x^{-s} \lambda_\nu^{(n)}(xt) dx \right\rangle . \square$$

Lemma 2

Let $\phi \in D(I)$ and denote

$$\psi(s) = \int_0^\infty y^{-s} \phi(y)\, dy.$$

Then

$$\int_{-R}^{R} \langle f(u), u^{\sigma+iw-1} \rangle \psi(\sigma + iw) dw = \left\langle f(u), \int_{-R}^{R} u^{\sigma+iw-1} \psi(\sigma + iw) dw \right\rangle$$

with $\sigma < 1$ and $R > 0$.

is continuous for every $u > 0$ and $t \neq 0$. Moreover

$$\frac{\partial}{\partial t}\left\{e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}\left(\phi(ue^{-t})\right)\right\}\Big|_{t=0} = \lim_{t\to 0}\frac{1}{t}\left\{e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}\left(\phi(ue^{-t})\right) - \frac{d^k}{du^k}\left(\phi(U)\right)\right\}$$

Hence the function defined by

$$G(t,u) = \begin{cases} H(t,u), & \text{if } t \neq 0 \text{ and } u > 0 \\ \frac{\partial}{\partial t}\left\{e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}\left(\phi(ue^{-t})\right)\right\}\Big|_{t=0}, & \text{if } t = 0 \text{ and } u > 0 \end{cases}$$

is continuous for $t \in \mathbb{R}$ and $u > 0$. Furthermore there exists a positive constant M such that $|G(t,u)| \leq M$ for every $t \in [-\delta, \delta]$ and $u > 0$, because $\phi \in D(I)$.

Therefore if $\varepsilon > 0$ then

$$\left|\frac{1}{\pi}\int_{-\delta}^{\delta} G(t,u)\sin(Rt)\,dt\right| < \varepsilon,$$

for some $\delta > 0$ and for each $R > 0$.

Write now

$$\frac{1}{\pi}\int_{-\infty}^{-\delta}\left\{e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}\left(\phi(ue^{-t})\right) - \frac{d^k}{du^k}\left(\phi(u)\right)\right\}\frac{\sin(Rt)}{t}\,dt = J_{1,R}(u) - J_{2,R}(u),$$

being

$$J_{1,R}(u) = \frac{1}{\pi}\int_{-\infty}^{-\delta} e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}\left(\phi(ue^{-t})\right)\frac{\sin(Rt)}{t}\,dt,$$

and

$$J_{2,R}(u) = \frac{1}{\pi}\frac{d^k}{du^k}\left(\phi(u)\right)\int_{-\infty}^{-R\delta}\frac{\sin z}{z}\,dz.$$

By partial integration in $J_{1,R}(u)$ we get

$$\begin{aligned} J_{1,R}(u) = \frac{1}{R\pi}\Bigg\{&\left[-e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}\left(\phi(ue^{-t})\right)\frac{\cos(Rt)}{t}\right]_{t=-\infty}^{t=-\delta} \\ &+\int_{-\infty}^{-\delta}\cos(Rt)\,\frac{\partial}{\partial t}\left(\frac{1}{t}\,e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}\left(\phi(ue^{-t})\right)\right)dt\Bigg\} \\ = {}& e^{(\sigma-1)\delta}\frac{d^k}{du^k}\left(\phi(ue^{-\delta})\right)\frac{\cos(R\delta)}{\pi R\delta} \\ &+\frac{1}{R\pi}\int_{-\infty}^{-\delta}\cos(Rt)\,\frac{\partial}{\partial u}\left(\frac{1}{t}e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}\left(\phi(ue^{-t})\right)\right)dt. \end{aligned}$$

Since $d^k/du^k(\phi(ue^{-t}))\cos(Rt)$ is bounded for $u \in (0,\infty)$ and $R \in \mathbb{R}$, it can be easily deduced that the first term of the last sum converges to zero as $R \to \infty$ uniformly for $u > 0$.

