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Norm fragmented weak* compact sets

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Abstract

A Banach space which is a Čech-analytic space in its weak topology has fourteen measure-theoretic, geometric and topological properties. In a dual Banach space with its weak-star topology essentially the same properties are all equivalent one to another.

1. Introduction

The development of the theory of properties like the Radon-Nikodým property has been complex and many mathematicians have made major contributions. Here we only highlight some of the developments that are most relevant to our work.

A Banach space X is said to have the *Radon-Nikodým property* if for each probability space $(\Omega, \mathcal{F}, \mu)$ and for each countably additive vector valued measure $\xi : \mathcal{F} \to X$ that is absolutely continuous with respect to μ and of bounded variation, there is a function $x : \Omega \to X$ that is Bochner-integrable with respect to μ and such that

$$\xi(E) = \int_E x \, d\mu$$

for all E in $\mathcal{F}$.

In 1967, Rieffel [29] introduced the concept of a dentable set in a Banach space and showed that a Banach space has the Radon-Nikodým property, if each of its bounded sets is dentable. Rather later, a series of papers by Maynard [24, 25], Davis and Phelps [3] and Huff [18] established the converse that, if a Banach space has the Radon-Nikodým property, then each of its bounded subsets is dentable.

We turn to consider a dual Banach space X^*. Below we list a number of conditions known to be equivalent to the condition that X^* has the Radon-Nikodým property. The equivalence of (α) with (β) is contained in the work of Rieffel, Maynard, Davis and Phelps, and Huff, referred to above. Schwartz [30] gives an explicit proof of the equivalence of (α) and (γ). Stegall proves the equivalence of (α) and (δ) in [33, Proposition 1.10]. The equivalence of (α) and (ε) follows from work of Stegall [32] and of van Dulst and Namioka [5]. Edgar [8, Theorem 1.5 and Proposition 1.7] using results of Stegall [32], proves the equivalence of (α) and (η).

Let X^* be the dual of a Banach space X. Then the following conditions (α) to (η) are equivalent in that each implies the others.

(α) *X^* has the Radon-Nikodým property.*

(β) *Each closed bounded convex subset of X^* is dentable.*

(γ) *Each Radon measure on the weak-star Borel subsets of X^* is supported by a countable union of norm compact sets.*

(δ) *For all weak-star compact convex sets K in X^*, for all separable Banach spaces Y, and for all bounded linear maps $T : Y \to X$, the image $T^*(K)$ of K under the dual map is norm separable in Y^*.*

(ε) *X^* contains no ε-tree.*

(η) *The subsets of X^* that are measurable for all Radon measures on the weak-star Borel subsets of X^* coincide with the subsets of X^* that are measurable for all Radon measures on the norm Borel subsets of X^*.*

A closed bounded convex set K in a Banach space X is said to have the Radon-Nikodým property (see, for example, [5, p. 15], or [4]), if for each probability space $(\Omega, \mathcal{F}, \mu)$ and for each countably additive vector-valued measure $\xi : \mathcal{F} \to X$ that is absolutely continuous with respect to μ with $\xi(E)/\mu(E) \in K$ for all E in $\mathcal{F}$ with $\mu(E) > 0$, there is a function $x : \Omega \to K$ that is Bochner-integrable with respect to μ and such that

$$\xi(E) = \int_E x \, d\mu$$

for all E in $\mathcal{F}$.

As in the Banach space case, the conditions (α) and (β) remain equivalent when X^* is replaced by any closed bounded convex set K in any Banach space. In [33, Proposition 1.10], Stegall proves that a weak-star compact convex set K in the dual Banach space X^* has the Radon-Nikodým property, if, and only if, the condition (δ) holds for the particular set K, rather than for all such sets in X^*.

More recently Radon-Nikodým compact spaces have been studied by Reynov [31] and Namioka [27, 28]; see [27, 28] where more information about their origin may be found. A compact Hausdorff space is said to be *Radon-Nikodým compact*, or *RN compact*, if it is homeomorphic to a weak-star compact set in a dual Banach space that has the Radon-Nikodým property. To explain some of this work we need several definitions.

A metric ρ on a topological space K is *lower semi-continuous*, if ρ is lower semi-continuous as a real-valued function on $K \times K$, *i.e.*, if the set

$$\{(x,y) : \rho(x,y) > t\}$$

is open in $K \times K$ for each real $t \geq 0$. For example, if K is a dual Banach space with its weak-star topology, the norm metric on K is lower semi-continuous.

Let ρ be a metric on a topological space K. The space K is said to be *fragmented by the metric* ρ, if, for each $\varepsilon > 0$, and for each non-empty subset H of K, there is a non-empty relatively open subset of H that has ρ-diameter less than ε. This concept, introduced in [20], is modelled on Rieffel's concept of a "dentable" set. When K is ρ-complete, it is easy to verify that K is fragmented by ρ, if, and only if, K has the *Point of Continuity Property* [20], that is, for each non-empty closed subset H of K, the identity map from H as a subset of K to H with its ρ-topology has a point of continuity. For a discussion of the elementary relationships of these and other similar concepts see [10] and [20]. Note that when we use topological notions without any qualifications we are always referring to the original topology on K (or later on Z); when we use the topology corresponding to the metric ρ we will always qualify the topological terms, and talk, for example, of a ρ-compact set or a ρ-open set.

The following characterizations of *RN* compact spaces are due to Namioka [27, 28], using results of Stegall [33].

Let K be a compact Hausdorff space. The following conditions are equivalent.

(1) K is RN compact.

(2) K is homeomorphic to a norm-fragmented weak-star compact subset of a dual Banach space.

(3) K is fragmented by some lower semi-continuous metric on K.

At a seminar in Paris in the spring of 1986, Namioka outlined a proof of the equivalence of (2) and (3). At the end of the talk, Nassif Ghoussoub mentioned the possibility of using the ideas of Ghoussoub-Maurey in [16, Theorem VII.1] to prove this equivalence.

