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# Classification of finite groups with many minimal subgroups and with the number of conjugacy classes of $G/S(G)$ equal to 8 †

Antonio Vera López, Jesús María Arregi Lizarraga

*Departamento de Matemáticas, Facultad de Ciencias, Universidad del País Vasco, Apartado 844, Bilbao, Spain*

and Francisco José Vera López

*Departamento de Matemática Aplicada, Escuela de Informática, Murcia, Spain*

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Abstract

In this paper we classify all the finite groups satisfying $r(G/S(G)) = 8$ and $\beta(G) = r(G) - \alpha(G) - 1$, where $r(G)$ is the number of conjugacy classes of $G$, $\beta(G)$ is the number of minimal normal subgroups of $G$, $S(G)$ the socle of $G$ and $\alpha(G)$ the number of conjugacy classes of $G$ out of $S(G)$. These results are a contribution to the general problem of the classification of the finite groups according to the number of conjugacy classes.

In this work $G$ will always denote a finite group and we shall follow the notation given in [4, 5, 6, 7]. Thus, a group satisfying the condition $\beta(G) = r(G) - \alpha(G) - 1$ will be named a $\Gamma$-group. The present paper is a continuation of [6] and [7], in which all finite $\Gamma$-groups with $r(G/S(G)) \leq 7$ are classified.

The condition $\beta(G) = r(G) - \alpha(G) - 1$ is equivalent to say that each conjugacy class of $G$ contained in $S(G) - \{1\}$ together with $\{1\}$ makes up a minimal normal

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subgroup of $G$. For these groups we have $|S(G)| = p^s$, $p$ being a prime number, and all the minimal normal subgroups of $G$ have the same cardinal. We shall often use the auxiliary lemmas of the referred papers [4, 5, 6, 7].

**Theorem**

*Let $G$ be a $\Gamma$-group such that $r(G/S(G))) = 8$. Then $G$ is isomorphic to one of the following groups:*

a) *If $G/S(G) = C_8$ then $G$ is one of the following groups:*

1) $C_{16} \times C_2^t$.
2) $C_3^{2t} \times_f C_8 = \left(\prod_{i=1}^t \langle x_i\rangle \times \langle y_i\rangle\right) \times_f \langle a\rangle$ *with* $x_i^a = y_i$, $y_i^a = x_i y_i$, $i = 1, \ldots, t$.

b) *If $G/S(G) = C_4 \times C_2$ then $G$ is one of the following groups:*

1) $C_8 \times C_4 \times C_2^t$.
2) $N \times C_2^t$ *with $N$ isomorphic to one of the following groups:*

$C_8 \times_\lambda C_2 = \langle a\rangle \times_\lambda \langle b\rangle$ *with* $a^b = a^5$,
$C_8 \times_\lambda C_4 = \langle a\rangle \times_\lambda \langle b\rangle$ *with* $a^b = a^5$,
$(C_2 \times C_8) \times_\lambda C_2 = (\langle a\rangle \times \langle b\rangle) \times_\lambda \langle c\rangle$ *with* $a^c = a$, $b^c = ab$,
$C_4 \times_\lambda C_8 = \langle a\rangle \times_\lambda \langle b\rangle$ *with* $a^b = a^{-1}$,
$(C_2 \times C_8) \times_\lambda C_4 = (\langle a\rangle \times_\lambda \langle b\rangle) \times \langle c\rangle$ *with* $a^c = a$, $b^c = ab$.

c) *If $G/S(G) = C_2^3$ then $G$ is one of the following groups:*

1) $N \times C_2^t$ *where $N$ is a 2-group isomorphic to one of the following groups:*

$C_4 \times D_8$,
$C_4 \times Q_8$,
$(C_4 \times C_2) \times_\lambda C_2 = (\langle a\rangle \times \langle b\rangle) \times_\lambda \langle c\rangle$ *with* $a^c = a$, $b^c = a^2 b$,
$C_4^2 \times_\lambda C_2 = (\langle a\rangle \times \langle b\rangle) \times_\lambda \langle c\rangle$ *with* $a^c = a$, $b^c = a^2 b$.

2) $C_2^4 \times_\lambda C_2 = \langle a_1, a_2, a_3, a_4\rangle \times_\lambda \langle b\rangle$ *with* $a_1^b = a_1$, $a_2^b = a_2$, $a_3^b = a_1 a_3$, $a_4^b = a_2 a_4$.
3) $C_4^2 \times_\lambda C_2 = (\langle a\rangle \times \langle b\rangle) \times_\lambda \langle c\rangle$ *with* $a^c = a^{-1}$, $b^c = b^{-1}$.
4) $C_4^2 \cdot C_4 = \langle a, b\rangle \cdot \langle c\rangle$ *with* $c^2 = a^2$, $a^c = a^{-1}$, $b^c = b^{-1}$.
5) $(C_2^2 \times C_4) \times_\lambda C_2 = (\langle a_1, a_2\rangle \times \langle a\rangle) \times_\lambda \langle b\rangle$ *with* $a_1^b = a_1$, $a_2^b = a_1 a_2$, $a^b = a^{-1}$.
6) $(C_2^2 \times C_4) \cdot C_4 = (\langle a_1, a_2\rangle \times \langle a\rangle) \cdot \langle b\rangle$ *with* $b^2 = a^2$, $a_1^b = a_1$, $a_2^b = a_1 a_2$, $a^b = a^{-1}$.
7) $C_4^2 \times_\lambda C_2 = (\langle a\rangle \times \langle b\rangle) \times_\lambda \langle c\rangle$ *with* $a^c = a^{-1}$, $b^c = a^2 b^{-1}$.
8) $(C_2^2 \times C_4) \times_\lambda C_2 = (\langle a_1, a_2\rangle \times \langle a\rangle) \times_\lambda \langle b\rangle$ *with* $a_1^b = a_1$, $a_2^b = a^2 a_2$, $a^b = a_1 a$.
9) $C_4^2 \times_\lambda C_2 = (\langle a\rangle \times \langle b\rangle) \times_\lambda \langle c\rangle$ *with* $a^c = b^2 a^{-1}$, $b^c = a^2 b$.

*10)* $C_4^2 \cdot C_4 = \langle a,b\rangle \cdot \langle c\rangle$ *with* $c^2 = (ab)^2$, $a^c = a^{-1}$, $b^c = a^2b^{-1}$.

*11) One of the following groups:*

$2^6\Gamma_2 m_1$,
$2^6\Gamma_2 m_2$,
$2^6\Gamma_2 n$,
$2^6\Gamma_2 t$,
$2^6\Gamma_4 g_1$,
$2^6\Gamma_4 g_2$,
$2^6\Gamma_4 h$,
$2^6\Gamma_4 i_k$, $k = 1,\ldots,5$,
$2^6\Gamma_4 j_1$,
$2^6\Gamma_4 k_1$,
$2^6\Gamma_4 k_2$,
$2^6\Gamma_4 l$,
$2^6\Gamma_9 a_1$,
$2^6\Gamma_9 a_2$,
$2^6\Gamma_9 b_1$,
$2^6\Gamma_9 b_2$,
$2^6\Gamma_9 b_4$,
$2^6\Gamma_9 c$,
$2^6\Gamma_9 d_2$,
$2^6\Gamma_9 e$.

*12)* $C_4 \times (C_4 \times C_2) \times_\lambda C_4 = C_4 \times (\langle a\rangle \times \langle z\rangle) \times_\lambda \langle b\rangle$ *with* $a^b = az$, $z^b = z$.

*13) The set of all* $N$ *such that* $N \in 2^7\Gamma_4 \cup 2^7\Gamma_9$, $Z(N) \cong C_2^4$, *and* $N$ *has no direct factors isomorphic to* $C_2$.

*14)* $(C_4^2 \times C_2^2) \times_\lambda C_4 = (\langle a_1,a_2\rangle \times \langle y,z\rangle) \times_\lambda \langle a_3\rangle$ *with* $a_1^{a_3} = a_1y$, $a_2^{a_3} = a_2z$, $y^{a_3} = y$, $z^{a_3} = z$.

*15)* $(C_4^2 \times_\lambda C_4) \times_\lambda C_4 = (\langle a_1,x,y,z\rangle \times_\lambda \langle a_2\rangle) \times_\lambda \langle a_3\rangle$ *with* $a_1^{a_2} = a_1x$, $a_1^{a_3} = a_1y$, $a_2^{a_3} = a_2z$, $x^{a_2} = x$, $y^{a_2} = y$, $z^{a_2} = z$, $x^{a_3} = x$, $y^{a_3} = y$, $z^{a_3} = z$.

*16)* $((C_4 \times C_2^3) \times_\lambda C_4) \cdot C_4 = (((\langle a_2\rangle \times \langle x,y,z\rangle) \times_\lambda \langle a_3\rangle) \cdot \langle a_1\rangle$ *with* $a_2^{a_3} = a_2z$, $a_2^{a_1} = a_2x$, $a_3^{a_1} = a_3y$, $x^{a_3} = x$, $y^{a_3} = y$, $z^{a_3} = z$, $x^{a_1} = x$, $y^{a_1} = y$, $z^{a_1} = z$, $a_1^2 = a_2^2$.

*17)* $((C_4 \times C_2^3) \times_\lambda C_4) \cdot C_4 = (((\langle a_2\rangle \times \langle x,y,z\rangle) \times_\lambda \langle a_3\rangle) \cdot \langle a_1\rangle$ *satisfying the same relations than the previous group but with* $a_1^2 = a_2^2a_3^2$ *instead of* $a_1^2 = a_2^2$.

*18)* $((C_4 \times C_2^2) \times_\lambda C_4) \times_\lambda C_4 = (((\langle b_1\rangle \times \langle y,z\rangle) \times_\lambda \langle b_2\rangle) \times_\lambda \langle b_3\rangle$ *with the relations* $b_1^{b_2} = b_1^{-1}$, $y^{b_2} = y$, $z^{b_2} = z$, $b_1^{b_3} = b_1y$, $b_2^{b_3} = b_2z$, $z^{b_3} = z$, $y^{b_3} = y$.

*19)* $((C_4 \times C_2^2) \times_\lambda C_4) \times_\lambda C_4 = (((\langle b_1\rangle \times \langle y,z\rangle) \times_\lambda \langle b_2\rangle) \times_\lambda \langle b_3\rangle$ *with* $b_1^{b_2} = b_1y$, $b_1^{b_3} = b_1z$, $b_2^{b_3} = b_2b_1^{b_2}$, $y^{b_3} = y$, $z^{b_3} = z$, $y^{b_2} = y$, $z^{b_2} = z$.

*20)* $(((C_4 \times C_2) \times_\lambda C_4) \times C_2^2) \times_\lambda C_4 = ((((\langle a_1\rangle \times \langle x\rangle) \times_\lambda \langle a_2\rangle) \times \langle y,z\rangle) \times_\lambda \langle a_3\rangle$ *with* $a_1^{a_2} = a_1x$, $x^{a_2} = x$, $a_1^{a_3} = a_1y$, $x^{a_3} = x$, $a_2^{a_3} = a_2z$, $y^{a_3} = y$, $z^{a_3} = z$.

*d) If* $G/S(G) \cong C_2 \times D_{10}$ *then there is no* $G$.

*e) If* $G/S(G) \cong C_5 \times_\lambda C_4$ *then there is no* $G$.

*f) If* $G/S(G) \cong C_2 \times A_4$ *then* $G$ *is one of the following groups:*

*1)* $((C_4^2 \times C_2^{2t}) \times_f C_3) \times_\lambda C_2 = (((\langle x,y\rangle \times \langle x_i,y_i\ i=1,\dots,t\rangle) \times_f \langle b\rangle) \times_\lambda \langle d\rangle$ *with relations* $x^b = y$, $y^b = x^{-1}y^{-1}$, $x_i^b = y_i$, $y_i^b = x_iy_i$, $x_i^d = x_i$, $y_i^d = y_i$, $b^d = b$, $x^d = xx_1$, $y^d = yy_1$.

*2)* $((C_4^2 \times C_2^{2t}) \times_f C_3) \times_\lambda C_2 = (((\langle x,y\rangle \times \langle x_i,y_i\ i=1,\dots,t\rangle) \times_f \langle b\rangle) \times_\lambda \langle d\rangle$ *with relations* $x^b = y$, $y^b = x^{-1}y^{-1}$, $x_i^b = y_i$, $y_i^b = x_iy_i$, $x_i^d = x_i$, $y_i^d = y_i$, $b^d = b$, $x^d = x^{-1}$, $y^d = y^{-1}$.

*3)* $((C_4^2 \times C_2^{2t}) \times_f C_3) \times_\lambda C_2 = (((\langle x,y\rangle \times \langle x_i,y_i\ i=1,\dots,t\rangle) \times_f \langle b\rangle) \times_\lambda \langle d\rangle$ *with relations* $x^b = y$, $y^b = x^{-1}y^{-1}$, $x_i^b = y_i$, $y_i^b = x_iy_i$, $x_i^d = x_i$, $y_i^d = y_i, b^d = b$, $x^d = xy^2$, $y^d = y^{-1}x^2$.

*4)* $((C_2^2 \times C_2^{2t}) \times_f C_3) \times_\lambda C_2 = (((\langle x,y\rangle \times \langle x_i,y_i\ i=1,\dots,t\rangle) \times_f \langle b\rangle) \times_\lambda \langle d\rangle$ *with relations* $x^b = xy$, $y^b = x$, $x_i^b = y_i$, $y_i^b = x_iy_i$, $x^d = xx_1$, $y^d = yy_1$, $x_i^d = x_i$, $y_i^d = y_i$, $b^d = b$.