Moreover, one has

$$\frac{\partial}{\partial t}\left(\frac{1}{t}\, e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}(\phi(ue^{-t}))\right)$$
$$= \frac{1}{t}\, e^{(\sigma-1)t}\left((\sigma - 1 - \frac{1}{t})\frac{\partial^k}{\partial u^k}(\phi(ue^{-t})) + \frac{\partial}{\partial t}\frac{\partial^k}{\partial u^k}(\phi(ue^{-t}))\right).$$

Also $\phi(ue^{-t}) = 0$ provided that $ae^t \leq u \leq be^t$ for some $0 < a < b < \infty$. Hence if $u > 1/m$ then there exist $t_0 < -\delta$ such that

$$\frac{\partial}{\partial t}\left(\frac{1}{t}e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}(\phi(ue^{-t}))\right) = 0 \qquad \text{for } t \leq t_0.$$

Then

$$\left|\int_{-\infty}^{-\delta}\cos(Rt)\,\frac{\partial}{\partial t}\left(\frac{1}{t}e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}(\phi(ue^{-t}))\right)dt\right| \leq M_1$$

for every $u > /m$ and for a certain $M_1 > 0$.

Thus we can conclude that $J_{1,R}(u) \longrightarrow 0$ as $R \longrightarrow \infty$ uniformly in $u > 1/m$.

On the other hand, since

$$\int_{-\infty}^{0}\frac{\sin z}{z}\,dz$$

is convergent and $\phi \in D(I)$, it follows that $\lim_{R\to\infty} J_{2,R}(u) = 0$, uniformly in $u > 1/m$.

By proceeding in a similar way we can also prove that

$$\lim_{R\to\infty}\int_{\delta}^{\infty}\left(e^{(\sigma-1)t}\frac{\partial^k}{\partial u^k}(\phi(ue^{-t})) - \frac{\partial^k}{\partial u^k}(\phi(u))\right)\frac{\sin(Rt)}{t}\,dt \longrightarrow 0$$

as $R \longrightarrow \infty$, uniformly in $u > 1/m$, provided that $\sigma < 1$.

Therefore the desired result is established. □

As a consequence of the three previous Lemmas we can now prove the following inversion formula.

Theorem 1

Let $f \in \mathcal{A}'$, $\phi \in D(I)$, $0 < \nu < 1/n$ and $\sigma < (n-1)\nu + 1/n - 1$. Then

$$\left\langle \lim_{R\to\infty} \int_{\sigma-iR}^{\sigma+iR} \frac{1}{K(s)} y^{-s} \int_0^\infty x^{-s} F(x) dx\, ds, \phi(y) \right\rangle = \langle f(t), \phi(t) \rangle,$$

where

$$F(x) = \left(\mathcal{L}_\nu^{(n)} f\right)(x), \quad \text{for } x > 0,$$

and

$$K(s) = (2\pi)^{(n-1)/2} n^{-n\nu-(1/2)} \frac{\Gamma(1+n\nu-s)\,\Gamma((1-s)/n))}{\Gamma(\nu+1-(s/n))}.$$

Proof. Let $f \in \mathcal{A}'$ and denote $F(x) = \langle f(t), \mathcal{L}_\nu^{(n)}(xt)\rangle$, for $x > 0$. It can be easily seen that the function

$$\varphi_R(y) = \frac{1}{2\pi i} \int_{\sigma-iR}^{\sigma+iR} \frac{1}{K(s)} y^{-s} \int_0^\infty x^{-s} F(x)\, dx\, ds$$

is continuous for $y > 0$, for every $R > 0$. Hence $\varphi_R(y)$ defines a regular distribution in $D'(I)$ being

$$\langle \varphi_R(y), \phi(y) \rangle = \frac{1}{2\pi i} \int_0^\infty \phi(y) \int_{\sigma-iR}^{\sigma+iR} \frac{1}{K(s)} y^{-s} \int_0^\infty x^{-s} F(x)\, dx\, ds\, dy,$$

for $\phi \in D(I)$.

