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The approximate solution of nonlinear singular integro-differential equations

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Abstract

This paper concerns the sufficient conditions for the applicability of the Newton-Kantorovich method to nonlinear singular integro-differential equation with Hilbert Kernel.

1. Introduction

Nonlinear singular integro–differential equations with Hilbert or Cauchy kernel have been treated by many author's, cf. Mal'sagov [5], Pogorzelski [6], Wolfersdorf [8] and the monograph [2] by Guseinov and Mukhtarov.

In this paper the class of nonlinear singular integro-differential equations with Hilbert kernel in the form

$$\begin{aligned} &P(u) = u - A(u) = 0\ , \\ &A(u) = \frac{1}{2\pi i}\int_0^{2\pi} \Psi\big(\sigma, u(\sigma), u'(\sigma)\big)\cot\frac{\sigma - s}{2}\, d\sigma \end{aligned} \tag{1}$$

is investigated by means of the Newton-Kantorovich method [2, 3, 5] in the generalized Hölder space $H_{\phi,m}$ [2, 7].

Lemma (1)

Let the function $\Psi(\sigma, u(\sigma), u'(\sigma))$ be defined on $u \in [-M, M]$, $u' \in [-R, R]$, periodic in σ (with period 2π) and have partial derivatives with respect to σ, u, u' up to order $(m-1)$. If the function $\Psi\ (\sigma, u(\sigma), u'(\sigma))$ satisfies the following condition for arbitrary $\sigma_n \in [0, 2\pi]$, $u_n \in [-M, M]$ and $u'_n \in [-R, R]$, $(n = 1, 2)$:

$$\left| \frac{\partial^k}{\partial\sigma^\alpha\, \partial u^\beta\, \partial u^\gamma}\, \Psi(\sigma_1, u_1, u'_1) - \frac{\partial_k}{\partial\sigma^\alpha\, \partial u^\beta\, \partial u^\gamma}\, \Psi(\sigma_2, u_2, u'_2) \right|$$

$$\leq C(k)\Big\{ f\big(|\sigma_1 - \sigma_2|\big) + |u_1 - u_2| + |u'_1 - u'_2| \Big\} \tag{2}$$

for $\alpha + \beta + \gamma = k$, $k = 0, 1, 2, \ldots, m-1$, where $f(\delta)$ is nondecreasing function from $(0, \pi]$ into R_+ and $f(0) = 0$. Then

$$\omega_\Psi^m(\delta) \leq C(m) \begin{cases} f(\delta) + \omega_u^1(\delta) + \omega_{u'}^1(\delta), & \text{at } m = 1 \\ \omega_u^m(\delta) + \delta f(\delta)\omega_u^{m-2}(\delta) + \omega_u^{m-\nu}(\delta)\omega_{u'}^{\nu}(\delta) & \\ +\delta f(\delta)\omega_{u'}^{m-2}(\delta) + \omega_{u'}^m(\delta), & \text{at } m \geq 2. \end{cases}$$

Where $\nu = 1, 2, \ldots, m-1$.

Proof. For $m = 1$, the Lemma is true.

For $m = 2$ we have

$$\begin{aligned} \Delta_h^2 \Psi(\sigma, u(\sigma), u'(\sigma)) &= \Psi(\sigma + 2h, u(\sigma + 2h), u'(\sigma + 2h)) \\ &\quad - 2\Psi(\sigma + h, u(\sigma + h), u'(\sigma + h)) + \Psi(\sigma, u(\sigma), u'(\sigma)). \end{aligned}$$

Using Lagrange's formula we get

$$\begin{aligned} |\Delta_h^2 \Psi(\sigma, u(\sigma), u'(\sigma))| = \Big| &\int_0^1 \Big[\Psi'_\sigma(\sigma + h + \theta h, u(\sigma + 2h), u'(\sigma + 2h)) \\ &- \Psi'_\sigma(\sigma + \theta h, u(\sigma + 2h), u'(\sigma + 2h)) \Big]\, h\, d\theta \\ &+ 2\int_0^1 \Big[\Psi'_u(\sigma + h, u(\sigma + h) + \theta\big(u(\sigma + 2h) \\ &- u(\sigma h)\big), u'(\sigma + 2h)) \\ &- \Psi'_u(\sigma + h, u(\sigma) + \theta\big(u(\sigma + 2h) - u(\sigma)\big), u'(\sigma + 2h)) \Big] \\ &\times \big(u(\sigma + 2h) - u(\sigma + h)\big) d\theta \end{aligned}$$