*5)* $(SL(2,3) \cdot C_4)_{C_2} \times C_2^t$.

*6)* $SL(2,3) \times C_4 \times C_2^t$

*g) If* $G/S(G) \cong D_{26}$ *then there is no* $G$.

*h) If* $G/S(G) \cong C_4^2 \times_f C_3$ *then* $G$ *is isomorphic to*

$$(C_8^2 \times C_2^{2t}) \times_f C_3 = ((\langle x,y\rangle \times \langle x_i,y_i : i=1,\dots,t\rangle) \times_f \langle b\rangle$$

*with relations* $x^b = y$, $y^b = x^{-1}y^{-1}$, $x_i^b = y_i$, $y_i^b = x_iy_i$, $i = i,\dots,t$.

*i) If* $G/S(G) \cong C_4^2 \times_f C_3$ *then* $G$ *is:*

*α) Equal to* $(P \times_\lambda C_3) \times C_2^t = (P \times_\lambda \langle b\rangle) \times C_2^t$, *where* $P \times_\lambda \langle b\rangle$ *is one of the following groups:*

$$(1)\qquad (C_2^4 \times_\lambda C_2^2) \times_\lambda C_3 = ((\langle a_1\rangle \times \langle a_2\rangle \times \langle w_1\rangle \times \langle w_2\rangle) \times_\lambda \langle a_3,a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1w_1$, $a_2^{a_3} = a_2w_1w_2$, $a_1^{a_4} = a_1w_2$, $a_2^{a_4} = a_2w_1$, $a_1^b = a_2$, $a_2^b = a_1a_2$, $a_3^b = a_4$, $a_4^b = a_3a_4$, $w_1^b = w_1$, $w_2^b = w_2$.

(2) $$(C_2^3 \times_\lambda C_2^2) \times_\lambda C_3 = (\langle a_1, a_2, w_1\rangle \times_\lambda \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1 w_1$, $a_2^{a_3} = a_2$, $a_1^{a_4} = a_1 w_1$, $a_2^{a_4} = a_2 w_1$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$, $w_1^b = w_1$.

(3) $$((Q_8 \times C_2^2) \times_\lambda C_2^2) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_1\rangle \times \langle w_2\rangle) \times_\lambda \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1 w_1$, $a_2^{a_3} = a_2 w_1 w_2$, $a_1^{a_4} = a_1 w_2$, $a_2^{a_4} = a_2 w_1$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$, $w_i^b = w_i = w_i^{a_j}$, $i = 1,2$, $j = 3,4$.

(4) $$((Q_8 \times C_2) \times_\lambda C_2^2) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_1\rangle) \times_\lambda \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1 w_1$, $a_2^{a_3} = a_2 w_1$, $a_1^{a_4} = a_1$, $a_2^{a_4} = a_2 w_1$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$, $w_1^b = w_1 = w_1^{a_j}$, $j = 3,4$.

(5) $$((Q_8 \times C_2) \times_\lambda C_2^2) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_2\rangle) \times_\lambda \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1^{-1}$, $a_2^{a_3} = a_2^{-1} w_2$, $a_1^{a_4} = a_1 w_2$, $a_2^{a_4} = a_2^{-1}$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$, $w_2^b = w_2 = w_2^{a_j}$, $j = 3,4$.

(6) $$(Q_8 \times C_2^2) \times_\lambda C_3 = (\langle a_1, a_2\rangle \times_\lambda \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1$, $a_2^{a_3} = a_2^{-1}$, $a_1^{a_4} = a_1^{-1}$, $a_2^{a_4} = a_2$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$.

(7) $$((Q_8 \times C_2^2) \times_\lambda Q_8) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_1\rangle \times \langle w_2\rangle) \times_\lambda \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1 w_1$, $a_2^{a_3} = a_2 w_1 w_2$, $a_1^{a_4} = a_1 w_2$, $a_2^{a_4} = a_2 w_1$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$, $w_i^b = w_i = w_i^{a_j}$, $i = 1,2$ $j = 3,4$.

(8) $$((Q_8 \times C_2^2) \cdot Q_8) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_1\rangle \times \langle w_2\rangle) \cdot \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with the same relations than the previous group and* $a_1^2 = a_2^2 = a_3^2 = a_4^2$.

$$(9) \qquad ((Q_8 \times C_2^2) \cdot Q_8) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_1\rangle \times \langle w_2\rangle) \cdot \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with the same relations than (7) and* $a_3^2 = w_1$.

$$(10) \qquad ((Q_8 \times C_2^2) \times_\lambda Q_8) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_1\rangle) \times_\lambda \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1 w_1$, $a_2^{a_3} = a_2$, $a_1^{a_4} = a_1 w_1$, $a_2^{a_4} = a_2 w_1$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$.

$$(11) \qquad ((Q_8 \times C_2) \times_\lambda Q_8) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_2\rangle) \times_\lambda \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1^{-1}$, $a_2^{a_3} = a_2^{-1} w_2$, $a_1^{a_4} = a_1 w_2$, $a_2^{a_4} = a_2^{-1}$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$, $w_2^b = w_2 = w_2^{a_j}$, $j = 3, 4$.

$$(12) \qquad ((Q_8 \times C_2) \cdot Q_8) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_1\rangle) \cdot \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1 w_1$, $a_2^{a_3} = a_2$, $a_1^{a_4} = a_1 w_1$, $a_2^{a_4} = a_2 w_1$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$, $w_1^b = w_1 = w_1^{a_j}$, $j = 3, 4$, $a_3^2 = a_1^2$.

$$(13) \qquad ((Q_8 \times C_2) \cdot Q_8) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_1\rangle) \cdot \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1 w_1$, $a_2^{a_3} = a_2$, $a_1^{a_4} = a_1 w_1$, $a_2^{a_4} = a_2 w_1$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$, $w_1^b = w_1 = w_1^{a_j}$, $j = 3, 4$, $a_3^2 = w_1$.

$$(14) \qquad ((Q_8 \times C_2) \times_\lambda Q_8) \times_\lambda C_3 = (((\langle a_1, a_2\rangle \times \langle w_2\rangle) \cdot \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1^{-1}$, $a_2^{a_3} = a_2^{-1} w_2$, $a_1^{a_4} = a_1 w_2$, $a_2^{a_4} = a_2^{-1}$, $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$, $a_3^2 = a_1^2$, $w_2^b = w_2 = w_2^{a_j}$, $j = 3, 4$.

$$(15) \qquad (Q_8 \times Q_8) \times_\lambda C_3 = ((\langle a_1, a_2\rangle \times \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^b = a_2$, $a_2^b = a_1 a_2$, $a_3^b = a_4$, $a_4^b = a_3 a_4$.

$$(16) \qquad (Q_8 \times_\lambda Q_8) \times_\lambda C_3 = ((\langle a_1, a_2\rangle \times_\lambda \langle a_3, a_4\rangle) \times_\lambda \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1$, $a_2^{a_3} = a_2^{-1}$, $a_1^{a_4} = a_1^{-1}$, $a_2^{a_4} = a_2$, $a_1^b = a_2$, $a_2^b = a_1a_2$, $a_3^b = a_4$, $a_4^b = a_3a_4$.

$$(17) \qquad (Q_8Q_8)_{c_2} \times_\lambda C_3 = ((\langle a_1, a_2\rangle\langle a_3, a_4\rangle)_{\langle a_1^2\rangle} \times_\lambda \langle b\rangle$$

*with relations* $a_1^b = a_2$, $a_2^b = a_1a_2$, $a_3^b = a_4$, $a_4^b = a_3a_4$, $a_ia_j = a_ja_i$, $i = 1,2$, $j = 3,4$, $a_1^2 = a_2^2 = a_3^2 = a_4^2$.

*β) Isomorphic to one of the following Frobenious groups:*

$$(18) \qquad (((C_4^2 \times C_2^2) \times_\lambda C_2^2) \times C_2^{2t}) \times_f C_3$$
$$= ((((\langle a_1, a_2\rangle \times \langle x_2, y_2\rangle) \times_\lambda \langle a_3, a_4\rangle) \times \langle z_1, w_1, \ldots, z_t, w_t\rangle) \times_f \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1x_2$, $a_1^{a_4} = a_1x_2y_2$, $a_2^{a_3} = a_2x_2y_2$, $a_2^{a_4} = a_2y_2$, $x_i^{a_3} = x_i = x_i^{a_4}$, $y_i^{a_3} = y_i = y_i^{a_4}$, $a_1^b = a_2$, $a_2^b = a_1^{-1}a_2^{-1}$, $x_2^b = y_2$, $y_2^b = x_2y_2$, $a_3^b = a_4$, $a_4^b = a_3^{-1}a_4^{-1}$, $z_j^b = w_j$, $w_j^b = z_jw_j$, $j = 1, \ldots, t$.

$$(19) \qquad (((C_2^2 \times C_2^2) \times_\lambda C_2^2) \times C_2^{2t}) \times_f C_3$$
$$= ((((\langle a_1, a_2\rangle \times \langle x_2, y_2\rangle) \times_\lambda \langle a_3, a_4\rangle) \times \langle z_1, w_1, \ldots, z_t, w_t\rangle) \times_f \langle b\rangle$$

*with the same relations than (18).*

$$(20) \quad ((C_4^2 \times C_2^2) \times C_2^{2t}) \times_f C_3 = (((\langle a_1, a_2\rangle \times_\lambda \langle a_3, a_4\rangle) \times \langle z_1, w_1, \ldots, z_t, w_t\rangle) \times_f \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1^{-1}$, $a_1^{a_4} = a_1^{-1}a_2^2$, $a_2^{a_3} = a_2^{-1}a_1^2$, $a_2^{a_4} = a_2^{-1}$, $a_1^b = a_2$, $a_2^b = a_1^{-1}a_2^{-1}$, $a_3^b = a_4$, $a_4^b = a_3a_4$, $z_j^b = w_j$, $w_j^b = z_jw_j$.

$$(21) \qquad (((C_4^2 \times C_2^2) \times_\lambda C_4^2) \times C_2^{2t}) \times_f C_3$$
$$= ((((\langle a_1, a_2\rangle \times \langle x_2, y_2\rangle) \times_\lambda \langle a_3, a_4\rangle) \times \langle z_1, w_1, \ldots, z_t, w_t\rangle) \times_f \langle b\rangle$$

*with the same relations than 18).*

(22 − 24) $$((C_4^2 \times C_2^2)\cdot C_4^2 \times C_2^{2t}) \times_f C_3$$
$$= ((((\langle a_1,a_2\rangle \times \langle x_2,y_2\rangle)\langle a_3,a_4\rangle) \times \langle z_1,w_1,\dots,z_t,w_t\rangle) \times_f \langle b\rangle$$

*with the same relations than (18) and* $a_3^2 = a_1^2$, $a_4^2 = a_2^2$ *or* $a_3^2 = a_1^2a_2^2x_2$, $a_4^2 = a_1^2y_2$ *or* $a_3^2 = a_1^2y_2$, $a_4^2 = a_2^2x_2y_2$.

(25) $$(C_4^4 \times C_2^{2t}) \times_f C_3 = (\langle a_1,a_2,a_3,a_4\rangle \times \langle z_1,w_1,\dots,z_t,w_t\rangle) \times_f \langle b\rangle$$

*with relations* $a_1^b = a_2$, $a_2^b = a_1^{-1}a_2^{-1}$, $a_3^b = a_4$, $a_4^b = a_3^{-1}a_4^{-1}$, $z_j^b = w_j$, $w_j^b = z_jw_j$.

(26) $$((C_4^2 \times_\lambda C_4^2) \times C_2^{2t}) \times_\lambda C_3 = ((\langle a_1,a_2\rangle \times_\lambda \langle a_3,a_4\rangle) \times \langle z_1,w_1,\dots,z_t,w_t\rangle) \times_f \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1^{-1}$, $a_1^{a_4} = a_1^{-1}a_2^2$, $a_2^{a_3} = a_2^{-1}a_1^2$, $a_2^{a_4} = a_2^{-1}$, $a_1^b = a_2$, $a_2^b = a_1^{-1}a_2^{-1}$, $a_3^b = a_4$, $a_4^b = a_3^{-1}a_4^{-1}$, $z_j^b = w_j$, $w_j^b = z_jw_j$.

(27) $$(((C_4^2)C_4^2) \times C_2^{2t}) \times_f C_3 = ((\langle a_1,a_2\rangle\langle a_3,a_4\rangle) \times \langle z_1,w_1,\dots,z_t,w_t\rangle) \times_f \langle b\rangle$$

*with relations* $a_1^{a_3} = a_1^{-1}$, $a_1^{a_4} = a_1^{-1}a_2^2$, $a_2^{a_3} = a_2^{-1}a_1^2$, $a_2^{a_4} = a_2^{-1}$, $a_3^2 = a_1^2a_2^2$, $a_4^2 = a_1^2$, $a_1^b = a_2$, $a_2^b = a_1^{-1}a_2^{-1}$, $a_3^b = a_4$, $a_4^b = a_3^{-1}a_4^{-1}$.

*j) If* $G/S(G) = GL(2,3)$ *then* $G$ *is* $\mathrm{Hol}(C_3^2)$.