By applying Fubini's theorem we can interchange the order of integration and we get

$$\begin{aligned} &\langle \varphi_R(y), \phi(y) \rangle \\ &\quad = \frac{1}{2\pi i} \int_{\sigma-iR}^{\sigma+iR} \frac{1}{K(s)} y^{-s} \left\{ \int_0^\infty x^{-s} \langle f(t), \mathcal{L}_\nu^{(n)}(xt) \rangle\, dx \right\} \int_0^\infty \phi(y)\, y^{-s}\, dy\, ds. \end{aligned}$$

By invoking now Lemma 1, it follows that

$$\begin{aligned} &\langle \varphi_R(y), \phi(y) \rangle \\ &\quad = \frac{1}{2\pi i} \int_{\sigma-iR}^{\sigma+iR} \frac{1}{K(s)} \left\langle f(t), t^{s-1} \int_0^\infty u^{-s} \mathcal{L}_\nu^{(n)}(u)\, du \right\rangle \int_0^\infty \phi(y)\, y^{-s} dy\, ds, \end{aligned}$$

and accordingly to (6),

$$\langle \varphi_R(y), \phi(y) \rangle = \frac{1}{2\pi i} \int_{\sigma - iR}^{\sigma + iR} \langle f(t), t^{s-1} \rangle \int_0^\infty \phi(y)\, y^{-s} dy\, ds.$$

Lemma 2 leads to

$$\langle \varphi_R(y), \phi(y) \rangle = \left\langle f(t), \frac{1}{2\pi i} \int_{\sigma - iR}^{\sigma + iR} t^{s-1} \int_0^\infty \phi(y)\, y^{-s}\, dy\, ds \right\rangle.$$

Finally, by interchanging again the order of integration and by using Lemma 3 we can establish

$$\langle \varphi_R(y), \phi(y) \rangle = \left\langle f(t), \frac{1}{\pi} \int_0^\infty \phi(y) \left(\frac{u}{y}\right)^\sigma \frac{\sin(R \log(u/y))}{u \log(u/y)}\, dy \right\rangle \longrightarrow \langle f(t), \phi(t) \rangle,$$

as $R \longrightarrow \infty$. Thus our theorem is proved. □

From Theorem 1 the following uniqueness theorem can be immediately proved:

Theorem 2

Let f and g be in $\mathcal{A}'$. If $(\mathcal{L}_\nu^{(n)} f)(x) = (\mathcal{L}_\nu^{(n)} g)(x)$, for $x > 0$, then $f = g$ in the sense of equality in $D'(I)$, provided that $0 < \nu < 1/n$.

Proof. It is sufficient to see that for every $\phi \in D(I)$

$$\begin{aligned} \langle f(t) - g(t), \phi(t) \rangle &= \left\langle \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\sigma - iR}^{\sigma + iR} \frac{1}{K(s)} y^{-s} \right. \\ &\quad \left. \times \int_0^\infty x^{-s} \{ (\mathcal{L}_\nu^{(n)} f)(x) - (\mathcal{L}_\nu^{(n)} g)(x) \} dx\, ds, \phi(y) \right\rangle \\ &= 0 \end{aligned}$$

where σ and $K(s)$ are as in Theorem 1. □

4. The generalized $\mathcal{L}_\nu^{(n)}$ transform of $E'(I)$

As it was mentioned in Section 2, $\mathcal{A}$ is contained in $E(I)$ and the topology of $\mathcal{A}$ is stronger than the one induced in it by $E(I)$. Hence, if $f \in E(I)$, then the restriction of f to $\mathcal{A}$ is also in $\mathcal{A}'$ and we can define the generalized $\mathcal{L}_\nu^{(n)}$ transform $\mathcal{L}_\nu^{(n)} f$ of f by

$$(\mathcal{L}_\nu^{(n)} f)(x) = \langle f(t), \lambda_\nu^{(n)}(xt) \rangle, \qquad \text{for } x > 0,$$

with $\nu > -1 + 1/n$ and $n \in \mathbb{N}$.