Theorem (Ghoussoub-Maurey [16, Theorem VII.1])

Let K be a compact metric space with metric d, and let d' be another bounded complete metric on K such that d' is lower semi-continuous on $K \times K$. There exists a Banach space Y and a map $\delta : K \to Y^*$ such that

(i) δ is a homeomorphism of K onto $\delta(K)$ with the relative weak-star topology on Y^*;

(ii) There exists $\alpha > 0$ such that for any $(x, y) \in K \times K$ we have

$$\|\delta(x) - \delta(y)\| \leq d'(x, y) \leq \alpha \|\delta(x) - \delta(y)\|.$$

The space Y is the space of all continuous real-valued functions on K which are d'-Lipschitz with the norm

$$\|f\| = \|f\|_{\mathrm{Lip}\, d'} + \|f\|_\infty$$

where $\|f\|_{\mathrm{Lip}\, d'}$ is the least constant C such that $|f(x) - f(y)| \leq Cd'(x, y)$ for all $x, y \in K$, and $\|f\|_\infty = \sup_{x \in K} |f(x)|$.

Our version, which follows their ideas, removes unnecessary hypotheses and slightly improves their theorem in that by changing the norm on Y we are able to obtain equality between the metric distance of two points and the norm distance of their images, rather than inequalities. We give full details of the proof here.

Theorem 2.1 (Ghoussoub-Maurey)

Let Z be a compact Hausdorff space and let ρ be a bounded lower semi-continuous metric on Z. Then there is a dual Banach space X^* and a homeomorphism φ mapping Z onto a subset of X^* taken with its weak-star topology, with

$$\|\varphi(\zeta) - \varphi(\zeta')\| = \rho(\zeta, \zeta')$$

for all ζ, ζ' in Z.

Our next objective is to obtain a series of equivalences like the equivalences of the conditions (β) to (η) that hold in much more general circumstances. We need to introduce a modified form of fragmentability [19]. Let ρ be a lower semi-continuous metric on a Hausdorff space Z. We say that Z is *σ-fragmented by the metric ρ*, if, for each $\varepsilon > 0$, it is possible to express Z as a countable union of sets, each with the property that each non-empty subset has a non-empty relatively open subset of ρ-diameter less than ε. We shall also need an apparently much weaker (although equivalent [19]) concept. Although the terminology may not seem very apt, we say that Z is *σ-separable with respect to ρ*, if, for each $\varepsilon > 0$, it is possible to express Z as a countable union of sets, each with the property that each non-empty subset has a non-empty relatively open subset that can be covered by a countable family of sets each of ρ-diameter less than ε.

Since we are to prove the equivalence of a large number of apparently very different conditions we need to explain many concepts, some rather unfamiliar, some so common that they are used by different authors in different ways.

For us, a *Borel measure* on a topological space Z will be a real-valued function μ, taking only finite non-negative values, defined initially on the Borel sets of Z and then extended to the *μ-measurable* sets of Z, that is to the sets E of Z for which there are Borel sets B_1, B_2 with $B_1 \subset E \subset B_2$ and $\mu(B_1) = \mu(B_2)$. A *Radon measure* on Z will be a Borel measure μ on Z, with the property that, for each μ-measurable subset M of Z and each $\varepsilon > 0$, there is a compact subset H of M with $\mu(H) > \mu(Z) - \varepsilon$. A measure μ is said to be *carried by* a μ-measurable subset M of Z if $\mu(M) = \mu(Z)$. A closed set S is said to be the *support* of the Borel measure μ if S is minimal among the closed sets that carry μ.

We shall say that a set in Z is an *ε-tree for the metric ρ*, if it is a countable set $\{x(s)\}$ of points of Z, indexed by the finite sequences s of 0's and 1's, and for each $l \geq 0$ and each sequence s of length l the sets

$$\mathrm{cl}\{x(t) : t|l+1 = s, 0 \text{ and has length } \geq l+1\}$$

and

$$\mathrm{cl}\{x(t) : t|l+1 = s, 1 \text{ and has length } \geq l+1\}$$

are separated by ρ-distance ε, where "cl" denotes topological closure. Here we allow s to be the empty sequence of length zero. Note that this concept of an ε-tree differs significantly from the concept, used in the condition (ε) above, a concept that only exists in a normed linear space.

A set B in the space Z is said to have the *Baire property* if, for some open set G

$$(G \backslash B) \cup (B \backslash G)$$

is a countable union of nowhere dense sets.

If $\mathcal{A}$ is a family of sets in a topological space Z, the *Souslin-$\mathcal{A}$* sets are the sets of the form

$$\bigcup\left\{\bigcap\{A(\sigma|n) : n \geq 1\} : \sigma \in \mathbb{N}^{\mathbb{N}}\right\},$$

with each set $A(\sigma|n)$ in $\mathcal{A}$, using $\sigma|n$ to denote

$$\sigma_1, \sigma_2, \ldots, \sigma_n,$$

when

$$\sigma = \sigma_1, \sigma_2, \sigma_3 \ldots$$

belongs to $\mathbb{N}^{\mathbb{N}}$, with $\mathbb{N} = \{1, 2, 3, \ldots\}$. We use $\mathcal{F}$ and $\mathcal{G}$ to denote the families of closed sets in Z and of open sets in Z. The family of Souslin-$(\mathcal{F} \cup \mathcal{G})$ sets includes the family $\mathcal{B}$ of all Borel sets in Z and indeed coincides with the family of Souslin-$\mathcal{B}$ sets.

Following Fremlin [14], we say that a space Z is *Čech-analytic* if it is a Souslin-$(\mathcal{F} \cup \mathcal{G})$ set in some compact Hausdorff space (see also [19]). In particular, a *Čech-complete space*, that is, a $\mathcal{G}_\delta$-set in a compact Hausdorff space, is Čech-analytic. Further, if Z is a dual Banach space with its weak-star topology, any Souslin-$(\mathcal{F} \cup \mathcal{G})$ set in Z is Čech-analytic, since Z is a countable union of compact sets in its Stone-Čech compactification.

These definitions enable us to state the main theorem.

Theorem 4.1

Let ρ be a lower semi-continuous metric on a Hausdorff space Z. Then the implications (a) $\Longleftrightarrow$ (b) and (b) $\Longrightarrow$ (c) and the equivalence of each of (c) to (n), hold among the following conditions. Further, if Z is Čech-analytic, all the conditions are equivalent.