*k) If* $G/S(G) = SL(2,3)C_4$ *then* $G$ *is:*

$$\mathrm{Hol}(C_7^2, SL(2,3)C_4) = \langle x,y\rangle \times_\lambda ((\langle\alpha,\beta\rangle \times_\lambda \langle\gamma\rangle)\langle\sigma\rangle),$$

*with* $x^\alpha = x^{-1}y^3$, $y^\alpha = x^{-3}y$, $x^\beta = xy^{-1}$, $y^\beta = x^2y^{-1}$, $x^\gamma = x^{-3}$, $y^\gamma = xy^2$, $x^\sigma = x^2y^{-1}$, $y^\sigma = x^{-2}y^{-2}$, $\langle\alpha,\beta\rangle \cong Q_8$, $\langle\gamma\rangle \cong C_3$, $\beta^\gamma = \alpha$, $\alpha^\gamma = \alpha\beta$, $\mathrm{o}(\sigma) = 4$, $\sigma^2 = \alpha^2$, $\alpha^\sigma = \alpha^{-1}$, $\beta^\sigma = \alpha\beta$, $\gamma^\sigma = \gamma^{-1}\beta\alpha^{-1} = \gamma^{-1}\alpha\beta$.

*l) If* $G/S(G) = C_2^3 \times_f C_7$ *then* $G$ *is one of the following groups:*

(1) $$(C_4^3 \times C_2^{3t}) \times_f C_7 = \left(\langle x_1,x_2,x_3\rangle \times \prod_{i=1}^{t} (\langle y_{1j}\rangle \times \langle y_{2j}\rangle \times \langle y_{3j}\rangle\right) \times_f \langle b\rangle$$

*with relations* $x_1^b = x_2$, $x_2^b = x_3$, $x_3^b = x_1x_3x_2^2$, $y_{1j}^b = y_{2j}$, $y_{2j}^b = y_{3j}$, $y_{3j}^b = y_{1j}y_{3j}$.

$$(P \times C_2^{3t}) \times_f C_7,$$

$P$ being a 2-group of order $2^6$ of type $Sz(8)$, $PC_7 = \mathrm{Hol}(P, C_7)$ and $C_2^{3t}C_7$ as above.

*m) If* $G/S(G) \cong C_{17} \times_f C_4$ *then there is no* $G$.

*n) If* $G/S(G) \cong C_{13} \times_f C_6$ *then there is no* $G$.

*o) If* $G/S(G) \cong C_2^4 \times_f C_5$ *then* $G$ *is one of the following groups:*

*1)* $\mathrm{Hol}(2^5\Gamma_5 a_2, C_5) \times C_2^t$.

*2)* $\mathrm{Hol}(2^6\Gamma_5 a_2, C_5) \times C_2^t$.

*3)* $\mathrm{Hol}(2^6\Gamma_{13} a_5, C_5) \times C_2^t$.

*p) If* $G/S(G) = \mathrm{Hol}(C_2^3, C_7 \times_f C_3)$ *then* $G$ *is one of the following groups:*

*1)* $\mathrm{Hol}(P, C_7 \times_f C_3)$ *with* $P$ *a 2-group of type* $Sz(8)$.

*2)* $C_4^3 \times_\lambda (C_7 \times_f C_3) = (\langle x_1, x_2, x_3\rangle) \times_\lambda (\langle a\rangle \times_f \langle b\rangle)$ *with relations* $a^b = a^2$, $x_1^a = x_2$, $x_2^a = x_3$, $x_3^a = x_1x_3x_2^2$, $x_1^b = x_1$, $x_2^b = x_3$, $x_3^b = x_1x_2^{-1}x_3^{-1}$.

*q) If* $G/S(G) = C_5^2 \times_f Q_8$ *then there is no* $G$.

*r) If* $G/S(G) = C_5^2 \times_f DC_3$ *then there is no* $G$.

*s) If* $G/S(G) = C_5^2 \times_f SL(2,3)$ *then there is no* $G$.

*t) If* $G/S(G) = PSL(2,11)$ *then* $G$ *is* $SL(2,11) \times C_2^t$.

*u) If* $G/S(G) = M_9$ *then* $G$ *is one of the following groups:*

*1)* $T_1\langle d\rangle \times C_2^t$ *with* $T_1 = SL(2,9)$ *being the only perfect extension of* $C_2$ *by* $A_6$ *and* $d$ *an element acting on* $T_1$ *in such a way that* $(T_1\langle d\rangle)/C_2 \cong M_9$.

*2)* $(T_2 \times C_3^t)\langle d\rangle$ *with* $T_2$ *being the only perfect central extension of* $C_3$ *by* $A_6$ *and* $d$ *acting on* $T_2$ *in such a way that* $(T_2\langle d\rangle)/C_3 \cong M_9$ *and* $x^d = x^{-1}$ *for all* $x \in C_3^t$.

*Proof.* Let $G$ be a $\Gamma$-group such that $r(G/S(G)) = 8$. Then $\bar{G} = G/S(G)$ is isomorphic to one of the following groups:

$C_8$,
$C_2 \times C_4$,
$C_2^3$,
$C_2 \times D_{10}$,
$C_5 \times_\lambda C_4 = \langle a\rangle \times_\lambda \langle b\rangle$, with $a^b = a^{-1}$,
$D_{26}$,
$C_{17} \times_f C_4$,
$C_{13} \times_f C_6$,
$C_2 \times A_4$,
$C_4^2 \times_f C_3$,
$C_2^4 \times_f C_3$,

$GL(2,3)$,
$SL(2,3)C_4$,
$C_2^3 \times_f C_7$,
$C_2^4 \times_f C_5$,
$\mathrm{Hol}(C_2^3, C_7 \times_f C_3)$,
$C_5^2 \times_f Q_8$,
$C_5^2 \times_f DC_3$,
$C_5^2 \times_f SL(2,3)$,
$PSL(2,11)$,
$M_9$.

In this proof $L$ will always denote a minimal normal subgroup of $G$. Set $|S(G)| = p^n$, $p$ a prime number. The cases: $C_8$ to $C_{13} \times_f C_6$ were already studied in [5].

A) Suppose $\bar{G} \cong C_2 \times A_4$. Then $\beta(\bar{G}) = 2$. There exists $C_2 \trianglelefteq \bar{G}$ so, if $p \neq 2$, $C_2$ acts f.p.f. on $S(G)$. Let $M/S(G)$ be the other minimal normal subgroup of $\bar{G}$. Since $C_2 \times C_2$ does not act f.p.f on $S(G)$, there exist $\bar{b} \in C_2^{2*}$ and $x \in L^*$ such that $x^b = x$, so $\beta(G) = 1$ (in any other case $b \in C_G(S(G))$ and

$$\bar{1} \neq C_G(S(G))/S(G) \trianglelefteq \bar{G},$$

so $M \leq C_G(S(G))$ and $M = S(G) \times C_2^2$. $C_2^2 \trianglelefteq G$ because $C_2^2$ char $M \trianglelefteq G$, then $C_2^2 \leq S(G)$, impossible) and $p^n - 1 = |Cl_G(x)|$ divides 24 so $S(G) \cong C_5^2$ or $C_3^2$.

If $S(G) \cong C_5^2$ then $C_2 \times A_4$ acts f.p.f. on $S(G)$ which is impossible. Therefore, $S(G) \cong C_3^2$. Let $\psi : \bar{G} \to \mathrm{Aut}(L)$ be the homomorphism induced by conjugation. We have $SD_{16} \in \mathrm{Syl}_2(GL(2,3))$ and $SD_{16}$ has no subgroups of type $C_2^3$. So, necessarily $2 \| \ker(\psi)|$. Again, as $\beta(\bar{G}) = 2$ and the central minimal normal subgroups act f.p.f. on $S(G)$ it follows that $\bar{M} \leq \ker(\psi)$, so $M \leq C_G(S(G))$ and $M = C_3^2 \times C_2^2$, impossible. Thus, $p = 2$ and $L \cong C_2$ or $C_2^2$ for each minimal normal subgroup $L$ of $G$. Set $E/S(G) = A_4 \trianglelefteq \bar{G}$ and $G = P \times_\lambda C_3$. We have $S(G) \leq Z(P)$ so $S(G) \leq Z(E)$ and $S(G) \leq S(E)$. We distinguish two cases:

1) $S(G) = S(E)$. Then, $E/S(E) \cong A_4$ and $E$ is a $\Gamma$-group. From [6] we get the following cases:
   1i) $E = (C_4^2 \times C_2^{2t}) \times_f C_3 = (\langle x, y\rangle \times \langle x_1, y_1, \ldots, x_t, y_t\rangle) \times_f \langle b\rangle$, with $x^b = y$, $y^b = x^{-1}y^{-1}$, $x_i^b = y_i$, $y_i^b = x_i y_i$, $i = 1, \ldots, t$.
   1ii) $E = SL(2,3) \times C_2^t$.
2) $S(G) < S(E)$. In this case there exists $c \in S(E) - S(G)$ such that $o(c) = 2$. Clearly $M = S(E) \cong C_2^{2t}$ and

$$E = (C_2^2 \times C_2^{2t}) \times_f C_3 = (\langle x, y\rangle \times \langle x_1, y_1, \ldots, x_t, y_t\rangle) \times_f \langle b\rangle,$$

with $x^b = y$, $y^b = xy$, $x_i^b = y_i$, $y_i^b = x_i y_i$.

We analize these two cases depending on $L \cong C_2$ or $L \cong C_2^2$.

Suppose $L \cong C_2^2$ and let

$$C_2 \cong N/S(G) = \langle \bar{d} \rangle \trianglelefteq \bar{G}.$$

We have $\bar{d}^{\bar{b}} = \bar{d}$, so $d^b = dz$ for some $z \in S(G)$. If $o(d) = 4$ then $(d^2)^b = d^2$ and $d^2 \in S(G)$ which is impossible as $C_3$ acts f.p.f. on $S(G)$. So, $o(d) = 2$ and $G = E \times_\lambda \langle d \rangle$. We have $\bar{b}^{\bar{d}} = \bar{b}$, so $b^d = bz_1$ for some $z_1 \in S(G)$. Assuming that $E$ verifies the relations of (1i) we have $\bar{x}^{\bar{d}} = \bar{x}$ so $x^d = xz_2$ for some $z_2 \in S(G)$. Therefore

$$(x^b)^{d^b} = (xz_2)^b = yz_2^b$$

implies $y^d = yz_2^b$. We have $(bd)^4 = bz_1^b$ and $(bd)^3 = z_1^b d$, so $o(bd) = 6$. Let $b_1 = (bd)^4$ and $d_1 = (bd)^3$. Then

$$b_1^{d_1} = b_1, \qquad o(b_1) = 3, \qquad o(d_1) = 2$$

and

$$x^{b_1} = x^b = y, \quad y^{b_1} = y^b = x^{-1}y^{-1}, \quad x^{d_1} = x^d = xz_2, \quad y^{d_1} = y^d = yz_2^b = yz_2z_2^{b_1}.$$

Besides we can choose generators $x_i, y_i$ such that

$$x_i^{b_1} = y_i, \qquad y_i^{b_1} = x_iy_i,$$

so we can suppose the following relations are satisfied:

$$x^b = y, \qquad y^b = x^{-1}y^{-1}, \qquad x^d = xw, \qquad y^d = yw^b,$$

$$x_i^b = y_i, \qquad y_i^b = x_iy_i, \qquad x_i^d = x_i, \qquad y_i^d = y_i, \qquad b^d = b$$

for some $w \in S(G)^*$.

Now we shall consider two cases:

a) $w \notin \langle x^2, y^2 \rangle$. Let
$$L = \{1\} \cup Cl_G(w) = \langle x_1, y_1 \rangle$$
with $w = x_1$. Then we get the group listed as (1).

b) $w \in \langle x^2, y^2 \rangle$. If $w = x^2$ we get the group (2). If $w = x^2y^2$ we exchange $(x, y, b, d)$ for $(x, (xy)^{-1}, b^2, d)$ and we obtain again the relations of the group (2). If $w = y^2$ we get the group (3). If $w = 1$, then $d \in Z(G)$ which is impossible.

Assume now that

$$E = \big(\langle x, y\rangle \times \langle x_1, y_1, \ldots, x_t, y_t\rangle\big) \times_f \langle b\rangle$$

with

$$x^b = y, \qquad y^b = xy, \qquad x_i^b = y_i, \qquad y_i^b = x_i y_i.$$

We reason as above observing that $\langle x^2, y^2\rangle = 1$. If $w \notin \langle x, y\rangle$ the group (4) appears. If $w \in \langle x, y\rangle$, then $\langle x, y\rangle$ is a minimal normal subgroup of $G$ which is impossible.