$$
\begin{aligned}
&+ 2 \int_0^1 \Psi'_u\big(\sigma + h, u(\sigma) + \theta\big(u(\sigma + h) - u(\sigma)\big), u'(\sigma_2 h)\big)\\
&\times \big(u(\sigma + 2h) - 2u(\sigma + h) + u(\sigma)\big) d\theta\\
&+ 4 \int_0^1 \Big[\Psi'_{u'}\big(\sigma + h, u(\sigma + h), u'(\sigma + h)\\
&+ \theta\big(u'(\sigma + 2h) - u'(\sigma + h)\big)\big)\\
&- \Psi'_{u'}\big(\sigma + h, u(\sigma + h), u'(\sigma + \theta\big(u'(\sigma + h) - u'(\sigma)\big)\big)\Big]\\
&\times u'(\sigma + 2h) - u'(\sigma + h)) d\theta\\
&+ 4 \int_0^1 \Psi'_{u'}\big(\sigma + h, u(\sigma + h), u'(\sigma) + \theta\big(u'(\sigma + h) - u'(\sigma)\big)\big)\\
&\times \big(u'(\sigma + 2h) - 2u'(\sigma + h) + u'(\sigma) d\theta\\
&- \int_0^1 \Big[\Psi'_u\big(\sigma, u(\sigma + h) + \theta\big(u(\sigma + 2h) - u(\sigma + h)\big), u'(\sigma + 2h)\big)\\
&- \Psi'_u\big(\sigma, u(\sigma) + \theta\big(u(\sigma + h) - u(\sigma)\big), u'(\sigma + 2h)\big)\Big]\\
&\times \big(u(\sigma + 2h) - u(\sigma + h)\big) d\theta\\
&- \int_0^1 \Psi'_u\big(\sigma, u(\sigma) + \theta\big(u(\sigma + h) - u(\sigma)\big), u'(\sigma + 2h)\big)\\
&\times \big(u(\sigma + 2h) - 2u(\sigma + h) + u(\sigma)\big) d\theta\\
&- 2 \int_0^1 \Big[\Psi'_{u'}\big(\sigma + h, u(\sigma), u'(\sigma + h) + \theta\big(u'(\sigma + 2h) - u'(\sigma + h)\big)\big)\\
&- \Psi'_{u'}\big(\sigma + h, u(\sigma), u'(\sigma) + \theta\big(u'(\sigma + h) - u'(\sigma)\big)\big)\Big]\\
&\times \big(u'(\sigma + 2h) - 2u'(\sigma + h)\big) d\theta\\
&- 2 \int_0^1 \Psi'_{u'}\big(\sigma + h, u(\sigma), u'(\sigma) + \theta\big(u'(\sigma + h) - u'(\sigma)\big)\big)\\
&\times \big(u'(\sigma + 2h) - u'(\sigma + h) + u'(\sigma)\big) d\theta\\
&+ \int_0^1 \Big[\Psi'_{u'}\big(\sigma, u(\sigma), u'(\sigma + h) + \theta\big(u'(\sigma + 2h) - u'(\sigma + h)\big)\big)\\
&- \Psi'_{u'}\big(\sigma, u(\sigma), u'(\sigma) + \theta\big(u'(\sigma + h) - u'(\sigma)\big)\big)\Big]\\
&\times \big(u'(\sigma + 2h) - u'(\sigma + h)\big) d\theta\\
&+ \int_0^1 \Psi'_{u'}\big(\sigma, u(\sigma), u'(\sigma) + \theta\big(u'(\sigma + h) - u'(\sigma)\big)\big)
\end{aligned}
$$