By proceeding as in Section 3 we can establish the following properties for the generalized $\mathcal{L}_\nu^{(n)}$ transformation in $E'(I)$. Notice that now we remove some restrictions for the parameter ν.

Proposition 6

Let $f \in E'(I)$. If $F(x) = (\mathcal{L}_\nu^{(n)} f)(x)$ for $x > 0$, then $F(x)$ is infinitely differentiable on $x > 0$ and

$$\frac{d^r}{dx^r} F(x) = \left\langle f(t), \frac{\partial^r}{\partial x^r} \lambda_\nu^{(n)}(xt) \right\rangle,$$

for $x > 0$ and $r \in \mathbb{N}$.

Proposition 7

If $f \in E'(I)$ and $F(x) = (\mathcal{L}_\nu^{(n)} f)(x)$, for $x > 0$, then there exist two positive numbers M and a such that

$$|F(x)| \leq M e^{-ax}, \qquad \text{for } x > 0,$$

provided that $\nu > 0$.

Proposition 8

Let P be a polynomial. If $f \in E'(I)$ then

$$(\mathcal{L}_\nu^{(n)} P(B_{\nu,n}^*) f)(x) = P((-x)^n)(\mathcal{L}_\nu^{(n)} f)(x), \qquad \text{for } x > 0.$$

Theorem 3

Let $f \in E'(I)$, $\phi \in D(I)$, $\nu > 0$ and $\sigma > 0$. Then

$$\lim_{R\to\infty} \left\langle \frac{1}{2\pi i} \int_{\sigma-iR}^{\sigma+iR} \frac{1}{K(s)} y^{-s} \int_0^\infty x^{-s} F(x)\, dx\, ds, \phi(y) \right\rangle = \langle f(t), \phi(t) \rangle,$$

where $F(x) = (\mathcal{L}_\nu^{(n)} f)(x)$, $x > 0$ and $K(s)$ is defined as in Theorem 1.

Theorem 4

Let f and g be in $E'(I)$ and $\nu > 0$. If $(\mathcal{L}_\nu^{(n)} f)(x) = (\mathcal{L}_\nu^{(n)} g)(x)$, for $x > 0$, then $f = g$.

In a previous paper [1] we establish a real inversion of the formula for the classical $\mathcal{L}_\nu^{(n)}$ transformation as follows.

Theorem 5

Let $0 < \nu < 1/n$, and $f(t)$, for $0 < t < \infty$, a real or complex function satisfying

i) $f(t) \in L_1([R^{-1}, R])$, *for every* $R > 1$,

ii) $f(t)e^{-ct} \in L_1(1, \infty)$, *for some* $c > 0$,

iii) $f(t)t^r \in L_1(0,1)$, *for some* $r > 1/n + (n-1)\nu - 2$,

and

iv)

$$\int_t^s \left|f_y(u) - f_y(t)\right| du = O\big(|s-t|\big), \qquad \text{as } t \longrightarrow s,$$

where

$$f_y(u) = \left|u^{(n-1)\nu+(1/n)-1} f(u) - y^{(n-1)\nu+(1/n)-1} f(y)\right|, \qquad \text{for } y > 0.$$

Then

$$\lim_{k\to\infty} A_{\nu,n,k} F\left(\frac{nk}{x}\right) = f(x), \qquad \text{for } x > 0$$

where $F(x) = \mathcal{L}_\nu^{(n)}\{f\}(x)$ *and*

$$A_{\nu,n,k} = \frac{n^{n\nu+(1/2)}(nk)^{2-(n-1)\nu-(1/n)}\,\Gamma\big(k + \frac{\nu+1}{n} - \frac{1}{n^2} + 1\big)}{(2\pi)^{(n-1)/2}\,\Gamma(nk + 2 + \nu - \frac{1}{n})\,\Gamma(k - (\frac{n-1}{n})\nu - \frac{1}{n^2} + \frac{2}{n})}\, x^{-nk-1} A_{\nu,n}^k,$$

being

$$A_{\nu,n} = x^{1-n\nu}\left(x^2 \frac{d}{dx}\right)^{n-1} x^{n\nu+1}\frac{d}{dx}.$$

We now prove a distributional version of the above inversion formula.