Suppose now that $L \cong C_2$ for all $L$. Then $E = SL(2,3) \times C_2$. $E' = Q_8$ char $E \trianglelefteq G$ implies $Q_8 \cong E' \trianglelefteq G$. Let $N/S(G) = C_2 \trianglelefteq \bar{G}$ and $H = SL(2,3)\langle d\rangle$ with $d \in N - S(G)$. Then $G = H \times C_2^s$ and $H \trianglelefteq G$, so if $G = P \times_\lambda C_3$, $H \cap P = D \trianglelefteq H$ and $H = D \times_\lambda C_3$ with $|D| = 16$ or $32$. The only non-abelian groups of order 16 which admit automorphisms of order 3 are $C_2 \times Q_8$ and

$$(C_4 \times C_2) \times_\lambda C_2 = \big(\langle a\rangle \times \langle b\rangle\big) \times_\lambda \langle c\rangle, \qquad a^c = a,\ b^c = a^2 b.$$

We have

$$\big|\mathrm{Aut}(C_2 \times Q_8)\big| = 2^6 \cdot 3,$$

so there exists a unique action of $C_3$ on $C_2 \times Q_8$ which corresponds to

$$(C_2 \times Q_8)C_3 = C_2 \times SL(2,3).$$

For this group $G/S(G) \cong C_2 \times A_4$ which is impossible. If

$$H \cap P \cong (C_4 \times C_2) \times_\lambda C_2,$$

then

$$\big|\mathrm{Aut}(H \cap P)\big| = 2^3 \cdot 6$$

and there is a unique action of $C_3$ on this group. We also notice that

$$(C_4 \times C_2) \times_\lambda C_2 \cong (Q_8 \cdot C_4)_{C_2} = \langle \alpha, \beta\rangle\langle\theta\rangle.$$

This group admits the following automorphism of order 3:

$$\begin{aligned} \alpha &\to \beta \\ \beta &\to \alpha\beta \\ \theta &\to \theta. \end{aligned}$$

So we get $H = (SL(2,3)C_4)_{C_2}$ and the group (5) appears.

Finally, if $|H \cap P| = 32$, then $d^2 \notin SL(2,3)$, so

$$G = \big(SL(2,3) \times_\lambda C_4\big) \times C_2^t.$$

Let $o(b) = 3$ $d^b = dz$ for some $z \in S(G)$, so, $d^{b^2} = dz^2 = d$ implies $d^b = d$. Let $Q_8 = \langle a_1, a_2\rangle$ with $a_1^b = a_2$, $a_2^b = a_1 a_2$; $\bar{a}_1^d = \bar{a}_1$ implies $a_1^d = a_1 z'$ with $z' \in Z(Q_8)$. Conjugating this last relation by $b$ we get $a_2^d = a_2 z'$. If we conjugate again by $b$ we obtain $(a_1 a_2)^b = a_1 a_2 z'$, so $a_1 z' a_2 z' = a_1 a_2 z'$ and $z' = 1$. Thus we get the group (6).

B) Suppose $\bar{G} \cong C_4^2 \times_f C_3$. If $p \neq 2$, since $C_2^2$ does not act f.p.f. on $L$ it follows that $x^b = x$ for some $x \in L^*$ and $b \in C_2^{2*}$. If $\beta(G) > 1$, then $b \in C_G(S(G))$ so

$$\bar{1} \neq C_G(S(G))/S(G) \trianglelefteq \bar{G}.$$

$\beta(\bar{G}) = 1$ implies that if $M/S(G) = C_2^2 \trianglelefteq \bar{G}$, then $M = S(G) \times C_2^2$ and $C_2^2$ char $M \trianglelefteq G$. This implies $C_2^2 \trianglelefteq G$ and $C_2^2 \leq S(G)$ which is impossible. Therefore $\beta(G) = 1$ and $(p^n - 1) | 16 \cdot 3$. Besides $2 \not| \, |C_G(S(G))/S(G)|$, so, $C_G(S(G)) = S(G)$ and $\bar{G} \leq \mathrm{Aut}(S(G))$. From these conditions we deduce $n = 2$ and $p^2 \in \{3^2, 5^2, 7^2\}$. Since $\bar{G}$ does not act f.p.f. on $S(G)$, $p^2 \neq 7^2$. $GL(2,5)$ has no subgroups isomorphic to $C_4^2 \times_f C_3$ so it follows that $p^2 = 3^2$. The Sylow 2-subgroups of $GL(2,3)$ are isomorphic to $SD_{16}$ so the case $p^2 = 3^2$ is also excluded. We conclude that $p = 2$ and $L \cong C_2$ or $L \cong C_2^2$ for each minimal normal subgroup $L$ of $G$.

Suppose $L \cong C_2^2$. Then $G = P \times_f C_3$ and $S(G) \leq Z(P)$. If $a_1 \in P - S(G)$, $[a_1, a_1^b] = 1$. Let $a_2 = a_1^b$, then $a_2^b = a_1^{-1} a_2^{-1}$ and $\langle a_1, a_2 \rangle$ is abelian. $P = \langle a_1, a_2 \rangle S(G)$ and $S(G) \leq Z(P)$ so $\langle a_1, a_2 \rangle \trianglelefteq G$ and $P = \langle a_1, a_2 \rangle \times T$. If $o(a_1) = 4$, then

$$\langle a_1^2, a_2^2 \rangle \cong C_2^2 \leq S(G)$$

which is impossible because $o(\bar{a}_1) = 4$. So

$$o(a_1) = 8 \qquad \text{and} \qquad (a_1^4)^b = a_2^4 \neq a_1^4.$$

This implies

$$\langle a_1, a_2 \rangle \cong C_8^2$$

getting the desired group.

If $L \cong C_2$, we have

$$S(G) = Z(G) \qquad \text{and} \qquad G/Z(G) \cong C_4^2 \times_f C_3.$$

Let $G = P \times_\lambda C_3$, $a_1 \in P - S(G)$ and $a_2 = a_1^b$. Then

$$P = \langle a_1, a_2 \rangle S(G) \qquad \text{and} \qquad P' \leq S(G) \leq Z(P).$$

Let $a_2^b = a_1^{-1} a_2^{-1} z_1$ with $z_1 \in S(G)$. We have

$$[a_1^2, a_2] = [a_1, a_2]^2 = 1$$

(since $\exp(S(G)) = 2$) and

$$[a_1, a_2^2] = 1$$

so $a_1^2, a_2^2 \in Z(P)$, $(a_1^2)^b = a_2^2$,

$$\begin{aligned}(a_2^2)^b &= (a_1^{-1}a_2^{-1})^2\\ &= a_1^{-1}(a_1^{-1})^{a_2}a_2^{-2}\\ &= a_1^{-1}a_1^{-1}z_1a_2^{-2}\\ &= a_1^{-2}a_2^{-2}z_1,\end{aligned}$$

so $(a_1^2z_1)^b = a_2^2z_1$ and

$$\begin{aligned}(a_2^2z_1)^b &= a_1^{-2}a_2^{-2}z_1^2\\ &= a_1^{-2}a_2^{-2}\\ &= (a_1^2z_1)^{-1}(a_2^2z_1)^{-1}.\end{aligned}$$

Therefore

$$D = \langle a_1^2z_1, a_2^2z_1\rangle \trianglelefteq G$$

and necessarily $o(a_1^2z_1) = 4$, so $o(a_1^2) = 4$, and $D$ is an abelian group of order at most 16. We have

$$a_1^4 \in Z(G) = S(G) \qquad \text{and} \qquad (a_1^4)^b = a_2^4.$$

So $a_1^4 = a_2^4$ and $|D| \leq 8$. As $D$ has elements of order 4 and has no elements of order 8, we have

$$D = \langle d\rangle \times \langle w\rangle \cong C_4 \times C_2$$

with $d = a_1^2z_1$, so $o(d^b) = 4$ and $d^b \in D$ implies $d^b = d^ew$ with $e = 1$ or $-1$. So

$$d^{b^2} = (d^ew)^ew = d^{e^2}w^2 = d$$

and $b^2 \in C_G(d)$, so $b \in C_G(d)$ which is impossible.

C) Assume $\bar{G} \cong C_2^4 \times_f C_3$. If $p \neq 2$, then we consider $\bar{N} = C_2^2 \trianglelefteq \bar{G}$ a minimal normal subgroup of $\bar{G}$. For each minimal normal subgroup $L$ of $G$ there exist $x \in L^*$ and $\bar{b} \in \bar{N}^*$ such that $x^b = x$. If $\beta(G) > 1$, $b \in C_G(S(G))$, so

$$C_G(S(G))/S(G) \neq \bar{1} \qquad \text{and} \qquad \bar{b} \in C_G(S(G))/S(G) \cap \bar{N}.$$

This implies

$$\bar{N} \leq C_G(S(G))/S(G)$$

so

$$N \leq C_G(S(G)) \qquad \text{and} \qquad N = C_2^2 \times S(G).$$

So, $C_2^2 \leq S(G)$ which is impossible. Thus $\beta(G) = 1$ and $p^n - 1 \mid 16.3$. Besides

$$2 \not| |C_G(S(G))/S(G)|,$$

so

$$C_2^4 \leq \mathrm{Aut}(S(G)).$$

Studying all the possible cases for $p$ and $n$ it is easily proved that this condition cannot hold. So $p = 2$. Let

$$P/S(G) = C_2^4 \trianglelefteq \bar{G}.$$

Then

$$S(G) \leq Z(P) \qquad \text{and} \qquad L \cong C_2 \text{ or } C_2^2$$

for each minimal normal subgroup $L$ of $G$. Suppose $L \cong C_2$ for all $L$. Then $S(G) \leq Z(G)$ but $Z(\bar{G}) = \bar{1}$ so

$$S(G) = Z(G) \qquad \text{and} \qquad G/Z(G) \cong C_2^4 \times_f C_3.$$

Let $a_1 \in P - S(G)$ and $a_2 = a_1^b$ with $o(b) = 3$. Condition $\bar{a}_1^{\bar{b}} = \bar{a}_2$ implies $\bar{a}_2^{\bar{b}} = \bar{a}_1 \bar{a}_2$ so

$$a_2^b = a_1 a_2 z \qquad \text{and} \qquad a_1 = a_1^{b^3} = a_2^{b^2} = (a_1 a_2 z)^b = a_2 a_1 a_2 z z = a_2 a_1 a_2.$$

Therefore $[a_1, a_2] = a_2^2$. Besides $\langle \bar{a}_1, \bar{a}_2 \rangle \trianglelefteq \bar{G}$ so

$$M_1 = \langle a_1, a_2 \rangle S(G) \trianglelefteq G.$$

Also,

$$(a_1 z)^b = a_2 z, \quad (a_2 z)^b = a_1 a_2 z z = a_1 a_2 = (a_1 z)(a_2 z) \quad \text{and} \quad \langle \bar{a}_1 \bar{z}, \bar{a}_2 \bar{z} \rangle = \langle \bar{a}_1, \bar{a}_2 \rangle.$$

We can choose $a_1, a_2$ so that

$$a_1^b = a_2, \qquad a_2^b = a_1 a_2 \qquad \text{and} \qquad [a_1, a_2] = a_2^2.$$

As $a_1^2 \in Z(G)$,

$$a_1^2 = (a_1^2)^b = a_2^2.$$

We can also choose $a_3, a_4 \in P - S(G)$ so that

$$M_2 = \langle a_3, a_4 \rangle S(G) \trianglelefteq G, \quad a_3^b = a_4, \quad a_4^b = a_3 a_4 \quad \text{and} \quad [a_3, a_4] = a_3^2 = a_4^2.$$

Then

$$P = M_1 M_2 \qquad \text{and} \qquad M_1 \cap M_2 = S(G) = Z(G).$$

Let

$$[a_1, a_3] = w_1 \in S(G).$$

Then

$$[a_2, a_4] = w_1^b = w_1.$$

If

$$[a_1, a_4] = w_2 \in S(G),$$

then

$$[a_2, a_3 a_4] = w_2^b = w_2 \qquad \text{and} \qquad [a_2, a_3][a_2, a_4] = w_2$$

(because $P' \leq Z(P)$) and

$$[a_2, a_3] = w_1 w_2.$$

Consider

$$E = \langle a_1, a_2, w_1, w_2 \rangle \trianglelefteq G.$$

We have

$$E \cap S(G) = \langle a_1^2, w_1, w_2 \rangle \leq C_2^3$$

and

$$E/(E \cap S(G)) \cong ES(G)/S(G) = \langle \bar{a}_1 \rangle \times \langle \bar{a}_2 \rangle \cong C_2^2.$$

Further, $\langle a_3, a_4 \rangle / \langle a_3^2 \rangle \cong C_2^2$ so $|\langle a_3, a_4 \rangle| = 4$ or $8$ and $|E| \,|\, 2^5$. Let

$$S(G) = \langle a_1^2, a_3^2, w_1, w_2 \rangle \times C_2^t.$$

Then

$$G = \big((E\langle a_3, a_4 \rangle) \times_\lambda C_3\big) \times C_2^t = \big((\langle a_1, a_2, w_1, w_2 \rangle \langle a_3, a_4 \rangle) \times_\lambda \langle b \rangle\big) \times C_2^t$$

and the following relations hold:

$$a_1^{a_3} = a_1 w_1, \qquad a_1^{a_4} = a_1 w_2, \qquad a_2^{a_3} = a_2 w_1 w_2,$$

$$a_2^{a_4} = a_2 w_1, \qquad a_1^b = a_2, \qquad a_2^b = a_1 a_2,$$

$$a_3^b = a_4, \qquad a_4^b = a_3 a_4, \qquad w_i^b = w_i = w_i^{a_j} \quad i = 1, 2 \ \forall j,$$

$$a_1^2 = a_2^2 = [a_1, a_2], \qquad a_3^2 = a_4^2 = [a_3, a_4].$$

We now distinguish two cases:

i) $P - S(G)$ has an element of order 2. We can suppose $a_3$ is one of these elements so that $\langle a_3, a_4 \rangle \cong C_2^2$.

ii) Every element of $P - S(G)$ has order 4 (in particular, $\langle a_1, a_2 \rangle \cong \langle a_3, a_4 \rangle \cong Q_8$).