$$\begin{aligned}
&\times \big(u'(\sigma+2h) - 2u'(\sigma+h) + u'(\sigma)\big)d\theta \\
&+ 2\int_0^1 \Big[\Psi'_{u'}\big(\sigma, u(\sigma+h), u'(\sigma) + \theta\big(u'(\sigma+h) - u'(\sigma)\big)\big) \\
&- \Psi'_{u'}\big(\sigma, u(\sigma+h), u'(\sigma) + \theta\big(u'(\sigma+h) - u'(\sigma)\big)\big)\Big] \\
&\times \big(u'(\sigma+h) - u'(\sigma)\big)d\theta \\
&+ 2\int_0^1 \Big[\Psi\prime_{u'}\big(\sigma+h, u(\sigma+h), u'(\sigma) + \theta\big(u'(\sigma+h) - u'(\sigma)\big)\big) \\
&- \Psi'_{u'}\big(\sigma, u(\sigma), u'(\sigma) + \theta\big(u'(\sigma+h) - u'(\sigma)\big)\big)\Big] \\
&\big(u'(\sigma+h) - u'(\sigma)\big)d\theta \\
&+ 2\int_0^1 \Big[\Psi'_{u'}\big(\sigma+h, u(\sigma+h), u'(\sigma) + \theta\big(u'(\sigma+h) - u'(\sigma)\big)\big) \\
&- \Psi'_{u'}\big(\sigma, u(\sigma+h), u'(\sigma) + \theta\big(u'(\sigma+2h) - u'(\sigma+h)\big)\big)\Big] \\
&\times \big(u'(\sigma+h) - u'(\sigma)\big)d\theta \\
&- 2\int_0^1 \Psi'_{u'}\big(\sigma, u(\sigma+h), u'(\sigma+h) + \theta\big(u'(\sigma+2h) - u'(\sigma+h)\big)\big) \\
&\times \big(u'(\sigma+2h) - 2u'(\sigma+h) + u'(\sigma)\big)d\theta \\
&+ 2\int_0^1 \Big[\Psi'_{\sigma}\big(\sigma+\theta h, u(\sigma+h), u'(\sigma)\big) \\
&- \Psi'_{\sigma}\big(\sigma+\theta h, u(\sigma), u'(\sigma+h)\big)\Big]\, hd\theta \\
&- 2\int_0^1 \Big[\Psi'_{\sigma}\big(\sigma+\theta h, u(\sigma), u'(\sigma+h)\big) \\
&- \Psi'_{\sigma}\big(\sigma+\theta h, u(\sigma), u'(\sigma)\big)\Big]\, h\, d\theta\Big|,
\end{aligned}$$

for $0 \le \theta \le 1$.

Using condition (2) we get

$$\left|\Psi'_u\big(\sigma, u(\sigma) + \theta\big(u(\sigma+h) - u(\sigma)\big), u'(\sigma+h)\big)\right| \le C(1)(M+R) + K_1,$$

and

$$\left|\Psi'_{u'}\big(\sigma, u(\sigma), u'(\sigma) + \theta\big(u'(\sigma+h) - u'(\sigma)\big)\big)\right| \le C(1)(M+R) + \tilde{K}_1$$

where

$$K_{m-1} = \max_{0\le\sigma\le 2\pi} \left|\Psi_u^{(m-1)}(\sigma, 0, 0)\right|,$$

and

$$\tilde{K}_{m-1} = \max_{0\leq\sigma\leq 2\pi} \left|\Psi_{u'}^{(m-1)}(\sigma, 0, 0)\right|$$

Then

$$\omega_{\Psi}^{2}(\delta) \leq C(2)\left[\delta f(\delta) + \omega_{u}^{1}(\delta)\,\omega_{u'}^{1}(\delta) + \omega_{u}^{2}(\delta) + \omega_{u'}^{2}(\delta)\right].$$

By induction we see that

$$\begin{aligned}\Delta_h^m \Psi\big(\sigma, u(\sigma), u'(\sigma)\big) = \sum_{n=0}^{m-1} \binom{m-1}{n} \int_0^1 &\Delta_h^n \Psi_u'\big(\sigma + h, u(\sigma) \\ &+ \theta\big(u(\sigma+h) - u(\sigma)\big), u'(\sigma+h)\big) \times \Delta_h^{m-n} u(\sigma + nh)d\theta \\ &+ \sum_{n=0}^{m-1} \binom{m-1}{n} \int_0^1 \Delta_h^n \Psi_{u'}'\big(\sigma + h, u(\sigma), u'(\sigma) \\ &+ \theta\big(u'(\sigma + h - u'(\sigma))\big)\big) \times \Delta_h^{m-n} u'(\sigma + nh)d\theta \\ &+ h\int_0^1 \Delta_h^{m-1}\Psi_\sigma'\big(\sigma + \theta h, u(\sigma), u'(\sigma)\big)d\theta.\end{aligned}$$