(a) *Z is σ-fragmented by ρ.*
(b) *Z is σ-separable with respect to ρ.*
(c) *Each compact subset of Z is fragmented by ρ.*
(d) *For each Radon measure μ on Z with $\mu(Z) > 0$, and for each $\delta > 0$, there is a compact subset H of Z with $\mu(H) > 0$ and ρ-diam $H < \delta$.*
(e) *For each Radon measure μ on Z with $\mu(Z) > 0$, there is a ρ-compact subset C of Z with $\mu(C) > 0$.*
(f) *For each Radon measure μ on Z and each $\varepsilon > 0$, there is a ρ-compact subset C of Z with $\mu(C) > \mu(Z) - \varepsilon$.*

(g) *The Radon measures on Z coincide with those Radon measures on Z taken with its ρ-topology that are carried by σ-compact sets of Z taken with its original topology.*

(h) *The subsets of Z that are measurable for every Radon measure on Z coincide with the subsets of Z that are measurable for those Radon measures on Z taken with its ρ-topology that are carried by σ-compact sets of Z taken with its original topology.*

(i) *Each ρ-closed subset of Z is measurable with respect to each Radon measure on Z.*

(j) *For no $\varepsilon > 0$ does Z contain a compact set H that admits a continuous map p onto the Cantor set $2^{\mathbb{N}}$ with the inverse images of distinct points of $2^{\mathbb{N}}$ separately by ρ-distance ε.*

(k) *For no $\varepsilon > 0$ does Z contain a relatively compact ε-tree for ρ.*

(l) *For each compact set K in Z, the points of continuity of the identity map from K to K with its ρ-topology are dense in K.*

(m) *For each compact set K in Z, the topology of K coincides with the ρ-topology of K on some dense $\mathcal{G}_\delta$ set of K.*

(n) *For each compact set K in Z, each ρ-Borel set of K has the Baire property in K.*

It is natural to enquire whether or not a set that is σ-fragmented by a lower semi-continuous metric ρ can always be expressed as a countable union of sets that are fragmented by ρ. In [19], we give two examples, one an $\mathcal{F}_{\sigma\delta}$-set in a compact metric space, the other the unit ball of c_0 taken with its weak topology, showing that such a decomposition is not always possible.

In connection with conditions (e) and (f) we should remark that a ρ-compact set C is necessarily closed and so is μ-measurable. Conditions (g) and (h) derive from work of Edgar [8, 9] and condition (n) derives from work of Talagrand [38, Proposition 10].

We do not require the phrase 'that are carried by σ-compact sets of Z taken with its original topology' in parts (g) and (h) of Theorem 4.1 when the ρ-topology is finer than the original topology, which is the case when the original topology is a weak or weak-star topology on a Banach space and ρ is the norm metric, since then ρ-compact sets are compact in the original topology. This is not the case in general, for if I denotes the closed unit interval with the discrete topology and $\rho(x, y) = |x-y|$ is the usual metric on the interval, then I is ρ-compact without being compact in the original discrete topology, ρ fragments I, and I is Čech-analytic.

In considering the implication that $(c) \Longrightarrow (a)$, there seems to be little scope for removing or weakening the hypothesis that Z is Čech-analytic. In [19], we show that it is consistent with the usual axioms of set theory (ZFC) that there is a co-analytic subset of $2^{\mathbb{N}}$, with each compact subset fragmented by the discrete metric, but not itself σ-fragmented by this metric. Again, in [19], we show that although each compact subset of the Banach space l^∞, taken with its weak topology, is fragmented by the norm metric, the space itself is not σ-fragmented by the norm metric. A further example in [19] shows that there is a Banach space such that the unit ball with its weak topology is not σ-fragmented by any lower semi-continuous metric.

Let ρ and ρ' be two lower semi-continuous metrics on Z giving the same metric topology. It is by no means obvious from the definitions that the properties of being σ-fragmented by ρ or of being σ-separable with respect to ρ are equivalent to the corresponding properties for ρ'; however, this is the case when Z is a Čech analytic space, since the condition (f) is stated in terms of the topology determined by ρ.

We note that Theorem 4.1 gives information about the subsets of Banach spaces taken with their weak topologies. This is because the norm is weakly lower semi-continuous, and because the weakly compact sets of a Banach space are fragmented by the norm. This latter fact canbe deduced from Troyanski's renorming work [42] (see [27] for a more elementary proof). Consequently, we have

Corollary 4.2

Let Z be a set in a Banach space taken with its weak topology. Take

$$\rho(\zeta, \zeta') = \|\zeta - \zeta'\|$$

for all ζ, ζ' in Z. Then Z and ρ also satisfy the conditions (c) to (n) of Theorem 4.1. If, in addition, Z is Čech-analytic, then Z and ρ also satisfy the conditions (a) and (b) of Theorem 4.1.

Many Banach spaces taken with their weak topologies are Čech-analytic, or at least σ-fragmented. Before describing these we recall some terminology. A Banach space is said to admit a *Kadec norm* if it has an equivalent norm $\|\cdot\|$ such that the weak and the norm topologies coincide on the set $\{x : \|x\| = 1\}$. A norm $\|\cdot\|$ is said to be *locally uniformly convex* if for every sequence $x_0, x_1, x_2, \ldots$ of points with norm one for which $|x_n + x_0|/2 \longrightarrow 1$ we have $|x_n - x_0| \longrightarrow 0$. A Banach space is said to be *weakly compactly generated* if it is the norm closure of the linear span of one of its weakly compact sets. A space Z is said to be *K-analytic* (respectively, *K-countably determined*) if it is a Hausdorff space and has a representation in the form

$$Z = \bigcup \{K(\sigma) : \sigma \in A\}$$

where $K(\sigma)$ is compact for each σ in $A = \mathbb{N}^{\mathbb{N}}$ (respectively, A is a subset of $\mathbb{N}^{\mathbb{N}}$) and K is *upper semi-continuous*, in that, the set

$$\{\sigma : \sigma \in A \text{ and } K(\sigma) \subset G\}$$

is open in A for each open set G in Z. A Banach space E is said to be K-analytic (respectively, K-countably determined) if E taken with its weak topology has this property.

We remark that a Banach space E taken with its weak topology and its norm metric ρ is σ-fragmented by ρ, and so E satisfies conditions (a) to (n) of Theorem 4.1, in the following two cases.

(i) E has the Point of Continuity (PC) Property. From the dentability characterization of Radon-Nikodým Property it follows immediately that such a space has the (PC) Property, and, indeed, it is easy to verify that the (PC) Property is equivalent to the unit ball in E, taken with its weak topology, being fragmented by the norm [20, pp. 54–55]. Bourgain and Rosenthal [2] have given an example of a Banach space with the (PC) Property, which fails to have the Radon-Nikodým Property.