We will study first the case (i).

i1) If $a_1^2 = 1$ then every element in $E^*$ has order 2 so, $E \cong C_2^4$ or $C_2^3$ (if $|E| = 4$, $E \le S(G)$ which is impossible). If $E \cong C_2^4$ we get the group (1). If $E \cong C_2^3$, $|E \cap S(G)| = 2$,

$$E \cap S(G) = \langle a_1^2, w_1, w_2 \rangle = \langle w_1, w_2 \rangle$$

so

$$\big|\langle w_1, w_2 \rangle\big| = 2 \qquad \text{and} \qquad E = \langle a_1, a_2 \rangle \times \langle w_1, w_2 \rangle.$$

There are three possibilities:

j) $w_2 = w_1 \neq 1$,

jj) $w_1 = 1 \neq w_2$,

jjj) $w_1 \neq 1 = w_2$.

If we assume case j) we get the group (2). Exchanging $(a_1, a_2, w_1)$ for $(a_2, a_1a_2, w_2)$ and $(a_1a_2, a_1, w_1)$ we get the groups corresponding to cases jj) and jjj) so these cases need not to be considered.

i2) If $a_1^2 \neq 1$, then $\langle a_1, a_2 \rangle \cong Q_8$ and we have the following possibilities:

i2a) $|E| = 2^5$. Then

$$E = \langle a_1, a_2 \rangle \times \langle w_1 \rangle \times \langle w_2 \rangle \cong Q_8 \times C_2 \times C_2.$$

i2b) $|E| = 2^4$. Then

$$|\langle w_1, w_2 \rangle \cap Q_8| = 2$$

or

$$\langle w_1, w_2 \rangle \cap Q_8 = \{1\}$$

and

$$E = Q_8 \times \langle w_1, w_2 \rangle$$

with $|\langle w_1, w_2 \rangle| = 2$.

i2c) $|E| = 2^3$. Then

$$E = \langle a_1, a_2 \rangle \cong Q_8 \qquad \text{and} \qquad \langle w_1, w_2 \rangle \le \langle a_1^2 \rangle.$$

In the case i2a) we get the group (3). In the case i2b) there are the following possibilities:

b1) $w_1 \neq 1 = w_2$,
b2) $w_1 = 1 \neq w_2$,
b3) $w_1 = w_2 \neq 1$,
b4) $a_1^2 = w_1$,
b5) $a_1^2 = w_2$,
b6) $a_1^2 = w_1 w_2$.

In the case b1) we get the group (4). Reasoning as in the cases j), jj) and jjj) we conclude that relations b1), b2) and b3) originate isomorphic groups. If we assume case b4) we get the group (5). Suppose now that relation b3) holds. Exchanging $(a_1, a_2, w_2)$ for $(a_2, a_1 a_2, a_2^2 w_2)$ we obtain a group isomorphic to (5).

In the case i2c) $\langle w_1, w_2\rangle \neq 1$ as in any other case $\langle a_3, a_4\rangle \leq S(G)$, which is impossible. So, we have the following three possibilities:

c1) $w_1 = 1$, $w_2 = a_1^2$,
c2) $w_2 = 1$, $w_1 = a_1^2$
c3) $w_1 = w_2 = a_1^2$.

In the case c1) we get the group (6). If we exchange $(a_3, a_4)$ for $(a_4, a_3 a_4)$ in the case c2) we obtain the group (6) again. This group also appears in the case c3) if we exchange $(a_1, a_2)$ for $(a_2, a_1 a_2)$.

Consider now the cases ii), that is, every element in $P - S(G)$ has order 4. There are three possibilities:

ii1) $|E| = 2^5$, so $E \cong Q_8 \times C_2^2$,
ii2) $|E| = 2^4$, so $E \cong Q_8 \times C_2$,
ii3) $|E| = 2^3$, so $E \cong Q_8$.

Assume case ii1). If $\langle a_3, a_4\rangle \cap E = \{1\}$, we get the group (7). If $\langle a_3, a_4\rangle \cap E \neq \{1\}$ then $a_3^2 \in \langle a_1^2, w_1, w_2\rangle^*$. Since every element of $P - S(G)$ has order 4, it follows that

$$(a_1 a_3)^2 = a_3^2 a_1^2 w_1 \neq 1$$

so the cases $a_2^2 \in \{1, a_1^2 w_1, a_2^2 w_2, a_1^2 w_1 w_2\}$ are excluded. Therefore,

$$a_3^2 = a_1^2 \qquad \text{or} \qquad a_3^2 \in \langle w_1, w_2\rangle^*.$$

In the first case we obtain the group (8). Suppose $a_3^2 \in \langle w_1, w_2\rangle^*$. If $a_3^2 = w_1$ we obtain the group (9). If we exchange $(a_1, a_2, w_1, w_2)$ for $(a_2, a_1 a_2, w_1 w_2, w_1)$ or $(a_1 a_2, a_1, w_2, w_1 w_2)$ we conclude that this group is isomorphic to the ones originated by the relations $a_3^2 = w_1 w_2$ or $a_3^2 = w_2$ respectively.

Suppose case ii2) holds. If $\langle a_3, a_4\rangle \cap E = \{1\}$ then we have the following possibilities:

d1) $a_1^2 \in \langle w_1, w_2\rangle^*$,
d2) $w_2 = 1 \neq w_1$,
d3) $w_2 \neq 1 = w_1$,
d4) $w_1 = w_2 \neq 1$.

The group (10) appears in cases d2), d3) and d4).

If $a_1^2 = w_1$ we obtain the group (11). This group also appears if $a_1^2 = w_1 w_2$ or if $a_1^2 = w_2$: we only have to exchange $(a_1, a_2, w_1, w_2)$ for $(a_2, a_1 a_2, w_1 w_2, w_1)$ or $(a_1 a_2, a_1, w_2, w_1 w_2)$ respectively. If $\langle a_3, a_4\rangle \cap E \neq 1$ then $a_3^2 \in \langle a_1^2, w_1, w_2\rangle^*$. As every element of $P - S(G)$ has order 4, it follows that $a_3^2 = a_1^2$ or $a_3^2 \in \langle w_1, w_2\rangle^*$. Besides, one of the cases d1) to d4) is verified.

Assume $w_1 = w_2$. Then

$$a_3^2 = a_1^2 \qquad \text{or} \qquad a_3^2 = w_1 = w_2$$

and we obtain the groups (12) and (13). If $w_1 = 1$,

$$a_3^2 = a_1^2 \qquad \text{or} \qquad a_3^2 = w_2 \neq 1.$$

If $w_2 = 1$, then $a_3^2 = a_1^2$ or $a_3^2 = w_1 \neq 1$. Finally, if $a_1^2 \in \langle w_1, w_2\rangle^*$ as

$$a_3^2 \notin a_1^2\{w_1, w_2, w_1 w_2\} \cup \{1\}$$

it follows that necessarily

$$a_3^2 = a_1^2 \in \langle w_1, w_2\rangle^*.$$

If $a_3^2 = a_1^2$ and $w_1 = 1$ or $w_2 = 1$, we have

$$(a_1 a_3)^2 = 1 \qquad \text{or} \qquad (a_1 a_4)^2 = 1$$

which is impossible.

The cases

$$a_3^2 = w_2 \neq 1, \qquad w_1 = 1$$

and

$$a_3^2 = w_1 \neq 1, \qquad w_2 = 1$$

originate the group (13).

The group (14) appears if $a_3^2 = a_1^2 = w_1$. It is obvious that cases $a_3^2 = a_1^2 = w_2$ and $a_3^2 = a_1^2 = w_1 w_2$ also originate the group (14).

Finally, we study the case ii3). Then $\langle w_1, w_2\rangle \le \langle a_1^2\rangle$. Set

$$\langle a_3, a_4\rangle \cap E = \{1\}.$$

If $w_1 = w_2 = 1$ we get the group (15) and if $w_1 = 1$, $w_2 = a_1^2$ the group (16) appears. If

$$\langle a_3, a_4\rangle \cap E \neq \{1\}$$

then

$$a_3^2 \in \langle a_1, a_2\rangle$$

so $a_3^2 = a_1^2$. If $w_1 = w_2 = 1$ we get the group (17). If $w_1 = 1$ and $w_2 = a_1^2$ then

$$(a_1a_3)^2 = a_3^2a_1^2w_1 = 1$$

which is impossible. Similarly, if $w_2 = 1$ and $w_1 = a_1^2$ we get the contradiction

$$(a_1a_4)^2 = a_4^2a_1^2w_2 = 1$$

and if $w_1 = w_2 = a_1^2$,

$$(a_2a_3)^2 = a_3^2a_2^2w_1w_2 = 1.$$

Suppose now that $L \cong C_2^2$ for each minimal normal subgroup $L$ of $G$. Then $G = P \times_f C_3$ and acording to [1, p.336], $[x, x^b] = 1$ for all $x \in P$ and $o(b) = 3$. Let $a_1 \in P - S(G)$ and $a_2 = a_1^b$. Then $[a_1, a_2] = 1$ and $\bar{a}_2^{\bar{b}} = \bar{a}_1\bar{a}_2$. This implies $a_2^b = a_1a_2x_1$ for some $x_1 \in S(G)$. Let $y_1 = x_1^b$. We have

$$a_1 = a_1^{b^3} = a_2^{b^2} = (a_1a_2x_1)^b = a_2a_1a_2x_1y_1,$$

so

$$1 = [a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2 = a_2^2x_1y_1$$

and $a_2^2 = x_1y_1$. This implies

$$a_1^2 = (a_2^2)^{b^2} = (x_1y_1)^{b^2} = (y_1x_1y_1)^b = y_1.$$

Let

$$\bar{G} = \left(\langle \bar{a}_1, \bar{a}_2\rangle \times \langle \bar{a}_3, \bar{a}_4\rangle\right) \times_f \langle \bar{b}\rangle$$

with $a_3^b = a_4$,

$$a_4^b = a_3a_4x_4,$$

with $x_4 \in S(G)$,

$$[a_1, a_3] = x_2 \in S(G), \qquad [a_1, a_4] = x_3 \in S(G).$$

Then

$$[a_1^b, a_3^b] = x_2^b = y_2$$

that is

$$[a_2, a_4] = y_2$$

and

$$[a_1^b, a_4^b] = [a_2, a_3 a_4] = x_3^b = y_3.$$

As $P' \leq Z(P)$,

$$[a_2, a_3 a_4] = [a_2, a_3][a_2, a_4]$$

so $[a_2, a_3] = y_2 y_3$. We have

$$a_1^b = a_2, \qquad a_2^b = a_1 a_2 x_1 = a_1 a_2 a_1^2 a_2^2 = a_1^{-1} a_2^{-1},$$
$$x_2^b = y_2, \qquad y_2^b = x_2 y_2, \qquad x_3^b = y_3, \qquad y_3^b = x_3 y_3,$$
$$a_1^{a_3} = a_1 x_2, \qquad a_1^{a_4} = a_1 x_3, \qquad a_2^{a_3} = a_2 y_2 y_3,$$
$$a_2^{a_4} = a_2 y_2, \qquad x_i^{a_3} = x_i = x_i^{a_4}, \qquad y_i^{a_4} = y_i = y_i^{a_3}.$$

Reasoning as with $x_1, y_1, a_1, a_2$ we conclude $a_3 = a_4^b$, $a_4^b = a_3^{-1} a_4^{-1}$. Besides,

$$(a_2^b)^{a_3^b} = (a_1^{-1} a_2^{-1})^{a_4} = a_1^{-1} x_3 a_2^{-1} y_2 = a_1^{-1} a_2^{-1} x_3 y_2 = a_2^b y_2^b y_3^b = a_1^{-1} a_2^{-1} x_2 y_2 x_3 y_3.$$

This implies $x_2 y_3 = 1$ so $x_2 = y_3$ and $y_2 = x_2^b = y_3^b = x_3 y_3$, that is, $x_3 = x_2 y_2$ and $y_3 = x_2$. The other relations

$$(a_1^b)^{a_3^b} = a_2^{a_4} = a_2 y_2 = a_1^b x_2^b,$$
$$(a_1^b)^{a_4^b} = a_2 y_3 = a_1^b x_3^b$$

and

$$(a_2^b)^{a_4^b} = a_1^{-1} a_2^{-1} x_2 y_2 = a_2^b y_2^b$$

are satisfied.

So we have

$$D = \langle a_1, a_2, x_2, y_2 \rangle \trianglelefteq G$$

and the following relations are satisfied:

$$a_1^{a_3} = a_1 x_2, \qquad a_1^{a_4} = a_1 x_2 y_2, \qquad a_2^{a_3} = a_2 x_2 y_2, \qquad a_2^{a_4} a_2 y_2,$$
$$x_2^{a_3} = x_2 = x_2^{a_4}, \qquad y_2^{a_3} = y_2 = y_2^{a_4}, \qquad a_1^b = a_2, \qquad a_2^b = a_1^{-1} a_2^{-1},$$
$$a_3^b = a_4, \qquad a_4^b = a_3^{-1} a_4^{-1}, \qquad x_2^b = y_2, \qquad y_2^b = x_2 y_2.$$

We consider now two cases:

a) $P - S(G)$ has an element of order 2. We can suppose $o(a_3) = 2$,
b) Every element of $P - S(G)$ is of order 4.