The functions

$$\Psi_u'\big(\sigma + h, u(\sigma) + \theta\big(u(\sigma+h) - u(\sigma)\big), u'(\sigma+h)\big),$$

$$\Psi_{u'}'\big(\sigma + h, u(\sigma), u'(\sigma) + \theta\big(u'(\sigma_h) - u'(\sigma)\big)\big)$$

and

$$\Psi_\sigma'\big(\sigma + \theta h, u(\sigma), u'(\sigma)\big)$$

have partial derivatives up to $n-1$, $n = 1, 2, \ldots, m-1$ and these derivatives satisfy condition (2), then by the hypothesis we get

$$\begin{aligned}\left|\Delta_h^m \Psi\big(\sigma, u(\sigma), u'(\sigma)\big)\right| \leq C(m)\Big[&\omega_u^m(h) + hf(h)\omega_u^{m-2}(h) \\ &+ \omega_u^{m-\nu}(h) \times \omega_{u'}^{\nu}(h) + hf(h)\omega_{u'}^{m-2}(h) + \omega_{u'}^{m}(h)\Big],\end{aligned}$$

that is

$$\begin{aligned}\omega_\Psi^m(\delta) \leq C(m)\Big[&\omega_u^m(\delta) + \delta f(\delta)\omega_u^{m-2}(\delta) + \omega_u^{m-\nu}(\delta)\omega_{u'}^{\nu}(\sigma) \\ &+ \delta f(\delta)\omega_{u'}^{m-2}(\delta) + \omega_{u'}^{m}(\delta)\Big].\end{aligned}$$

Remark. We say that $(f;\phi) \in EH\Phi^m$ if

$$\delta^{m-1} f(\delta) \int_\delta^\pi \frac{\phi(\zeta)}{\zeta^{m-1}}\, d\zeta O(\phi(\delta)).$$

Lemma (2).

If the condition (2) is satisfied and if $(f;\phi) \in EH\Phi^m$, $u(\sigma), u'(\sigma) \in H_{\phi,m}$, *then* $\Psi\big(\sigma, u(\sigma), u'(\sigma)\big) \in H_{\phi,m}$.

Proof. It follows from the above remark, Lemma (1) and Marshoud's theorem [4, 7].

Lemma (3).

If the function $\Psi\big(\sigma, u(\sigma), u'(\sigma)\big)$ *satisfies condition (2) then the operator* $P(u)$ *is Fréchet differentiable in the space* $H_{\phi,m}$, *moreover*

$$P'(u)h = h(s) - \frac{1}{2\pi i} \int_0^{2\pi} \Big\{ \Psi'_u\big(\sigma, u(\sigma), u'(\sigma)\big)\, h(\sigma) + \Psi'_{u'}\big(\sigma, u(\sigma), u'(\sigma)\big) h'(\sigma) \Big\} \cot \frac{\sigma - s}{2}\, d\sigma \ \ldots \tag{3}$$

Also $P'(u)$ *satisfies Lipschitz condition:*

$$\big|P'(u) - P'(u_0)\big|_{\phi,m} \le C(m) \big|u - u_0\big|_{\phi,m},$$

$C(m)$ *is a constant, in the sphere* $\big|u - u_0\big|_{\phi,m} \le r$.

Proof. Let $u(s)$ be a fixed and $h(s)$ an arbitrary element in $H_{\phi,m}$. Then we have

$$P(u+h) - P(u) = h(s) - \frac{1}{2\pi i} \int_0^{2\pi} \Big\{ \Psi\big(\sigma, u(\sigma) + h(\sigma), u'(\sigma)\big) h'(\sigma) - \Psi\big(\sigma, u(\sigma), u'(\sigma)\big) \Big\} \cot \frac{\sigma - s}{2}\, d\sigma.$$

According to the following identity

$$\begin{aligned} \Psi(\sigma, u+h, u'+h') - \Psi_u(\sigma, u, u') = {} & \Psi'_u(\sigma, u, u')h + \Psi'_{u'}(\sigma, u, u')h' \\ & + \int_0^1 (1-\theta) \Big\{ \Psi''_{u^2}(\sigma, u+\theta h, u'+\theta h')h^2 \\ & + 2\Psi''_{uu'}(\sigma, u+\theta h, u'+\theta h, u'+\theta h')hh' \\ & + \Psi''_{u'^2}(\sigma, u+\theta h, u'+\theta h')h,^2 \Big\}\, d\theta, \end{aligned}$$

where $0 \le \theta \le 1$, we have

$$P(u+h) - P(u) = h(s) - \frac{1}{2\pi i}\int_0^{2\pi} \Big\{\Psi'_u(\sigma, u(\sigma), u'(\sigma))h(\sigma) + \Psi'_{u'}(\sigma, u(\sigma), u'(\sigma))h'(\sigma)\Big\} \times \cot\frac{\sigma - s}{2}\, d\sigma + \Omega_1(\sigma, h, h') + \Omega_2(\sigma, h, h') + \Omega_3(\sigma, h, h'),$$