(ii) E is Čech-analytic. This is the case if E admits a Kadec norm, and then, as W. Schachermayer has noted, E is even a Borel subset of its second dual E^{**}, and so is a Borel subset of its Stone-Čech compactification (see [8, p. 676] and [9, Cor. 2.3]). It is also the case that every weakly compactly generated Banach space is K-analytic [36] (see also [19, p. 51]), and it is immediate that every K-analytic space is K-countably determined. Now L. Vašák [43, Cor. 3(b)] has shown, among other things, that every K-countably determined Banach space admits an equivalent locally uniformly convex norm. It is not difficult to see that a locally uniformly convex norm is a Kadec norm. Thus not only all K-analytic Banach spaces, but also all K-countably determined Banach spaces are Čech-analytic. Consequently any Banach space which is K-countably determined, but not K-analytic, is an example of a Banach space which is Čech-analytic but not K-analytic; Talagrand [4] has given such an example. M. Fabian and G. Godefroy [12] have recently shown that every dual Banach space with the Radon-Nikodým Property has an equivalent locally uniformly convex norm; and so such dual Banach spaces are Čech-analytic.

As we have already mentioned, the space ℓ^∞, taken with its weak topology, is not σ-fragmented by the norm metric, and so, by Corollary 4.2, ℓ^∞ is not Čech-analytic in its weak topology. Thus ℓ^∞ is not a Souslin-$(\mathcal{F} \cup \mathcal{G})$ set in its second dual with its weak-star topology. This improves on a result of Talagrand [37] who shows that ℓ^∞ is not a Borel set in its second dual with its weak-star topology. Note, however, that A. Tortrat has shown that, when a Banach space, taken with its weak topology, is regarded as a subset of its second dual, with its weak-star topology, it is measurable there for all Radon measures [31, 40].

2. Compact spaces that admit lower semi-continuous metrics

In this section we consider a compact Hausdorff space Z that admits a bounded lower semi-continuous metric ρ. After proving some simple lemmas we use ideas from a paper by Ghoussoub and Maurey [16] to show that it is possible to embed Z in a dual Banach space with its weak-star topology and ensure that ρ coincides with the norm distance on the Banach space.

Lemma 2.1

Let ρ be a lower semi-continuous metric on a Hausdorff space Z. Then each compact subset of Z is ρ-closed and ρ-complete in the ρ-metric. Further, each ρ-compact set in Z is closed.

We give an example to show that closed sets in Z are not necessarily ρ-closed. Take Z to be the set of numbers $\{0, 1/2, 1/3, \ldots\}$ with the discrete topology. Take ρ to be the modulus of the differences between the numbers. Then ρ is a continuous metric on Z. The set $\{1/2, 1/3, \ldots\}$ is closed in Z but is not ρ-closed.

Proof of Lemma 2.1. The first statement is proved in [19, Lemma 3.3].

Now suppose that C is ρ-compact in Z. Let e be a point of Z not in C. For each c in C write

$$V_c = \left\{ z : \rho(z,c) < \frac{1}{2}\rho(e,c) \right\},$$

$$U_c = \left\{ z : \rho(z,c) > \frac{1}{2}\rho(e,c) \right\}.$$

Then V_c is a ρ-open neighbourhood of c, and, since ρ is lower semi-continuous, U_c is an open neighbourhood of e. Further $U_c \cap V_c = \emptyset$. Since C is ρ-compact it is covered by a finite family of the sets V_c, $c \in C$. Now the intersection of the corresponding finite family of the sets U_c is an open neighbourhood of e that does not meet C. Hence C is closed. □

If A is any set in Z and $\varepsilon > 0$, we use

$$B(A,\varepsilon) = \{z \in A : \inf\{\rho(z,a) : a \in A\} \le \varepsilon\}$$

to denote the ε-neighbourhood of the set A.

Lemma 2.2

Let Z be a compact Hausdorff space and let ρ be a lower semi-continuous metric on Z. If A is a non-empty closed set in Z and $\varepsilon > 0$, then the ε-neighbourhood $B(A, \varepsilon)$ is closed in Z.

Proof. Consider any point e not in $B(A, \varepsilon)$. Then we can choose η with

$$\inf \{\rho(e, a) : a \in A\} > \eta > \varepsilon.$$

By the lower semi-continuity of ρ, for each point a of A, we can choose open sets $U(a)$, $V(a)$ with

$$a \in U(a), \quad e \in V(a),$$

so that

$$\rho(u, v) > \eta \qquad \text{for } u \in U(a),\ v \in V(a).$$

Since A is compact, we can choose points $a_1, a_2, \ldots, a_n$ in A so that

$$A \subset \bigcup_{i=1}^{n} U(a_i).$$

If there were a point v common to the sets

$$V = \bigcap_{i=1}^{n} V(a_i) \qquad \text{and} \qquad B = B\left(\bigcup_{i=1}^{n} U(a_i), \varepsilon\right),$$

we could find a u in

$$U = \bigcup_{i=1}^{n} U(a_i)$$

with $\rho(u, v) < \eta$. For at least one i, $1 \leq i \leq n$ this would contradict the choice of $U(a_i)$ and $V(a_i)$. Thus V is an open set containing e that does not meet the set

$$B\left(\bigcup_{i=1}^{n} U(a_i), \varepsilon\right)$$

that contains $B(A, \varepsilon)$. Hence $B(A, \varepsilon)$ is closed in Z. □

Lemma 2.3

Let Z be a compact Hausdorff space and let ρ be a lower semi-continuous metric on Z. If F_0 and F_1 are non-empty closed subsets of Z, the function ρ attains its infimum on $F_0 \times F_1$.

Proof. Write

$$\delta = \inf \{\rho(z_0, z_1) : z_0 \in F_0,\ z_1 \in F_1\}.$$

Then, for each integer $n \geq 1$, the set

$$K_n = \{(z_0, z_1) : \rho(z_0, z_1) \leq \delta + (1/n)\} \cap (F_0 \times F_1)$$

in a non-empty subset of $Z \times Z$. Since ρ is a lower semi-continuous metric, each set K_n is compact in $Z \times Z$. Hence

$$\bigcap_{n=1}^{\infty} K_n$$

is non-empty and ρ attains its infimum δ at each point of this set. □

The next lemma improves a lemma of Ghoussoub and Maurey [16, Lemma VII.2]. The proof, which is modelled as in [16] on that of Urysohn's Lemma (see, for example, [21, pp. 114–115], is a bit simpler than that given in [16].