Consider the case a). We have $G = ((D \times_\lambda C_2^2) \times C_2^{2t}) \times_f C_3$ with:

$$D \times_\lambda C_2^2 = \langle a_1, a_2, x_2, y_2 \rangle \times_\lambda \langle a_3, a_4 \rangle.$$

$|D| \mid 4^2 \cdot 4$ as

$$D/(D \cap S(G)) \cong C_2^2 \qquad \text{and} \qquad D \cap S(G) = \langle a_1^2, a_2^2, x_2, y_2 \rangle.$$

If $|D| = 4^2 \cdot 4$ we get the group (18) and if $|D| = 4^2$ then we have the following posibilities:

If $a_1^2 = 1$ we get the group (19).

If $a_2^2 \neq 1$ and $x_2 = y_2 = 1$ then $\langle a_3, a_4 \rangle \leq S(G)$ which is impossible. So, $x_2 \neq 1$ and

$$\langle a_1^2, a_2^2 \rangle = \langle x_2, y_2 \rangle \cong C_2^2.$$

If $a_1^2 = x_2$ we get the group (20). If we change $(a_1, a_2, x_2, y_2)$ for

$$(a_2, a_1^{-1} a_2^{-1}, x_2 y_2, x_2) \qquad \text{or} \qquad (a_1 a_2, a_1^{-1}, y_2, x_2 y_2),$$

we conclude that the groups originated by the relations $a_1^2 = x_2 y_2$ and $a_1^2 = y_2$ are isomorphic to the group (20).

Suppose now that every element of $P - S(G)$ has order 4. Then

$$(a_1 a_3)^2 = a_3^2 a_1^2 x_2 \neq 1, \qquad (a_1 a_4)^2 = a_4^2 a_1^2 x_2 y_2 \neq 1, \qquad (a_2 a_3)^2 = a_3^2 a_2^2 x_2 y_2 \neq 1,$$

$$(a_2 a_4)^2 = a_4^2 a_2^2 y_2 \neq 1, \qquad (a_1 a_2 a_3)^2 = a_1^2 a_2^2 a_3^2 y_2 \neq 1.$$

So

$$a_3^2 \notin \{1, a_1^2 x_2, a_2^2 x_2 y_2, a_1^2 a_2^2 y_2\}.$$

If $|D| = 4^3$ there are two possible cases:

$$\langle a_3, a_4 \rangle \cap D = 1 \qquad \text{or} \qquad \langle a_3, a_4 \rangle \cap D \neq 1.$$

In the first case we get the group (21) whereas in the second one we have

$$\langle a_3^2, a_4^2 \rangle \leq \langle a_1^2, a_2^2 \rangle \times \langle x_2 \rangle \times \langle y_2 \rangle.$$

Assume $\langle a_3^2, a_4^2 \rangle = \langle a_1^2, a_2^2 \rangle$. If $a_3^2 = a_1^2$, then

$$a_4^2 = (a_3^2)^b = (a_1^2)^b = a_2^2 \qquad \text{and} \qquad (a_2 a_4)^2 = y_2.$$

Let

$$a_3' = a_2a_4 \qquad \text{and} \qquad a_4' = (a_2a_4)^b = a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1}.$$

We have

$$a_1^{a_3'} = a_1^{a_2a_4} = a_1^{a_4} = a_1x_2y_2 = a_1x_2'$$

with $x_2' = x_2y_2$;

$$a_1^{a_4'} = a_2^{a_3a_4} = a_2x_2 = a_2y_2' \qquad \text{and} \qquad a_3'^2 = y_2 = x_2'y_2'.$$

So, relations $a_3^2 = x_2y_2$ and $a_3^2 = a_1^2$ originate isomorphic groups. Suppose again that $a_3^2 = a_1^2$ originate isomorphic groups. Suppose again that $a_3^2 = a_1^2$ and let

$$a_3' = a_1a_4, \qquad a_4' = (a_1a_4)^b = a_2a_3^{-1}a_4^{-1}.$$

Then

$$a_1^{a_3'} = a_1^{a_4} = a_1x_2y_2 = a_1x_2'$$

with $x_2' = x_2y_2$;

$$a_1^{a_4'} = a_1y_2 = a_1x_2'y_2'$$

with $y_2' = x_2$;

$$(a_1a_4)^2 = a_4^2a_1^2x_2y_2 = a_1^2a_2^2x_2',$$

so

$$a_3'^2 = a_1^2a_2^2x_2y_2 = a_1^2a_2^2x_2'.$$

We conclude that relations

$$a_3^2 = a_1^2, \qquad a_3^2 = x_2y_2 \qquad \text{and} \qquad a_3^2 = a_1^2a_2^2x_2$$

originate isomorphic groups. Suppposse $a_3^2 = a_1^2$ and let

$$a_3' = a_1a_2a_4, \qquad a_4' = a_1^{-1}a_3^{-1}a_4^{-1}, \qquad x_2' = x_2y_2, \qquad y_2' = x_2.$$

Then

$$a_3'^2 = a_4^2a_1^2a_2^2x_2 = a_1^2x_2 = a_1^2y_2',$$

so we can add relation $a_3^2 = a_1^2y_2$ to the previous ones.

Assume $a_3^2 = x_2y_2$. Then

$$a_4^2 = y_2x_2y_2 = x_2.$$

Let

$$a_3' = a_3 a_4, \qquad a_4' = a_3^{-1}, \qquad a_3'^2 = A_3^2 a_4^2 = y_2.$$

If $x_2' = y_2$ and $y_2' = x_2 y_2$ we have

$$a_1^{a_3'} = a_1 x_2', \qquad a_1^{a_4'} = a_1 x_2' y_2', \qquad a_3'^2 = y_2'.$$

So relations $a_3^2 = x_2$ and $a_3^2 = y_2$ can be added to the four previous ones.

Let $a_3^2 = a_1^2$. Taking $(a_3 a_4, a_3^{-1})$ instead of $(a_3, a_4)$ we have

$$a_3'^2 = a_3^2 a_4^2 = a_1^2 a_2^2$$

and taking $(a_4, a_3^{-1} a_4^{-1})$,

$$a_3'^2 = a_4^2 = a_2^2.$$

So the group (22) is originated by the relations $a_3^2 = a_2^2$ and $a_3^2 = a_1^2 a_2^2$.

We know that

$$a_3^2 \notin \{a_1^2 x_2, a_2^2 x_2 y_2, a_1^2 a_2^2 y_2\}.$$

If $a_3^2 = a_1^2 a_2^2 x_2$, then $a_4^2 = a_1^2 y_2$. Let $a_3' = a_3 a_4$ and $a_4' = a_3^{-1}$. Then

$$(a_3')^2 = a_3^2 a_4^2 = a_2^2 x_2 y_2 = a_2^2 y_2'$$

with $y_2' = x_2 y_2$,

$$a_1^{a_3'} = a_1 x_2'$$

with $x_2' = y_2$, and

$$a_1^{a_4'} = a_1 x_2 = a_1 x_2' y_2'.$$

If we take now

$$a_3'' = a_4, \qquad a_4'' = a_3^{-1} a_4^{-1}, \qquad x_2'' = x_2 y_2, \qquad y_2'' = x_2,$$

we have

$$a_3''^2 = a_4^2 = a_1^2 y_2 = a_1^2 x_2'' y_2'',$$

so the three relations

$$a_3^2 = a_1^2 a_2^2 x_2, \qquad a_2^2 y_2 \qquad \text{or} \qquad a_1^2 x_2 y_2$$

originate the group (24).

Finally, suppose $|D| = 4^2$. If $\langle a_3, a_4\rangle \cap D = 1$ and $x_2 = y_2 = 1$ we get the group (25). If $x_2 \neq 1$ then $\langle a_1^2, a_2^2\rangle = \langle x_2, y_2\rangle$ and we can suppose $a_1^2 = x_2$ (so, $a_2^2 = y_2$) getting the group (26). If $\langle a_3, a_4\rangle \cap D \neq 1$, we have

$$\langle a_3^2, a_4^2\rangle \leq \langle a_1^2, a_2^2, x_2, y_2\rangle.$$

If $x_2 = y_2 = 1$, $\langle a_1 a_2, a_3, a_4\rangle$ is an abelian group and if $a_3^2 = a_1^{2i} a_2^{2j}$, $o(a_3 a_1^i a_2^j) = 2$ which is impossible. So, $x_2 \neq 1$ and

$$\langle a_3^2, a_4^2\rangle = \langle x_2, y_2\rangle = \langle a_1^2, a_2^2\rangle.$$

If $a_1^2 = x_2$, as

$$a_3^2 \notin \{a_1^2 x_2 = 1, a_2^2 x_2 y_2 = x_2 = a_1^2, a_2^2 = y_2\}$$

we have $a_3^2 = x_2 y_2 = a_1^2 a_2^2$ and the group (27) appears. The cases $a_1^2 = y_2$ and $a_1^2 = x_2 y_2$ originate isomorphic groups to the last one.

D) Assume $\bar{G} \cong GL(2,3)$. In this case $\beta(\bar{G}) = 1$ and $S(\bar{G}) = C_2$. If $p \neq 2$, then $S(\bar{G}) \cong C_2$ acts f.p.f. on $S(G)$. Besides, $Q_8$ char $SL(2,3) \trianglelefteq \bar{G}$ so, $Q_8 \trianglelefteq \bar{G}$ and $Q_8$ acts f.p.f. on each minimal normal subgroup $L$ of $G$. So $|L| - 1 = 8k$ divides 48. As $\mathrm{Aut}(L)$ has no subgroups isomorphic to $Q_8$, $L \ncong C_p$, for every prime number $p$. The only possible cases are $|L| = 3^2, 5^2$ or $7^2$. If $|L| = 7^2$ then $GL(2,3)$ acts f.p.f. on $L$ and

$$C_2^2 \leq SD_{16} \leq GL(2,3)$$

which is impossible. If $|L| = 5^2 = 1 + 24$, the each subgroup of $GL(2,3)$ of type $C_3$ acts f.p.f. on $L$, so $SL(2,3)$ also acts f.p.f. on each $L$. Then there exists $d \in G - S(G)$ such that $o(d) = 2^8$ and $x^d = x$ for some $x \in L^*$. If $\beta(G) > 1$, [5] implies $d \in C_G(S(G))$, so $2 \mid |C_G(S(G))|$ and

$$C_G(S(G)) = S(G) \times T$$

being $T$ a 2-group contained in a minimal normal subgroup $L$, impossible. Thus, $\beta(G) = 1$ and $S(G) \cong C_5^2$. We have

$$G = C_5^2 \times_\lambda SL(2,3)\langle d\rangle$$

and

$$GL(2,3) \leq GL(2,5),$$

impossible. Assume now $L \cong C_3^2$ and let

$$H/S(G) = Q_8 \trianglelefteq GL(2,3).$$

$H$ is a $\Gamma$-group such that $H/S(H) \cong Q_8$ so $\beta(H) = 1$ and $S(G) \cong C_3^2$. Let

$$N/S(G) = SL(2,3), \qquad Q_8 \in \mathrm{Syl}_2(N, \qquad H \trianglelefteq N.$$

Then

$$N = HN_N(Q_8) = S(G)N_N(Q_8)$$

and

$$S(G) \cap N_N(Q_8) = S(G) \cap Q_8 = \{1\}.$$

So,

$$N_N(Q_8) \cong SL(2,3) \qquad \text{and} \qquad N = [C_3^2]SL(2,3).$$

Therefore, $G - S(G)$ has elements of order 3 and the Sylow 3-subgroup splits on $S(G) \cong C_3^2$. We conclude from the Sylow's theorems of Gaschtz that $\bar{G}$ splits on $S(G)$ and $G = \mathrm{Hol}(C_3^2)$.

If $p = 2$ then $L \cong C_2$ or $C_2^2$ for each minimal subgroup $L$ of $G$. Let

$$H/S(G) = Q_8 \trianglelefteq \bar{G}, \qquad S(G) \leq Z(H).$$

If $S(G) = Z(H)$, $H/Z(H) \cong Q_8$ which is impossible and if $S(G)\langle Z(H)$,

$$H/Z(H) \cong C_2^2$$

so $H' \cong C_2 \trianglelefteq G$ implies $H' \leq S(G)$ and $H/S(G)$ is abelian, impossible.

E) Assume $\bar{G} \cong SL(2,3).C_4$. In this case we reason as in D). The cases

$$C_3^2 \times_\lambda \left(SL(2,3).C_4\right) \qquad \text{and} \qquad C_5^2 \times_\lambda \left(SL(2,3).C_4\right)$$

are excluded because $GL(2,3)$ and $GL(2,5)$ have no subgroups of type $SL(2,3).C_4$.