where

$$\Omega_1(\sigma, h, h') = \frac{1}{2\pi i}\int_0^{2\pi}\left\{\int_0^1 (1-\theta)\Psi''_{u^2}(\sigma, u+\theta h, u'+\theta h')h^2\, d\theta\right\} \times \cot\frac{\sigma - s}{2}\, d\sigma,$$

$$\Omega_2(\sigma, h, h') = -\frac{1}{\pi i}\int_0^{2\pi}\left\{\int_0^1 (1-\theta)\Psi''_{uu'}(\sigma, u+\theta h, u'+\theta h')hh'\, d\theta\right\} \times \cot\frac{\sigma - s}{2}\, d\sigma$$

and

$$\Omega_3(\sigma, h, h') = -\frac{1}{2\pi i}\int_0^{2\pi}\left\{\int_0^1 (1-\theta)\Psi''_{u'^2}(\sigma, u+\theta h, u'+\theta h')h'^2\, d\theta\right\} \times \cot\frac{\sigma - s}{2}\, d\sigma.$$

Since

$$\left|\frac{1}{2\pi i}\int_0^{2\pi} \rho(\sigma)\cot\frac{\sigma - s}{2}\, d\sigma\right|_{\phi,m} \le C(m)\,|\rho|_{\psi,m}, \text{ [2]} \tag{4}$$

and

$$|uv|_{\phi,m} \le C(m)\,|u|_{\phi,m}\,|v|_{\phi,m}.$$

Then

$$\left|\Omega_1(\sigma, h, h'\right|_{\phi,m} \le C(m)|h|^2_{\phi,m}\left|\rho_1\right|_{\phi,m},$$

where

$$\rho_1(\sigma) = \int_0^1 (1-\theta)\,\Psi''_{u^2}(\sigma, u+\theta h, u'+\theta h')\, d\theta$$

in $H_{\phi,m}$, $h(\sigma) \in H_{\phi,m}$, and

$$\frac{\left|\Omega_1(\sigma, h, h')\right|_{\phi,m}}{|h|_{\phi,m}} \le C(m)\,|h|_{\phi,m}\left|\rho_1\right|_{\phi,m}.$$

Therefore, we obtain

$$\lim_{|h|_{\phi,m}\to 0} \frac{\left|\Omega_1(\sigma,h,h')\right|_{\phi,m}}{\left|h\right|_{\phi,m}} = 0.$$

Similarly

$$\lim_{|h'|_{\phi,m}\to 0} \frac{\left|\Omega_2(\sigma,h,h')\right|_{\phi,m}}{|h|_{\phi,m}} = 0,$$

and

$$\lim_{|h'|_{\phi,m}\to 0} \frac{\left|\Omega_3(\sigma,h,h')\right|_{\phi,m}}{|h'|_{\phi,m}} = 0,$$

which proves the differentiability of $P(u)$ in the sense of Fréchet and its derivative is given by

$$P'(u)h = h(s) - \frac{1}{2\pi i}\int_0^{2\pi} \Big\{ \Psi'_u\big(\sigma,u(\sigma),u'(\sigma)\big)h(\sigma) + \Psi'_{u'}\big(\sigma,u(\sigma),u'(\sigma)\big)h'(\sigma)\Big\} \cot\frac{\sigma-s}{2}\,d\sigma.$$

Moreover, using condition (2) and inequality (4) we get

$$\left|P'(u) - P'(u_0)\right|_{\phi,m} \le C(m)\left|u-u_0\right|_{\phi,m}.$$

Then Lemma (3) is proved. Consider the following linear operator

$$L_0\,h = h(s) - \frac{a}{2\pi i}\int_0^{2\pi} h(\sigma)\cot\frac{\sigma-s}{2}\,d\sigma - \frac{b}{2\pi i}\int_0^{2\pi} h'(\sigma)\cot\frac{\sigma-s}{2}\,d\sigma, \tag{5}$$

where a and b are nonzero real constants.

To find the operator L_0^{-1}, we investigate the solvability of the equation

$$L_0 h = g(s) \tag{6}$$

in the space $H_{\phi,m}$ for an arbitrary $g(s) \in H_{\phi,m}$.