Lemma 2.4

Let Z be a compact Hausdorff space and let ρ be a lower semi-continuous metric on Z. If F_0 and F_1 are disjoint non-empty closed subsets of Z, with

$$\rho(z_0, z_1) > 1 \qquad \text{for } z_0 \in F_0 \text{ and } z_1 \in F_1,$$

then there is a continuous function $f : Z \to [0,1]$, taking the value 0 on F_0 and the value 1 on F_1, and satisfying

$$|f(\zeta) - f(\zeta')| \leq \rho(\zeta, \zeta')$$

for all ζ, ζ' in Z.

Proof. Let Q be the set of all the rational numbers in [0,1]. We construct a family $\{U_r : r \in Q\}$ of open sets in Z such that:

(a) $F_0 \subset U_0$ and $U_1 = Z \backslash F_1$; and
(b) if s, t belong to Q and $s < t$, then $\rho(\zeta, \zeta') > |s - t|$, whenever $\zeta \in \mathrm{cl}\, U_s$ and $\zeta' \notin U_t$.

Using Lemmas 2.2 and 2.3, the closed set F_0 does not meet the closed set $B(F_1,1)$. Since Z is normal, there is an open set U_0 in Z with

$$F_0 \subset U_0 \qquad \text{and} \qquad \mathrm{cl}\, U_0) \cap B(F_1,1) = \emptyset.$$

Take $U_1 = Z \backslash F_1$. This ensures that the condition (a) is satisfied. Further, since $\rho(z,z_1) > 1$ whenever $z \in \mathrm{cl}\, U_0$ and $z_1 \in F_1$, it is clear that the condition (b) is satisfied when $s = 0$ and $t = 1$.

Enumerate Q as $r_0 = 0$, $r_1 = 1$, $r_2, \ldots$. Suppose that for some $n \geq 1$, the sets U_{r_i}, $0 \leq i \leq n$, have been chosen so that U_0, U_1 satisfy (a) and so that (b) holds for all choices of s, t from $\{r_0, r_1, \ldots, r_n\}$. We describe how the next set $U_{r_{n+1}}$ is to be chosen. Write $r = r_{n+1}$ and

$$S = \{r_j : 0 \leq j \leq n,\ r_j < r\},$$

$$T = \{r_j : 0 \leq j \leq n,\ r < r_j\}.$$

Note that $0 \in S$ and $1 \in T$. Now, if $s \in S$, $t \in T$ and

$$\zeta \in \mathrm{cl}\, U_s, \qquad \zeta' \in Z \backslash U_t,$$

the condition (b) ensures that

$$\rho(\zeta, \zeta') > |s - t| = (r - s) + (t - r),$$

so that

$$B(\mathrm{cl}\, U_s, r - s) \cap B(Z \backslash U_t, t - r) = \emptyset.$$

Thus the sets

$$\bigcup_{s \in S} B(\mathrm{cl}\, U_s, r - s)$$

$$\bigcup_{t \in T} B(Z \backslash U_t, t - r)$$

are disjoint closed sets. Since Z is normal, there is an open set $U_r = U_{r_{n+1}}$ with

$$\bigcup_{s \in S} B(\mathrm{cl}\, U_s, r - s) \subset U_r,$$

$$(\mathrm{cl}\, U_r) \cap \bigcup_{t \in T} B(Z \backslash U_t, t - r) = \emptyset.$$

These last two conditions are equivalent to the condition (b) when $s \in \{r_i : 0 \leq i \leq n\}$ and $t = r_{n+1}$ and when $s = r_{n+1}$ and $t \in \{r_i : 0 \leq i \leq n\}$. The construction now follows by induction.

We now define a function f on Z by taking f to be 1 on F_1, and

$$f(z) = \inf \{r : z \in U_r \text{ and } r \in Q\},$$

when $z \in Z \backslash F_1 = U_1$. Just as in the proof of Urysohn's Lemma (see, for example, [21, p. 114]), it follows that f is a continuous map from Z to [0,1]. Clearly f takes the values 0 and 1 on F_0 and on F_1. Finally, suppose that ζ, ζ' belong to Z and that $f(\zeta) = a < b = f(\zeta')$. Then for any s, t in Q with $a < s < t < b$, we have $\zeta \in U_s$ and $\zeta' \notin U_t$, so that, by (b), $\rho(\zeta, \zeta') > |s - t|$. Hence

$$|f(\zeta) - f(\zeta')| = b - a \leq \rho(\zeta, \zeta'). \square$$

Theorem 2.1

Let Z be a compact Hausdorff space and let ρ be a bounded lower semi-continuous metric on Z. Then there is a dual Banach space X^* and a homeomorphism φ mapping Z onto a subset of X^* taken with its weak-star topology, with

$$\|\varphi(\zeta) - \varphi(\zeta')\| = \rho(\zeta, \zeta') \tag{2.1}$$

for all ζ, ζ' in Z.

Proof. We suppose, as we may after a change of scale, that 1 is a bound for ρ. Let X be the space of all continuous real-valued functions f on Z that satisfy a uniform Lipschitz condition of order 1 with respect to ρ. Then $\|f\|_{\text{Lip}}$, defined to be the least constant K such that

$$|f(z_1) - f(z_2)| \leq K\rho(z_1, z_2), \qquad \text{for all } z_1, z_2 \in Z,$$

is a semi-norm on X. It is easy to verify that the function p, defined by

$$p(f) = \max \{\|f\|_{\text{Lip}}, \|f\|_\infty\}, \qquad \|f\|_\infty = \sup_{z \in Z} |f(z)|,$$

is a norm on X and that X is a Banach space with this norm. We define a map $\varphi : Z \to X^*$. For each z in Z, let $\varphi(z)$ be the linear functional $\varphi(z) : X \to \mathbb{R}$ in X^* defined by $\varphi(z)(f) = f(z)$ for each f in X. Note that $\varphi(z)$ does belong to X^*, since