For $S(G) \cong C_7^2$ we get the group:

$$C_7^2 \times_\lambda \left(SL(2,3).C_4\right) = \langle x,y\rangle \times_\lambda \left((\langle \alpha,\beta\rangle \times_\lambda \langle\gamma\rangle)\langle\sigma\rangle\right),$$

with

$$x^\alpha = x^{-1}y^3, \qquad y^\alpha = x^{-3}y, \qquad x^\beta = xy^{-1},$$
$$y^\beta = x^2y^{-1}, \qquad x^\gamma = x^{-3}, \qquad y^\gamma = xy^2,$$
$$x^\sigma = x^2y^{-1}, \qquad y^\sigma = x^{-2}y^{-2}, \qquad \langle\alpha,\beta\rangle \cong Q_8,$$
$$\langle\gamma\rangle \cong C_3, \qquad \beta^\gamma = \alpha, \qquad \alpha^\gamma = \alpha\beta,$$

$$o(\sigma) = 4, \qquad \sigma^2 = \alpha^2, \qquad \alpha^\sigma = \alpha^{-1},$$
$$\beta^\sigma = \alpha\beta, \qquad \gamma^\sigma = \gamma^{-1}\beta\alpha^{-1} = \gamma^{-1}\alpha\beta.$$

F) Suppose $\bar{G} \cong C_2^3 \times_f C_7$. Then, $\beta(\bar{G}) = 1$. If $p \neq 2$ as $C_2^3$ does not act f.p.f. on $S(G)$, there exist $x \in L^*$, $\bar{b} \in C_2^{3*}$ such that $x^b = x$. So, necessarily $\beta(G) = 1$ as, if $\beta(G) > 1$ then $b \in C_G(S(G))$. This would imply $C_2^3 \leq C_G(S(G))$. But, $C_2^3 \trianglelefteq G$, so $C_2^3 \leq S(G)$ which is impossible.

Thus $S(G) = L$ and $p^n - 1$ divides 7.8. Further, $n > 1$ because $C_2^3 \leq \mathrm{Aut}(L)$ (if $2 \mid |C_G(S(G))|$ we would reason as above). So, $p = 3$ and $n = 2$. Then

$$S(G) = \{1\} \cup Cl_G(x)$$

and

$$|Cl_G(x)| = 8 \qquad \forall x \in S(G)^*.$$

This implies $C_2^3$ acts f.p.f. on $S(G)$, impossible. Thus $p = 2$. If

$$P/S(G) = C_2^3 \trianglelefteq \bar{G}$$

then

$$S(G) \leq Z(P) \qquad \text{and} \qquad L \cong C_2 \text{ or } C_2^3.$$

If $P - S(G)$ has an element $x$ of order 2, as $\bar{P}^* = Cl_{\bar{G}}(\bar{x})$, it follows that every element of $P - S(G)$ has order 2, that is $P \cong C_2^m$. Then

$$S(G) \trianglelefteq P \qquad \text{and} \qquad 7 \nmid |G/C_G(S(G))|$$

so $P = N \times S(G)$ for some $N \trianglelefteq G$ and $N \cong C_2^3$ is necessarily a minimal normal subgroup. It follows $N \leq S(G)$ which is impossible. We conclude that every element of $P - S(G)$ has order 4.

Suppose $L \cong C_2$ for all $L$. Then $S(G) \leq Z(G)$. Let $N \trianglelefteq G$ such that

$$N/S(G) = \langle \bar{x}_1, \bar{x}_2, \bar{x}_3 \rangle \qquad \text{and} \qquad G = N \times_\lambda C_7.$$

Exchanging $b$ by $b^{-1}$ if necessary, we can suppose

$$\bar{x}_1^{\bar{b}} = \bar{x}_2, \qquad \bar{x}_2^{\bar{b}} = \bar{x}_3, \qquad \bar{x}_3^{\bar{b}} = \bar{x}_1\bar{x}_2.$$

Further, we can suppose

$$x_1^b = x_2, \qquad x_2^b = x_3, \qquad x_3^b = x_1 x_2 z'.$$

for some $z' \in S(G)$ Let $y_i = x_i z'$. Then,

$$y_1^b = y_2, \qquad y_2^b = y_3, \qquad y_3^b = y_1 y_2.$$

As $y_1^2 = z$ for some $z \in S(G)$ and $z^b = z$, it follows that

$$y_1^2 = y_2^2 = y_3^3 = z.$$

$z_1 = [y_1, y_2] \in S(G)$, so, if we conjugate by $b$ succesiively, we get

$$[y_1, y_2] = [y_2, y_3] = [y_3, y_1 y_2] = [y_3, y_2][y_3, y_1]$$

and $[y_1, y_3] = 1$. If we conjugate this relation by $b$ again, we get

$$1 = [y_2, y_1 y_2] = [y_2, y_1].$$

So, $\langle y_1, y_2, y_3 \rangle$ is an abelian normal subgroup of $G$. As $y_3^2 = z = z^2$, we have:

$$z = z^b = \left(y_3^b\right)^2 = y_1 y_2 y_1 y_2 = y_1^2 y_2^2 = z^2 = 1.$$

So,

$$M = \langle y_1, y_2, y_3 \rangle \cong C_2^3$$

is a minimal normal subgroup of $G$ such that $M \cap S(G) = \{1\}$ which is impossible. Thus, the case $L \cong C_2$ is excluded.

Assume now that $L \cong C_2^3$. Then

$$G = P \times_f C_7 = P \times_f \langle b \rangle.$$

We have

$$P/S(G) \cong C_2^3$$

and

$$S(G) = Z(P) \cong C_2^{3t}.$$

Further, every element of $P - S(G)$ has order 4. Suppposse there exist

$$L_1 = \langle w_1 \rangle \times \langle w_2 \rangle \times \langle w_3 \rangle, \qquad L_2 = \langle z_1 \rangle \times \langle z_2 \rangle \times \langle z_3 \rangle$$

such that

$$w_1^b = w_2, \qquad w_2^b = w_3, \qquad w_3^b = w_1 w_3, \qquad z_1^b = z_2, \qquad z_2^b = z_3, \qquad z_3^b = z_1 z_2.$$

Then

$$L_3 = \{1\} \cup Cl_G(w_1 z_1)$$

is a minimal normal subgroup such that

$$(w_1 z_1)^b = w_2 z_2,$$

$$(w_2 z_2)^b = w_3 z_3,$$

$$(w_3 z_3)^b = (w_1 z_1)(w_3 z_3) \text{ or } (w_1 z_1)(w_2 z_2)$$

which is different from $w_1 w_3 z_1 z_2$. Thus, the minimal polynomial of $b$ is the same on each minimal normal subgroup $L$ of $G$. If

$$\text{min. pol}(b) = x^3 + x + 1$$

then

$$\text{min. pol}(b^3) = x^3 + x^2 + 1,$$

so, exchanging $b$ for $b^3$, we can suppose that

$$\text{min. pol}(b) = x^3 + x^2 + 1.$$

Let

$$x_1 \in P - S(G), \qquad x_2 = x_1^b, \qquad x_3 = x_2^b.$$

Then

$$(x_1^2)^b = x_2^2, \qquad (x_2^2)^b = x_3^2, \qquad (x_3^2)^b = x_1^2 x_3^2.$$

If $\bar{x}_3^{\bar{b}} = \bar{x}_1 \bar{x}_3$, then $x_3^b = x_1 x_3 y_1$ for some $y_1 \in S(G)$. We have

$$(x_3^2)^b = (x_1 x_3 y_1)^2 = (x_1 x_3)^2,$$

so

$$(x_1 x_3)^2 = x_1^2 x_3^2 \qquad \text{and} \qquad [x_1, x_3] = 1.$$

Let $[x_1, x_2] = z_1$. Then $[x_1^{b^2}, x_2^{b^2}] = z_1^{b^2}$, that is $[x_3, x_1 x_3] = z_1^{b^2}$, so

$$[x_3, x_1][x_3, x_3] = z_1^{b^2} \qquad \text{and} \qquad z_1^{b^2} = 1.$$

It follows that $z_1 = 1$, $[x_1, x_2] = 1$. Now $[x_1^b, x_2^b] = 1$ that is, $[x_2, x_3] = 1$. Therefore $\langle x_1, x_2, x_3 \rangle$ is an abelian group and $P = \langle x_1, x_2, x_3 \rangle Z(P)$ is also abelian. As

$$Z(P) = S(G) = \Omega_1(P)$$

we have

$$P \cong C_4^3 \times C_2^{3t}.$$

Further, $x_1 = x_1^{b^7}$ implies $y_1 = x_2^2$ and therefore $G$ is isomorphic to the group (1). If $(\bar{x}_3)^{\bar{b}} = \bar{x}_1\bar{x}_2$, then $x_3^b = x_1x_2w_1$ for some $w_1 \in S(G)$.

Let $w_2 = w_1^b$, $w_3 = w_2^b$. Then $w_3^b = w_1w_3$. If we take as new representatives of $\bar{x}_i$ the elements

$$x_1' = x_1w_2, \qquad x_2' = {x_1'}^b = x_2w_3, \qquad x_3' = {x_2'}^b = x_3w_1w_3,$$

then we have

$$(x_3')^b = x_1x_2w_1w_2w_1w_3 = x_1w_2x_2w_3 = x_1'x_2'.$$

So we can suppose $w_i = 1$ and $x_3^b = x_1x_2$. Let $z_i = x_i^2$. We have

$$z_1^b = z_2, \qquad z_2^b = z_3, \qquad z_3^b = z_1z_3.$$

But

$$z_1z_3 = z_3^b = (x_3^b)^2 = x_1x_2x_1x_2 = x_1^2x_2^2[x_1, x_2] = z_1z_2[x_1, x_2].$$

So,

$$[x_1, x_2] = z_2z_3 = x_2^2x_3^2.$$

Conjugating succesively by $b$, we get

$$[x_2, x_3] = z_1 = x_1^2, \qquad [x_1, x_3] = z_1z_2 = x_1^2x_2^2.$$

So, $M = \langle x_1, x_2, x_3\rangle$ is a normal subgroup of $G$. If $z_1 = 1$, then $z_2 = z_3 = 1$ and $M \cong C_2^3$ would be a minimal normal subgroup of $G$ not contained in $S(G)$. So $z_1 \neq 1$ and

$$N = \langle z_1, z_2, z_3\rangle \cong C_2^3$$

is a minimal normal subgroup of $G$. Thus, $M$ is a group of order $2^6$ generated by the elements $x_i$ of order 4 and with the conmutators $[x_i, x_j]$ given by the above relations. This group is a 2–subgroup of type $Sz(8)$. If $T \cong C_2^{3t}$ is a complement of $N$ in $S(G)$, we get the second group in the table.

G) Assume $\bar{G} \cong C_2^4 \times_f C_5$. If $p \neq 2$ as there exist $x \in L^*$ and $\bar{b} \in C_2^{4*}$ such that $x^b = x$ it follows that $\beta(G) = 1$ and $p^n - 1$ divides $2^4.5$. This implies $|S(G)| = 3^2$. Since $\mathrm{Aut}(C_3^2)$ has no subgroups isomorphic to $C_2^4$, it follows that $2 \mid |C_G(S(G))|$. Condition $\beta(\bar{G}) = 1$ implies then

$$C_G(S(G)) = C_3^2 \times C_2^4$$

and $C_2^4$ contains a minimal normal 2-subgroup. Thus $p = 2$ and if $G = P \times_\lambda C_5$, $S(G) \leq Z(P)$ and, necessarily, $L \cong C_2$ for each minimal normal subgroup $L$ of $G$, so $S(G) = Z(G)$. Let $b \in G$ such that $o(b) = 5$ and

$$x_1 \in P - S(G), \qquad x_2 = x_1^b, \qquad x_3 = x_2^b, \qquad x_4 = x_3^b.$$

Then

$$\bar{x}_4^b = \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4$$

and

$$x_4^b = x_1x_2x_3x_4z_1$$

for some $z_1 \in S(G)$. As $x_1^2 \in Z(G)$, we have

$$x_1^2 = x_2^2 = x_3^2 = x_4^2.$$

Let $[x_1, x_2] = w_1 \in Z(G)$, then

$$[x_2, x_3] = [x_1, x_2]^b = w_1^b = w_1 \qquad \text{and} \qquad [x_3, x_4] = [x_2, x_3]^b = w_1.$$

If $[x_1, x_3] = w_2 \in Z(G)$, then

$$[x_2, x_4] = w_2 \qquad \text{and} \qquad [x_3, x_4]^b = w_1.$$

This implies

$$[x_4, x_1x_2x_3x_4] = w_1,$$

so

$$[x_4, x_1][x_4, x_2][x_4, x_3] = w_1 \qquad \text{and} \qquad [x_4, x_1] = [x_4, x_2]^{-1} = w_2.$$

Thus we have

$$[x_1, x_2] = [x_2, x_3] = [x_3, x_4] = w_1 \qquad \text{and} \qquad [x_1, x_3] = [x_1, x_4] = [x_2, x_4] = w_2.$$

Further,

$$x_1 = x_1^{b^5} = x_2^{b^4} = x_3^{b^3} = x_4^{b^2} = (x_1x_2x_3x_4z_1)^b = x_2x_3x_4x_1x_2x_3x_4z_1,$$

so

$$\begin{aligned} z_1 =& x_1^{-1}(x_2x_3x_4)x_1(x_2x_3x_4) \\ &= x_1^{-1}x_1^{x_2x_3x_4}(x_2x_3x_4)^2 \\ &= [x_1, x_2x_3x_4](x_2x_3x_4)^2 \\ &= [x_1, x_2][x_1, x_3][x_1, x_4](x_2x_3x_4)^2 \\ &= [x_1, x_2](x_2x_3x_4)^2, \end{aligned}$$

but

$$(\bar{x}_2\bar{x}_3\bar{x}_4)^{\bar{b}} = \bar{x}_3\bar{x}_4\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4 = \bar{x}_1\bar{x}_2.$$