By introducing the new variables $\tau = e^{i\sigma}$ and $t = e^{is}$, equation (6) is transformed into the form

$$y(t) - \frac{a}{\pi i}\int_L iy(\tau)\left[\frac{1}{\tau-t} - \frac{1}{2\tau}\right]d\tau - \frac{b}{\pi i}\int_L iy'(\tau)\left[\frac{1}{\tau-t} - \frac{1}{2\tau}\right]d\tau = G(t) \tag{7}$$

where L-unit circle, $h(s) = y(e^{is})$ and $g(s) = G(e^{is})$.

Now we introduce the following piecewise holomorphic function of variable z running in the complex plane.

$$Y(z) = \frac{1}{2\pi i}\int_L iy(\tau)\left[\frac{1}{\tau - z} - \frac{1}{2\tau}\right] d\tau$$
$$= \frac{1}{2\pi i}\int_L \frac{iy(\tau)}{\tau - z}\, d\tau - C, \tag{8}$$

where

$$C = \frac{1}{4\pi}\int_L \frac{y(\tau)}{\tau}\, d\tau.$$

By differentiating the function y we get

$$Y'(z) = \frac{1}{2\pi i}\int_L \frac{iy'(\tau)}{\tau - z}\, d\tau = \frac{1}{2\pi i}\int_L iy'(\tau)\left[\frac{1}{\tau - z} - \frac{1}{2\tau}\right] d\tau \tag{9}$$

where

$$-\frac{1}{4\pi i}\int_L iy'(\tau)\frac{d\tau}{\tau} = \frac{-i}{4\pi}\int_0^{2\pi} y'(\sigma)\, d\sigma = 0.$$

At $z \to \infty$ we get

$$Y(\infty) = \frac{-i}{4\pi}\int_0^{2\pi} h(\sigma)\, d\sigma, \tag{10}$$

and

$$Y'(\infty) = 0. \tag{11}$$

Hence, according to the Sokhotskii-plemelj formulae [1], we have

$$Y^{+}(t) - Y^{-}(t) = iy(t), \tag{12}$$

$$Y^{+(k)}(t) - Y^{-(k)}(t) = \frac{1}{\pi i}\int_L \frac{iy^{(k)}(\tau)}{\tau - t}\, d\tau - 2C,\ (k = 0, 1) \tag{13}$$

Substituting from (12) and (13) into (7) we establish the generalized Riemann boundary value problem:

$$\left[Y^{+\prime}(t) + \frac{a+i}{b}Y^{+}(t)\right] + \left[Y^{-\prime}(t) + \frac{a-i}{b}Y^{-}(t)\right] = -\frac{G(t)}{b}. \tag{14}$$

Let

$$A(z) = \begin{cases} Y'(z) + \dfrac{a+i}{b}Y(z), & \text{for } z \in D^{+} \\ Y'(z) + \dfrac{a-i}{B}Y(z), & \text{for } z \in D^{-}, \end{cases} \tag{15}$$

where D^+ and D^- are, respectively, the interior and exterior of unit circle in the complex plane.

According to the limitting values of $Q(z)$ given in (15) we obtain

$$Q^+(t) = Y^{+'}(t) + \frac{a+i}{b} Y^+(t), \tag{16}$$

$$Q^-(t) = Y^{-'}(t) + \frac{a-i}{b} Y^-(t). \tag{17}$$

From (14), (16) and (17) we have

$$Q^+(t) + Q^-(t) = -\frac{G(t)}{b}. \tag{18}$$

It is obviously, that the index of Riemann problem (18) is zero. From the theory of linear singular integral equations [1], the solution of problem (18) is given by

$$Q(z) = -\frac{X(z)}{2ib\pi} \int_L \frac{G(\tau)}{(\tau - z) X^+(\tau)} d\tau + CX(z) \tag{19}$$

where C is an arbitrary constant, which can be written as $Q(\infty) = C$.

From (10), (11) and (15) we get

$$Q(\infty) = \frac{a-i}{b} Y(\infty),$$

$$X(z) = e^{\Gamma(z)}$$

and

$$\Gamma(z) = \int_L \frac{d\tau}{\tau - z} = \begin{cases} \pi i & z \in D \\ 0 & z \in D^-. \end{cases}$$

Thus

$$X^+(t) = -1 \qquad \text{and} \qquad X^-(t) = 1.$$

From (19), according to the limitting values of $Q(z)$ we have

$$Q^+(s) - Q^-(t) = -\frac{1}{b\pi i} \int_L \frac{G(\tau)}{\tau - t} d\tau - 2C,$$

therefore

$$Q^+(s) - Q^-(s) = \frac{1}{2b\pi} \int_0^{2\pi} g(\sigma) \cot \frac{\sigma - s}{2} d\sigma - \frac{1}{2b\pi} \int_0^{2\pi} g(\sigma) d\sigma - 2\frac{a-i}{2} Y(\infty).$$