$$\|\varphi(z)\| = \sup \{|f(z)| : p(f) \leq 1\} \leq 1.$$

By Lemma 2.4, applied to a rescaled ρ, the space X separates the points of Z. Hence φ maps distinct points of Z to distinct points of X^*. Clearly φ is continuous as a function from Z to X^* with its weak-star topology. Since Z is compact, φ maps Z homeomorphically to $\varphi(Z)$ taken with its weak-star topology in X^*. Now consider distinct points ζ, ζ' of Z. Write $r = \rho(\zeta, \zeta')$. On one hand

$$\begin{aligned}\|\varphi(\zeta) - \varphi(\zeta')\| &= \sup\{|f(\zeta) - f(\zeta')| : f \in X \text{ and } p(f) \leq 1\}\\ &\leq \sup\{|f(\zeta) - f(\zeta')| : f \in X \text{ and } \|f\|_{\mathrm{Lip}} \leq 1\}\\ &\leq \rho(\zeta, \zeta'). \end{aligned} \tag{2.2}$$

On the other hand, for any $\varepsilon > 0$, we can apply Lemma 2.4, with the rescaled lower semi-continuous metric $r^{-1}(1+\varepsilon)\rho$, to construct a continuous function g on Z with $g(\zeta) = 0$, $g(\zeta') = 1$, $0 \leq g \leq 1$, and

$$|g(z_1) - g(z_2)| \leq r^{-1}(1+\varepsilon)\,\rho(z_1, z_2)$$

for all z_1, z_2 in Z. Thus

$$\begin{aligned} p(rg) &= rp(g)\\ &= r\max\{\|g\|_{\mathrm{Lip}}, \|g\|_\infty\}\\ &\leq r\max\{r^{-1}(1+\varepsilon), 1\}\\ &= \max\{1+\varepsilon, r\}\\ &= \max\{1+\varepsilon, \rho(\zeta,\zeta')\}\\ &= 1+\varepsilon. \end{aligned}$$

Hence

$$\begin{aligned} \rho(\zeta,\zeta') &= |rg(\zeta) - rg(\zeta')|\\ &= |\varphi(\zeta)(rg) - \varphi(\zeta')(rg)|\\ &\leq \|\varphi(\zeta) - \varphi(\zeta')\|p(rg)\\ &\leq (1+\varepsilon)\|\varphi(\zeta) - \varphi(\zeta')\|. \end{aligned}$$

On letting ε tend to zero, we obtain

$$\rho(\zeta,\zeta') \leq \|\varphi(\zeta) - \varphi(\zeta')\|. \tag{2.3}$$

Now (2.2) and (2.3) yield (2.1). □

Remark. The equivalence of the conditions (2) and (3), discussed in the Introduction, follows immediately from Theorem 2.1, since the norm distance in a dual Banach space is lower semi-continuous for the weak-star topology.

3. Measure-theoretic arguments

In this section we develop some measure-theoretic arguments and prove some of the implications needed to establish Theorem 4.1. We prove a number of lemmas.

Lemma 3.1

Let Z and W be compact Hausdorff spaces and let p be a continuous map of Z onto W. Then, for any Radon measure μ on W, there is a Radon measure ν on Z with $\nu(p^{-1}(B)) = \mu(B)$ for each Borel subset B of W.

Proof. Let $C(Z)$ be the Banach space of continuous real-valued functions on Z, with the supremum norm. Let $C(Z)^*$ be the Banach space dual of $C(Z)$, regarded as the space of signed measures of the form $\nu_1 - \nu_2$ with ν_1, ν_2 Radon measures on Z. In particular, for each z in Z the Dirac measure δ_z assigning measure 1 to $\{z\}$ and measure 0 to $Z\backslash\{z\}$ belongs to $C(Z)^*$. The map $\delta : z \mapsto \delta_z$ embeds Z topologically in $C(Z)^*$ taken with its weak-star topology. We use similar concepts and notation for W.

Let $T : C(W) \to C(Z)$ be continuous linear map defined by $T(f) = f \circ p$ for each f in $C(W)$. Let $T^* : C(Z)^* \to C(W)^*$ be the adjoint map with $T^*(\nu) = \mu$ characterized, for each ν in $C(Z)^*$, by the formula

$$\int_Z f \circ p \, d\nu = \int_W f \, d\mu, \qquad \text{for all } f \in C(W).$$

These relations are summarized in the following diagram

$$\begin{array}{lccccccc} p: & Z & \longrightarrow & W: & z & \longmapsto & p(z), \\ T: & C(W) & \longrightarrow & C(Z): & f & \longmapsto & f \circ p, \\ T^*: & C(Z)^* & \longrightarrow & C(W)^*: & \nu & \longmapsto & \mu. \end{array}$$

Here T^* is weak-star to weak-star continuous and

$$T^*(\delta_z) = \delta_{p(z)}$$

for each z in Z.

Let $M_1^+(Z)$ be the space of all non-negative measures ν with $\nu(Z) = 1$ in $C(Z)^*$, and let $M_1^+(W)$ be the corresponding subset of $C(W)^*$. Then $M_1^+(Z)$ is the weak-star closed convex hull of

$$\delta(Z) = \{\delta_z : z \in Z\}$$

and $M_1^+(W)$ is the weak-star closed convex hull of $\delta(W)$. Since T^* is weak-star to weak-star continuous and linear it maps weak-star compact convex sets to weak-star compact convex sets. Since $T^*(\delta(Z)) = \delta(W)$, if follows that $T^*(M_1^+(Z)) = M_1^+(W)$. Now, if μ is any Radon measure on W, with $\mu(W) > 0$, then $(\mu(W))^{-1}\mu$ belongs to $M_1^+(W)$ and there is a λ in $M_1^+(Z)$ with $T^*(\lambda) = (\mu(W))^{-1}\mu$. Take $\nu = \mu(W)\lambda$. Then $T^*(\nu) = \mu$ so that

$$\int_z f \circ p \, d\nu = \int_W f \, d\mu$$

for all f in $C(W)$. This ensures that

$$\nu(p^{-1}(B)) = \mu(B)$$

for each Borel set B in W. □

Throughout the rest of this section, Z will be a Hausdorff space and ρ will be a lower semi-continuous metric on Z. Under this assumption we prove lemmas establishing a chain of implications (see page 23) between the following conditions.