This implies

$$(x_2x_3x_4)^2 = (x_1x_2)^2$$

and

$$\begin{aligned} [x_1, x_2](x_1x_2)^2 &= x_1^{-1}x_2^{-1}(x_1x_2)^3 \\ &= x_1^{-1}x_2^{-1}(x_1x_2)^{-1} \\ &= x_1^{-1}x_2^{-1}x_1^{-1} \\ &= x_2^{-2}x_1^{-1} \\ &= 1. \end{aligned}$$

Thus $x_1^2 = x_2^2$ and $z_1 = 1$. Let $M = \langle x_1, x_2, x_3, x_4\rangle \trianglelefteq G$. Then

$$M \cap S(G) = \langle x_1^2, w_1, w_2\rangle \leq C_2^3,$$

$$M/(M \cap S(G)) \cong C_2^4$$

and

$$G = (M \times_\lambda C_5) \times C_2^t.$$

We have

$$[x_1, x_2] = w_1 \qquad \text{and} \qquad [x_1, x_2](x_1x_2)^2 = 1,$$

so $w_1 = (x_1x_2)^2$. Besides,

$$\begin{aligned} w_1 &= (x_1x_2)^2 \\ &= (x_2x_3x_4)^2 \\ &= x_2x_3x_4x_2x_3x_4 \end{aligned}$$

$$= (x_2x_3)^2(x_2x_3)^{-1}x_4^2x_4^{-1}\ (x_2x_3)x_4$$
$$= (x_2x_3)^2x_4^2[x_2x_3, x_4]$$
$$= (x_2x_3)^2x_4^2[x_2, x_3][x_2, x_4]$$
$$= (x_2x_3)^2x_4^2w_1w_2$$
$$= [x_2, x_3]x_2^2x_3^2x_4^2w_1w_2$$
$$= w_1x_2^2x_3^2x_4^2w_1w_2$$
$$= x_4^2w_2 = x_1^2w_2,$$

as $x_1^2 = x_2^2 = x_3^2 = x_4^2$. So $w_1 = x_1^2w_2$ and

$$M \cap S(G) = \langle x_1^2, w_1, w_2\rangle \leq C_2^2.$$

Then, $|M| = 2^5$ or $2^6$. Further

$$M/Z(M) \cong C_2^4 \quad \text{and} \quad |Z(M)| \leq 4$$

so $b$ acts trivially on $Z(M)$. It is enough to determine the 2-groups of order $2^n$, $5 \leq n \leq 6$, such that

$$5 \mid |\mathrm{Aut}(M)| \quad \text{and} \quad M/Z(M) \cong C_2^4$$

(because for such a 2-group $\bar{b}$ acts trivially on $M/Z(M)$ or acts f.p.f. on $C_2^4$. In the first case $\bar{x}^{\bar{b}} = \bar{x}$, so $x^b = xz$ and $x^{b^2} = x$, that is $b = 1$, impossible. Consequently, $\bar{b}$ acts f.p.f. on $C_2^4$ and

$$(M/Z(M))\langle\bar{b}\rangle \cong C_2^4 \times_f C_5).$$

$M/Z(M) \cong C_2^4$ implies $M = 2^n\Gamma_k$ with $k = 5, 10, 11$ or $13$ and $n = 5$ or $6$. The groups $2^5\Gamma_5a_2$, $2^6\Gamma_5a_2$ and $2^6\Gamma_{13}a_5$ satisfy the additional condition $5 \mid |\mathrm{Aut}(M)|$ [3]. For these groups $|\mathrm{Aut}(M)| = 2^5 \cdot 384 \cdot 5$, $2^5 \cdot 144 \cdot 5$ and $2^6 \cdot 2^2 \cdot 3 \cdot 5$ respectively. So the desired groups appear.

H) Assume $\bar{G} \cong \mathrm{Hol}(C_2^3, C_7 \times_f C_3)$. If $p \neq 2$, there exist $x \in L^*$, $\bar{b} \in C_2^{3*}$ such that $x^b = x$. Then $\beta(G) = 1$ and necessarily, $C_2^3 \leq \mathrm{Aut}(L)$ as $2 \nmid |C_G(S(G))|$. Besides, $p^n - 1$ divides $8 \cdot 7 \cdot 3$, so $n = 2$ and $p \in \{3, 5\}$. In any case $2 \nmid |C_G(x)|$ for all $x \in S(G)^*$ so $C_2^3$ acts f.p.f. on $S(G)$ which is impossible. Thus $p = 2$ and $S(G) \leq Z(P)$ with

$$G = P \times_\lambda (C_7 \times_f C_3).$$

Consequently, $L \cong C_2$, $C_2^2$ or $C_2^3$. If $L \cong C_2^3$, we consider

$$H = S(G)(C_7 \times_f C_3).$$

$H$ is a $\Gamma$-group such that

$$H/S(H) \cong C_7 \times_f C_3.$$

Then $\beta(H) = 1$, that is, $S(G) \cong C_2^3$. Therefore, $|P| = 2^6$. As the sectional rank of $P$ is $\leq 4$, it follows that $P \cong C_4^3$ or $P$ is of the type $Sz(8)$, obtaining the two listed groups. If $L \cong C_2^2$, $C_7$ acts trivially on $L$ and

$$C_G(S(G)) = S(G) \times C_7,$$

so $C_7 \trianglelefteq G$, impossible. If $L \cong C_2$, then

$$S(G) = Z(G) = \Omega_1(G)$$

and every element of $P - S(G)$ is of order 4 according to Maschke's Theorem. Let

$$N/S(G) = C_2^3 \times_f C_7.$$

Then

$$S(N) = \Omega_1(Z(N)) = S(G)$$

and $N$ is a $\Gamma$-group such that

$$N/S(N) \cong C_2^3 \times_f C_7 \qquad \text{and} \qquad S(N) = Z(N),$$

which is impossible (if $S(G) \langle S(N)$ then $P - S(G)$ has elements of order 2, so $P \cong C_2^5$ and according to Maschke's Theorem $P = S(G) \times T$ with $T \trianglelefteq G$, impossible).

I) Suppose

$$\bar{G} \in \{C_5^2 \times_f Q_8,\ C_5^2 \times_f DC_3,\ C_5^2 \times_f SL(2,3)\}.$$

Then $\beta(\bar{G}) = 1$ and $S(\bar{G}) = C_5^2$. Consider the homomorphism $\psi : \bar{G} \to \mathrm{Aut}(L)$. Then, $\ker(\psi) \trianglelefteq G$ and $\ker(\psi) = \{1\}$ or $C_5^2 \leq \ker(\psi)$. In the first case $\bar{G} \leq \mathrm{Aut}(L)$ and $5^2 \cdot 4$ divides $|GL(m,p)|$ and $p^m - 1$ divides $5^2 \cdot 3 \cdot 4$. If $L \cong C_p^m$ then $|L| = 11^2$ and $11^2 - 1 = 5 \cdot 24$ so, necessarily

$$\bar{G} \cong C_5^2 \times_f SL(2,3)$$

and $SL(2,3)$ acts f.p.f. on $S(G)$. In this case we have

$$G = N \times_f SL(2,3)$$

and $C_2$ acts f.p.f. on $N$. This implies that $N$ is abelian and $C_5^2 \trianglelefteq G$, impossible. If $C_5^2 \leq \ker(\psi)$, then $C_5^2 \leq C_G(L)$ and $p = 5$. If $|L| = 5^m$, $5^m - 1$ is a divisor of 24, so $m = 1$ or 2. If $m = 2$,

$$G = N \times_f SL(2,3).$$

Let

$$H = S(G) \times_f SL(2,3).$$

$H$ is a $\Gamma$-group such that

$$H/S(H) \cong SL(2,3),$$

so $\beta(H) = 1$, that is $S(G) = C_5^2$. Besides, $N$ is abelian because it has automorphisms of order 2 that act f.p.f. on $N$. So $\Omega_1(N) = S(G) = C_5^2$ implies $N \cong C_{25}^2$ and

$$G = C_{25}^2 \times_f SL(2,3),$$

impossible because such holomorph does not exist. If $L = \langle x \rangle \cong C_5$ and we suppose that there exists $Q_8 \leq \bar{G}$, then $x^{a^2} = x$ with $\langle a^2 \rangle = Z(Q_8)$, so $x^a = x^{-1} = x^b$ with $Q_8 = \langle a, b \rangle$. Then $x^{ab} = x$ and $o(ab) = 4$ which is impossible as $|Cl_G(x)| = 4$. We conclude $\bar{G}$ has no subgroups isomorphic to $Q_8$ and

$$\bar{G} \cong C_5^2 \times_f DC_3.$$

Then $G = N \times_f C_4$ so $N$ is abelian and $C_3 \leq C_G(S(G))$. This implies $C_3 \trianglelefteq G$ which is impossible.

J) Assume $\bar{G} = PSL(2,11)$. We have

$$|PSL(2,11)| = 660 = 11 \cdot 5 \cdot 3 \cdot 2^2.$$

If there exists $L$ such that $C_G(L)/L \neq \bar{1}$, then $C_G(L)/L = \bar{G}$ and $L \cong C_2$, $S(G) = Z(G)$. Consequently

$$G/Z(G) \cong PSL(2,11)$$

and

$$G'Z(G)/Z(G) \cong PSL(2,11).$$

So,

$$G'/(G' \cap Z(G')) \cong PSL(2,11),$$

that is

$$G'/Z(G') \cong PSL(2,11)$$

and, in general

$$G^{\langle s\rangle}/Z(G^{\langle s\rangle}) \cong PSL(2,11).$$

Let $s \in \mathbb{N}$ be such that

$$G^{\langle s\rangle\prime} = G^{\langle s\rangle}.$$

Then $G^{\langle s\rangle}$ is a perfect central extension of $PSL(2,11)$ and $Z(G^{\langle s\rangle})$ is a subgroup of the Schur multiplier of $PSL(2,11)$, that is, $Z(G^{\langle s\rangle}) \leq C_2$. If $Z(G^{\langle s\rangle}) = 1$, then

$$G^{\langle s\rangle} \cong PSL(2,11)$$

is a minimal normal subgroup of $G$, which is impossible. So $Z(G^{\langle s\rangle}) \cong C_2$ and

$$G = G^{\langle s\rangle} Z(G) = G^{\langle s\rangle} \times C_2^t$$

$G^{\langle s\rangle}$ being the only perfect central extension of $C_2$ by $PSL(2,11)$. Suppose, on the other hand, that $C_G(L) = L$ for all $L$, so $\bar{G} \leq \mathrm{Aut}(L)$. If $p = 2$, as 2 has order 10 modulo 11 and $C_{11}$ is not a subgroup of $C_G(L)$ it follows that $|L| = 2^{s+10e}$ with $2^s = |C_L(b)|$ and $2^{10e} = |[L,b]|$ being $b$ an element of order 11. But $|L| - 1$ divides $11 \cdot 5 \cdot 3 \cdot 2^2$ which is impossible. If $L \cong C_p^m$ and $p \neq 2$, then $p^m - 1$ is a divisor fo 660 and $\bar{G} \leq \mathrm{Aut}(L)$, impossible.

K) Finally, assume $\bar{G} \cong M_9$. We have

$$\beta(\bar{G}) = 1 \qquad \text{and} \qquad |\bar{G}| = 720 = 16 \cdot 9 \cdot 5.$$

Besides $M_9/A_6 \cong C_2$. Let $M/S(G) = A_6 \trianglelefteq \bar{G}$. If there exists $L$ such that $C_G(L)/L \neq \bar{1}$ then $M \leq C_G(L)$ and therefore $L \cong C_2$ or $C_3$. If $L \cong C_2$, then

$$S(G) = Z(G) \cong C_2^t \qquad \text{and} \qquad G/Z(G) \cong M_9.$$

$M/Z(M) \cong A_6$ so $M = T \times C_2^s$ being $T$ the only perfect extension of $C_2$ by $PSL(2,9) \cong A_6$. Let $d \in G - M$ such that $o(d) = 2^e$. So,

$$G = T\langle d\rangle \times C_2^s \qquad \text{and} \qquad T\langle d\rangle/C_2 \cong M_9.$$

If $L \cong C_3$, we have

$$M^{\langle s\rangle\prime} = M^{\langle s\rangle}$$

for some $s \in \mathbb{N}$. This implies $M = D \times C_3^s$ with $D$ the only perfect central extension of $C_3$ by $A_6$ and

$$G = (D \times C_3^8)\langle e\rangle$$

with $o(e) = 2^\lambda$ and $x^e = x^{-1}$ for all $x \in C_3^s$, being $D\langle e\rangle$ such that

$$D\langle e\rangle/Z(D) \cong M_9, \qquad Z(D) \cong C_3.$$

If $C_G(L) = L$, then $\bar{G} \leq \mathrm{Aut}(L)$ and $|L| - 1$ divides 720. It follows $|L| \in \{2^4, 3^4\}$. $A_8$ has no subgroups isomorphic to $M_9$ so the case $|L| = 2^4$ is excluded. If $|L| = 3^4$, $|L^*| = 5 \cdot 16$, so the 2-subgroups of $\bar{G}$ act f.p.f. on $L$ which is impossible as these subgroups are isomorphic to $SD_{16}$ and $C_2^2 \leq SD_{16}$.

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