Theorem 6

Let $f \in E'(I)$ and $\nu > -1 + 1/n$. If $F(x)$ denotes the generalized $\mathcal{L}_\nu^{(n)}$ transform of f then

$$\lim_{k\to\infty} \left\langle A_{\nu,n,k} F\left(\frac{nk}{x}\right), \phi(x)\right\rangle = \langle (f(t), \phi(t) \rangle,$$

for $\phi \in D(I)$, where $A_{\nu,n,k}$ is defined as in Theorem 5.

Proof. Let $f \in E'(I)$ and denote $F(x) = \langle f(t), \lambda_\nu^{(n)}(xt)\rangle$, for $x > 0$.

By virtue of Proposition 3 and by (5) we can write

$$A_{\nu,n,k}F\left(\frac{nk}{x}\right) = M_{\nu,n,k}x^{-1-nk}\left\langle \frac{1}{t}f(t), t^{nk+1}\lambda_\nu^{(n)}\left(\frac{nkt}{x}\right)\right\rangle$$

where

$$M_{\nu,n,k} = \frac{n^{n\nu+(1/2)}(nk)^{nk+2-(n-1)\nu-(1/n)}\,\Gamma(k+\frac{\nu+1}{n}-\frac{1}{n^2}+1)}{(2\pi)^{(n-1)/2}\,\Gamma(nk+2+\nu-\frac{1}{n})\,\Gamma(k-(\frac{n-1}{n})\nu-\frac{1}{n^2}+\frac{2}{n})}$$

for each $k \in \mathbb{N}$.

Also $A_{\nu,n,k}F(nk/x)$ defines a regular distribution in $D'(I)$ by

$$\left\langle A_{\nu,n,k}F\left(\frac{nk}{x}\right), \phi(x)\right\rangle = \int_0^\infty A_{\nu,n,k}F\left(\frac{nk}{x}\right)\phi(x)\,dx, \qquad \text{for } \phi \in D(I).$$

Moreover,

$$\left\langle A_{\nu,n,k}F\left(\frac{nk}{x}\right), \phi(x)\right\rangle = \left\langle \frac{1}{t}\,f(t), M_{\nu,n,k}t^{nk+1}\int_0^\infty x^{-1-nk}\lambda_\nu^{(n)}\left(\frac{nky}{x}\right)\phi(x)\,dx\right\rangle \tag{13}$$

for every $\phi \in D(I)$.

In effect, let $\phi \in D(I)$. We choose $0 < a < b < \infty$ such that $\phi(x) = 0$, for every $x \in [a,b]$. If $\{x_{m,l}\}_{m=0}^{l}$ denotes a partition of $[a,b]$ being $d_l = x_{m,l} - x_{m,l-1}$, $m = 1, 2, \ldots, l$, then we get

$$\int_0^\infty A_{\nu,n,k}F\left(\frac{nk}{x}\right)\phi(x)\,dx =$$
$$= \lim_{l\to\infty} d_l \sum_{m=0}^{l} A_{\nu,n,k}F\left(\frac{nk}{x_{m,l}}\right)\phi(x_{m,l})$$
$$= \lim_{l\to\infty}\left\langle \frac{1}{t}\,f(t), t^{nk+1}M_{\nu,n,k}d_l\sum_{m=0}^{l} x_{m,l}^{-nk-1}\lambda_\nu^{(n)}\left(\frac{nkt}{x_{m,l}}\right)\phi(x_{m,l}\right\rangle$$

Hence we must show that

$$\lim_{l\to\infty} d_l\sum_{m=0}^{l} x_{m,l}^{-nk-1}\lambda_\nu^{(n)}\left(\frac{nkt}{x_{m,l}}\right)\phi(x_{m,l}) = \int_a^b x_{-nk-1}\lambda_\nu^{(n)}\left(\frac{nkt}{x_{m,l}}\right)\phi(x)\,dx,$$

in the sense of convergence in $E(I)$.