(c) *Each compact subset of Z is fragmented by ρ.*

(d) *For each Radon measure μ on Z with $\mu(Z) > 0$, and for each $\delta > 0$, there is a compact subset H of Z with $\mu(H) > 0$ and ρ-diam $H < \delta$.*

(e) *For each Radon measure μ on Z with $\mu(Z) > 0$, there is a ρ-compact subset C of Z with $\mu(C) > 0$*

(f) *For each Radon measure μ on Z and each $\varepsilon > 0$, there is a ρ-compact subset C of Z with $\mu(C) > \mu(Z) - \varepsilon$.*

(g) *The Radon measures on Z coincide with those Radon measures on Z taken with its ρ-topology that are carried by σ-compact sets of Z taken with its original topology.*

(h) *The subsets of Z that are measurable for every Radon measure on Z coincide with the subsets of Z that are measurable for those Radon measures on Z taken with its ρ-topology that are carried by σ-compact sets of Z taken with its original topology.*

(i) *Each ρ-closed subset of Z is measurable with respect to each Radon measure on Z.*

(j) *For no $\varepsilon > 0$ does Z contain a compact set H that admits a continuous map p onto the Cantor set $2^{\mathbb{N}}$ with inverse images of distinct points of $2^{\mathbb{N}}$ separated by ρ-distance ε.*

Lemma 3.2

(c) $\Longrightarrow$ (d).

Proof. Let μ be a Radon measure on Z with $\mu(Z) > 0$ and let $\delta > 0$ be given. Then we can choose a compact set K in Z with $\mu(K) > 0$. Let $\mu_{|K}$, defined by

$$\mu_{|K}(B) = \mu(B \cap K), \qquad \text{for } B \text{ a Borel set,}$$

be the restriction of μ to K. Then $\mu_{|K}$ has compact support, say L, contained in K. This support L is a compact subset of K with

$$\mu_{|K}(L) = \mu_{|K}(K) = \mu(K) > 0,$$

and with

$$\mu_{|K}(L \cap G) > 0,$$

whenever G is an open subset of Z with $L \cap G \neq \emptyset$. By our assumption (c), L is fragmented by ρ and we can choose an open set G with $L \cap G \neq \emptyset$ and ρ-diam$(L \cap G) < \delta$. Choose a point l in $L \cap G$. Since Z is a Hausdorff space and l does not lie in the compact set $L \backslash G$, we can choose an open set U in Z, containing l, and with the closure

$$H = \operatorname{cl} U$$

disjoint from $L \backslash G$. Now $L \cap H$ is a compact subset of Z contained in $L \cap G$ and so of ρ-diameter less than δ. Further

$$\mu(L \cap H) \geq \mu_{|K}(L \cap U) > 0,$$

since U is an open set meeting the support L of $\mu_{|K}$. This proves the lemma. □

Lemma 3.3

(d) $\Longrightarrow$ (e).

Proof. Let μ be a Radon measure on Z with $\mu(Z) > 0$. Fix a positive number δ. Using condition (d), the set Z contains a compact set H with $\mu(H) > 0$ and ρ-diam $H < \delta$. Let $\mathcal{H}$ be a maximal disjoint family of such compact subsets H of Z with $\mu(H) > 0$ and ρ-diam $H < \delta$. Clearly $\mathcal{H}$ is non-empty but countable. Write

$$J = \bigcup \{H : H \in \mathcal{H}\}.$$

We prove that $\mu(J) = \mu(Z)$. Otherwise, we must have $\mu(J) < \mu(Z)$ and $Z\backslash J$ is a μ-measurable set of positive measure. Hence we can choose a compact subset L of $Z\backslash J$ having positive μ-measure. Applying (d) to the measure ν defined on L by

$$\nu(E) = \mu(L \cap E), \qquad E \text{ Borel},$$

we can choose a compact subset H' of L with $\mu(H') = \nu(H') > 0$, and ρ-diam $H' < \delta$. Now H' is a compact subset of Z disjoint from the union J of the family $\mathcal{H}$, with $\mu(H') > 0$ and ρ-diam $H' < \delta$. This is contrary to the maximality of $\mathcal{H}$. Hence $\mu(J) = \mu(Z)$ as required.

For each integer $n \geq 1$, we apply the result of the last paragraph with $\delta = 1/n$; and, by taking a suitable finite sub-family from the family constructed, we obtain a compact subset C_n of Z such that

(1) $\mu(C_n) > \mu(Z)(1 - 2^{-n-1})$,

and

(2) C_n is the union of a finite number of disjoint compact sets, each of ρ-diameter less than $1/n$.

Write

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Then C is a compact subset of Z with $\mu(C) > (1/2)\mu(Z) > 0$. Since ρ is lower semi-continuous, Lemma 2.1 shows that C is ρ-closed and indeed ρ-complete. The condition (2) ensures that C is totally bounded in the metric ρ. Thus C is compact in Z and ρ-compact and the condition (e) holds. □

Lemma 3.4

(e) $\Longrightarrow$ (f).

Proof. Let μ be a Radon measure on Z and let $\varepsilon > 0$ be given. We assume, as we may, that $\mu(Z) \geq \varepsilon$. Using condition (e) the space Z contains a set D that is ρ-compact and has $\mu(D) > 0$. Let $\mathcal{D}$ be a maximal disjoint family of such ρ-compact subsets D of Z with $\mu(D) > 0$. Clearly $\mathcal{D}$ is non-empty but countable. Just as in the proof of Lemma 3.3, it follows that

$$\mu\left(\bigcup\{D : D \in \mathcal{D}\}\right) = \mu(Z).$$

Hence, by taking C to be a suitable finite union of sets from $\mathcal{D}$, we obtain a ρ-compact set C with $\mu(C) > \mu(Z) - \varepsilon$. □

The next lemma will be a step towards the proof of the implication (f) $\Longrightarrow$ (g).

Lemma 3.5

Let $\mathcal{S}$ and $\mathcal{T}$ be two Hausdorff topologies on a space X. Let $\mathcal{A}$ be the family of all subsets of X on which the topologies inherited from $\mathcal{S}$ and $\mathcal{T}$ coincide and are compact. If μ is a Radon measure for $\mathcal{S}$ carried by a countable union of sets in $\mathcal{A}$, then μ is also a Radon measure for $\mathcal{T}$.