Using (10) we get

$$Q^+(s) - Q^-(s) = \frac{i}{2b\pi}\int_0^{2\pi} g(\sigma)\cot\frac{\sigma - s}{2}\,d\sigma + \frac{ia}{2b\pi}\int_0^{2\pi} g(\sigma)d\sigma$$
$$= \frac{i}{b}[\zeta + ag_0], \tag{20}$$

where

$$\zeta = \frac{1}{2\pi}\int_0^{2\pi} g(\sigma)\cot\frac{\sigma - s}{2}\,d\sigma,$$

$$g_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\sigma)d\sigma.$$

Also

$$Q^+(s) + Q^-(s) = -\frac{g(\sigma)}{b} \tag{21}$$

The equations (16) and (17) can be rewritten in terms of the real variable s as follows.

$$Y'^+(s) + \frac{a+i}{b}Y^+(s) = Q^+(s), \tag{22}$$

$$Y'^-(s) + \frac{a-i}{b}Y^-(s) = Q^-(s). \tag{23}$$

We shall find the periodic solution (with period 2π) of differential equations (22) and (23). Assuming that the right hand side is periodic known function (with period 2π). Then the solution has the form

$$Y^+(s) = Y^+(0)\exp\left(-\frac{a+i}{b}s\right) + \int_0^s \exp\left[\frac{a+i}{b}(\sigma - s)\right]Q^+(\sigma)d\sigma, \tag{24}$$

$$Y^-(s) = Y^-(0)\exp\left(-\frac{a-i}{b}s\right) + \int_0^s \exp\left[\frac{a-i}{b}(\sigma - s)\right]Q^-(\sigma)d\sigma, \tag{25}$$

where

$$Y^+(0) = \frac{\int_0^{2\pi}\exp((a+i/b)\sigma)Q^+(\sigma)d\sigma}{1 - \exp(-(a+i/b)2\pi)},$$

r

$$Y^-(0) = \frac{\int_0^{2\pi}\exp(-(a-i/b)\sigma)Q^-(\sigma)d\sigma}{1 - \exp(-(a-i/b)2\pi)}.(26)$$

From (24), (25) and by virtue of (12) we have

$$
\begin{aligned}
ih(s) = {} & \left[Y^{+}(0) - Y^{-}(0)\right] e^{-(a/b)s} \cos(s/b) \\
& - i\left[Y^{+}(0) + Y^{-}(0)\right] e^{-(a/b)s} \sin(s/b) \\
& + \int_0^s e^{(a/b)(\sigma - s)} \cos\left(\frac{\sigma - s}{b}\right) \left[Q^{+}(\sigma) - Q^{-}(\sigma)\right] d\sigma \\
& + i\int_0^s e^{(a/b)(\sigma - s)} \sin\left(\frac{\sigma - s}{b}\right) \left[Q^{+}(\sigma) + Q^{-}(\sigma)\right] d\sigma. \qquad (27)
\end{aligned}
$$

From (26) we have

$$
\begin{aligned}
Y^{+}(0) + Y^{-}(0) = {} & \frac{1}{2e^{-(a/b)2\pi}[\cos h(2\pi(a/b)) - \cos(2\pi/b)]} \\
& \times \left\{ \int_0^{2\pi} \left[e^{(a/b)\sigma} \cos\left(\frac{\sigma}{b}\right) - e^{(a/b)(\sigma - 2\pi)} \cos\left(\frac{\sigma + 2\pi}{b}\right)\right] \right. \\
& \times \left[Q^{+}(\sigma) + Q^{-}(\sigma)\right] d\sigma \\
& + i\int_0^{2\pi} \left[e^{(a/b)\sigma} \sin\left(\frac{\sigma}{b}\right) - e^{(a/b)(\sigma - 2\pi)} \sin\left(\frac{\sigma + 2\pi}{b}\right)\right] \\
& \left. \times \left[Q^{+}(\sigma) - Q^{-}(\sigma)\right] d\sigma \right\} \qquad (28)
\end{aligned}
$$