Let K be a compact subset of $(0,\infty)$ and $r \in \mathbb{N}$. One has

$$\frac{d^r}{dt^r}\left(d_l \sum_{m=0}^{l} x_{m,l}^{-nk-1} \lambda_\nu^{(n)}\left(\frac{nkt}{x_{m,l}}\right)\phi(x_{m,l}) - \int_a^b x_{-nk-1}\lambda_\nu^{(n)}\left(\frac{nkt}{x}\right)\phi(x)\,dx\right)$$

$$= d_l \sum_{m=0}^{l} x_{m,l}^{-nk-1} \frac{\partial^r}{\partial t^r}\left(\lambda_\nu^{(n)}\left(\frac{nkt}{x_{m,l}}\right)\right)\phi(x_{m,l})$$

$$- \int_a^b x_{-nk-1}\frac{\partial^r}{\partial t^r}\left(\lambda_\nu^{(n)}\left(\frac{nkt}{x}\right)\right)\phi(x)\,dx.$$

Hence, since the function

$$x_{-nk-1}\frac{\partial^r}{\partial t^r}\left(\lambda_\nu^{(n)}\left(\frac{nkt}{x}\right)\right)\phi(x)$$

is uniformly continuous for $(x,y) \in [a,b] \times K$, then

$$\lim_{l\to\infty} d_l \sum_{m=0}^{l} x_{m,l}^{-nk-1} \frac{\partial^r}{\partial t^r}\left(\lambda_\nu^{(n)}\left(\frac{nkt}{x_{m,l}}\right)\right)\phi(x_{m,l})$$

$$= \int_a^b x_{-nk-1}\frac{\partial^r}{\partial t^r}\left(\lambda_\nu^{(n)}\left(\frac{nkt}{x}\right)\right)\phi(x)\,dx$$

uniformly in $x \in K$. Thus (13) is proved.

On the other hand by making single changes of variables we can write

$$\int_0^\infty A_{\nu,n,k}F\left(\frac{nk}{x}\right)\phi(x)\,dx = \left\langle g(t), M_{\nu,n,k}t^{-nk-1}\int_0^\infty \lambda_\nu^{(n)}\left(\frac{nkx}{t}\right)\psi(x)\,x^{nk}\,dx\right\rangle,$$

where $g(t) = (1/t)f(t)$ and $\psi(x) = (1/x)f(1/x)$.

To complete the proof we have to show that

$$\lim_{k\to\infty} M_{\nu,n,k}t^{-nk-1}\int_0^\infty \lambda_\nu^{(n)}\left(\frac{nkx}{t}\right)\psi(x)\,x^{nk}\,dx = \psi(t), \tag{14}$$

in the sense of convergence in $E(I)$.

By using (6) we derive

$$\frac{d^r}{dt^r}\left(M_{\nu,n,k}t^{-nk-1}\int_0^\infty \lambda_\nu^{(n)}\left(\frac{nkx}{t}\right)\psi(x)\,x^{nk}\,dx - \psi(t)\right)$$

$$= M_{\nu,n,k}t^{-nk-1}\int_0^\infty x^{nk-(n-1)\nu-(1/n)+1}\lambda_\nu^{(n)}\left(\frac{nkx}{t}\right)$$

$$\times\left(x^{(n-1)\nu+(1/n)-1}\left(\frac{x}{t}\right)^r\frac{d^r}{dx^r}(\psi(x)) - t^{(n-1)\nu+(1/n)-1}\cdot\frac{d^r}{dt^r}(\psi(t))\right)dx$$

for every $r \in \mathbb{N}$.

Hence, according to (3), if K is a compact subset of $(0,\infty)$, then there exists a positive constant $M > 0$ such that

$$\begin{aligned}\left|\frac{d^r}{dt^r}\left(M_{\nu,n,k}t^{-nk-1}\int_0^\infty \lambda_\nu^{(n)}\left(\frac{nkx}{t}\right)\psi(x)\,x^{nk}\,dx - \psi(t)\right)\right| \\ \leq M\frac{1}{(nk)}\left(\frac{nk}{t}\right)^{nk+1}\int_0^\infty e^{-nkx/t}x^{nk}\left|x^{(n-1)\nu+(1/n)-1}\left(\frac{x}{t}\right)^r\frac{d^r}{dx^r}(\phi(x))\right. \\ \left. - t^{(n-1)\nu+(1/n)-1}\frac{d^r}{dx^r}(\psi(t))\right|dx\end{aligned}$$

for every $t \in K$.

We divide the last integral as follows

$$\begin{aligned}M\frac{1}{(nk)!}\left(\frac{nk}{t}\right)^{nk+1}\int_0^\infty &= M\frac{1}{(nk)!}\left(\frac{nk}{t}\right)^{nk+1}\left\{\int_0^{t(1-\eta)} + \int_{t(1-\eta)}^{t(1+\eta)} + \int_{t(1+\eta)}^\infty\right\} \\ &= I_1(t,k) + I_2(t,k) + I_3(t,k)\end{aligned}$$

with $\eta > 0$.

For every $t \in K$,

$$\begin{aligned}\left|I_1(t,k)\right| &\leq M\frac{1}{(kn)}\left(\frac{nk}{t}\right)^{nk+1}\left\{\int_0^{t(1-\eta)} e^{-nkx/t}x^{nk+(n-1)\nu+(1/n)-1}\left|\frac{d^r}{dx^r}(\psi(x))\right|dx\right. \\ &\quad \left. + \int_0^{t(1-\eta)} e^{-nkx/t}x^{nk}\,dx\,\left|t^{(n-1)\nu+(1/n)-1}\frac{d^r}{dt^r}(\psi(t))\right|\right\} \\ &\leq M_1\frac{(nk)^{nk+1}}{(nk)!}\int_0^{1-\eta} e^{-nku}\,u^{nk}\,du,\end{aligned}$$

for some $M_1 > 0$. By invoking [15, (17)] we obtain

$$I_1(t,k) \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty, \text{ uniformly in } t \in K. \tag{15}$$

By proceeding in a similar way we can prove that

$$I_3(t,k) \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty, \text{ uniformly in } t \in K. \tag{16}$$

Finally we analize $I_2(t,k)$. From the mean value Theorem we deduce

$$\left| x^{(n-1)\nu+(1/n)-1}\left(\frac{x}{t}\right)^r \frac{d^r}{dx^r}(\psi(x)) - t^{(n-1)\nu+(1/n)-1}\frac{d^r}{dt^r}(\psi(t))\right| \leq M_2\,|t-x|$$

for $x,t \in (0,\infty)$ and M_2 being a suitable positive constant.

Hence, by using [15, (16)], if $x \in (t(1-\eta), t(1+\eta))$

$$|I_2(t,k)| \leq M_3\eta\,\frac{(nk)^{nk+1}}{(nk)!}\int_{1-\eta}^{1+\eta} e^{-nku}\,u^{nk}\,du \leq M_4\eta, \qquad \text{for every } k \in \mathbb{N}, \quad (17)$$

for certain $M_i > 0$, $i = 3,4$.

Result (14) follows from (15)–(17).

Therefore we can conclude that

$$\lim_{k\to\infty}\left\langle A_{\nu,n,k}F\left(\frac{nk}{x}\right), \phi(x)\right\rangle = \langle g(t),\psi(t)\rangle = \langle f(t),\phi(t)\rangle, \qquad \text{for } \phi \in D(I). \; \square$$

From Theorem 5 it is inmediately deduced the following uniqueness theorem

Theorem 6

Let f and g be in $E'(I)$. If $(\mathcal{L}_\nu^{(n)}f)(x) = (\mathcal{L}_\nu^{(n)}g)(x)$, for $x > 0$, then $f = g$, provided that $\nu > -1 + 1/n$.

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