and

$$
\begin{aligned}
Y^{+}(0) - Y^{-}(0) = {} & \frac{1}{2e^{-(a/b^{b})2\pi}[\cos h(2\pi a/b) - \cos(2\pi/b)]} \\
& \times \left\{ \int_0^{2\pi} \left[e^{(a/b)\sigma} \cos\left(\frac{\sigma}{b}\right) - e^{(a/b)(\sigma - 2\pi)} \cos\left(\frac{\sigma + 2\pi}{b}\right)\right] \right. \\
& \times \left[Q^{+}(\sigma) - Q^{-}(\sigma)\right] d\sigma \\
& + i\int_0^{2\pi} \left[e^{(a/b)\sigma} \sin\left(\frac{\sigma}{b}\right) - e^{(a/b)(\sigma - 2\pi)} \sin\left(\frac{\sigma + 2\pi}{b}\right)\right] \\
& \left. \times \left[Q^{+}(\sigma) + Q^{-}(\sigma)\right] d\sigma \right\}. \qquad (29)
\end{aligned}
$$

Substituting from (20) and (21) into (27), (28) and (29), finally we get

$$
h(s) = \frac{e^{(a/b)(2\pi - s)} \cos(s/b)}{2b[\cos h(2\pi a/b) - \cos(2\pi/b)]}
$$

$$
\begin{aligned}
&\times \left\{ \int_0^{2\pi} \left[e^{(a/b)\sigma} \cos\left(\frac{\sigma}{b}\right) - e^{(a/b)(\sigma-2\pi)} \cos\left(\frac{\sigma+2\pi}{b}\right)\right] (\zeta + a g_0) d\sigma \right. \\
&\left. - \int_0^{2\pi} \left[e^{(a/b)\sigma} \sin\left(\frac{\sigma}{b}\right) - e^{(a/b)(\sigma-2\pi)} \sin\left(\frac{\sigma+2\pi}{b}\right)\right] g(\sigma) d\sigma \right\} \\
&+ \frac{e^{(a/b)(2\pi-s)} \sin(s/b)}{2b\left[\cos h(2\pi a/b) - \cos(2\pi/b)\right]} \\
&\times \left\{ \int_0^{2\pi} \left[e^{(a/b)\sigma} \cos\left(\frac{\sigma}{b}\right) - e^{(a/b)(\sigma 2\pi)} \cos\left(\frac{\sigma+2\pi}{b}\right)\right] g(\sigma) d\sigma \right. \\
&\left. + \int_0^{2\pi} \left[e^{(a/b)\sigma} \sin\left(\frac{\sigma}{b}\right) - e^{(a/b)(\sigma-2\pi)} \sin\left(\frac{\sigma+2\pi}{b}\right)\right] (\zeta + a g_0) d\sigma \right\} \\
&+ \frac{1}{b} \int_0^{s} e^{(a/b)(\sigma-s)} \cos\left(\frac{\sigma-s}{b}\right) (\zeta + a g_0) d\sigma \\
&- \frac{1}{b} \int_0^{s} e^{(a/b)(\sigma-s)} \sin\left(\frac{\sigma-s}{b}\right) g(\sigma) d\sigma \\
= {} & L_0^{-1} g(s)
\end{aligned}
\tag{30}
$$

From (30) it is easy to see that

$$
\left|L_0^{-1}\right|_{\phi,m} \leq \text{const.} \overset{\text{def.}}{\equiv} \alpha_0 \tag{31}
$$

which is the proof of the following Lemma:

Lemma (4)

If conditions of Lemma (3) are satisfied, then the linear operator (5), effects from $H_{\phi,m}$ into $H_{\phi,m}$, has a bounded inverse L_0^{-1} satisfying (31).

Therefore, all conditions of applicability and convergence of modified Newton's method are satisfied.

Hence, the following Theorem is valid.

Theorem

If the conditions of Lemma (3) are satisfied and $u_0 \in H_{\phi,m}$ is the initial approximation for the equation (1), and if

$$
\left|L_0^{-1} P(u_0)\right|_{\phi,m} \leq \zeta_0;
$$

$$C(m)\alpha_0\zeta_0 < 1/2,$$

then equation (1) has a unique solution u^* *in the sphere* $|u - u_0|_{\phi,m} < r$ $(r = 2\zeta_0)$, *to which the successive approximations*

$$u_{n+1} = u_n - L_0^{-1}P(u_0), \qquad (n = 0, 1, 2, \ldots)$$

of the modified Newton's method converges with rate of convergence given by the inequality

$$|u_n - u^*| \leq \frac{\gamma^n}{1-\gamma}\zeta_0;$$

$$\gamma = 1 - \sqrt{1 - 2C(m)\alpha_0\zeta_0}\,.$$

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