Proof. Suppose that μ is carried by a set $C = \bigcup_{n=1}^{\infty} C_n$, where $C_n \in \mathcal{A}$ for each $n \geq 1$. Let $\mathcal{B}$ and $\mathcal{D}$ be the $\mathcal{S}$-Borel sets and the $\mathcal{T}$-Borel sets respectively. If F is a $\mathcal{T}$-closed set, then $F \cap C_n$ is $\mathcal{S}$-closed, and so $F \cap C \in \mathcal{B}$. Therefore, $E \cap C \in \mathcal{B}$, whenever $E \in \mathcal{D}$. Similarly $E \cap C \in \mathcal{D}$, whenever $E \in \mathcal{B}$.

If $E \in \mathcal{D}$, we have

$$E \cap C \subset E \subset (E \cap C) \cup (X \backslash C),$$

with $E \cap C \in \mathcal{B}$, $X\backslash C \in \mathcal{B}$ and $\mu(X\backslash C) = 0$. Hence E is μ-measurable. Define a $\mathcal{T}$-Borel measure by taking

$$\hat{\mu}(E) = \mu(E) \qquad \text{for} \qquad E \in \mathcal{D},$$

and forming the completion. If E is a $\hat{\mu}$-measurable set, there are sets A and B in $\mathcal{D}$ with $A \subset E \subset B$ and $\hat{\mu}(B\backslash A) = 0$. But then $\mu(B\backslash A) = 0$. Hence E is μ-measurable and

$$\hat{\mu}(E) = \hat{\mu}(A) = \mu(A) = \mu(E).$$

Furthermore, given $\varepsilon > 0$, there is an $\mathcal{S}$-compact set $K \subset A$ with $\mu(K) > \mu(A) - \varepsilon = \hat{\mu}(E) - \varepsilon$. Now

$$\mu(K) = \mu(C \cap K) = \mu\left(\bigcup_{n=1}^{\infty} C_n \cap K\right).$$

Hence there is an $m \geq 1$ with

$$\mu\left(\bigcup_{n=1}^{m} C_n \cap K\right) > \hat{\mu}(E) - \varepsilon.$$

Each set $C_n \cap K$ is $\mathcal{T}$-compact. Thus $L = \bigcup_{n=1}^{m} C_n \cap K$ is a $\mathcal{T}$-compact subset of E with $\hat{\mu}(L) = \mu(L) > \hat{\mu}(E) - \varepsilon$. This shows that $\hat{\mu}$ is a Radon measure for $\mathcal{T}$. To prove that $\mu = \hat{\mu}$ it remains to show that each μ-measurable set is $\hat{\mu}$- measurable. Suppose that E is μ-measurable, so that $A \subset E \subset B$, with $\mu(B\backslash A) = 0$, for some A and B in $\mathcal{B}$. Then

$$A \cap C \subset E \subset (B \cap C) \cup (X\backslash C),$$

and $A \cap C$, $B \cap C$ and $X\backslash C$ belong to $\mathcal{D}$. Now

$$\hat{\mu}\big([(B \cap C) \cup (X\backslash C)]\backslash(A \cap C)\big) = \mu\big([(B\backslash A) \cap C] \cup (X\backslash C)\big) = 0.$$

Thus E is $\hat{\mu}$-measurable as required. $\square$

Lemma 3.6

(f) $\Longrightarrow$ *(g).*

Proof. Let $\mathcal{A}$ be the family of all compact subsets of Z that are also ρ-compact. If A is any such set, it follows, by Lemma 2.1, that the ρ-topology on A coincides with the topology inherited from Z. Thus $\mathcal{A}$ is just the family introduced in Lemma 3.5 for the space Z and its two topologies.

(1) Suppose that μ is a Radon measure on Z. Then, by (f), for each $\varepsilon > 0$, there is a ρ-compact subset C of Z with $\mu(C) > \mu(Z) - \varepsilon$. Further, by Lemma 2.1, C is closed and so is μ-measurable. Since μ is a Radon measure, there is a compact subset K of C with $\mu(K) > \mu(Z) - \varepsilon$. By Lemma 2.1, K is ρ-closed and so is ρ-compact as C is ρ-compact. Thus K, being compact and ρ-compact, belongs to $\mathcal{A}$. It follows that μ is carried by a set that is a countable union of members of $\mathcal{A}$. By Lemma 3.5, μ is a Radon measure for ρ carried by a σ-compact set of Z.

(2) Suppose that $\hat{\mu}$ is a Radon measure for ρ carried by a σ-compact set K of Z. Write $K = \bigcup_{n=1}^{\infty} K_n$, with each set K_n compact. Note that each set K_n is ρ-closed by Lemma 2.1. Since $\hat{\mu}$ is a Radon measure for ρ, it is also carried by a set $C = \bigcup_{m=1}^{\infty} C_m$, with each set C_m ρ-compact. By Lemma 2.1 each set C_m is closed. Now $\hat{\mu}$ is carried on

$$K \cap C = \bigcup \{K_n \cap C_m : n \geq 1,\ m \geq 1\},$$

and each set $K_n \cap C_m$, being both compact and ρ-compact, belongs to $\mathcal{A}$. By Lemma 3.6, it follows that $\hat{\mu}$ is a Radon measure on Z. □

Lemma 3.7

(g) $\Longrightarrow$ *(h).*

Proof. By (g) the family of Radon measures on Z coincides with the family of those ρ-Radon measures on Z that are carried by some σ-compact set in Z. Hence the sets that are measurable for all measures of one family coincide with the sets that are measurable for all measures of the other family. □

Lemma 3.8

(h) $\Longrightarrow$ *(i).*

Proof. (i) is just a very special case of (h). □

Lemma 3.9

(i) $\Longrightarrow$ (j).

Proof. We suppose that (j) fails and seek a contradiction to (i). Since (j) fails we can choose $\varepsilon > 0$ and a compact set H in Z and a continuous surjective map p onto the Cantor set $2^{\mathbb{N}}$ with the inverse images of distinct points of $2^{\mathbb{N}}$ separated by ρ-distance ε.

Take μ to be the standard probability measure on $2^{\mathbb{N}}$ and let μ^* denote the corresponding outer measure. By Lemma 3.1 we can choose a Radon measure ν on Z carried by H with $\nu(p^{-1}(B)) = \mu(B)$ for each Borel subset B of $2^{\mathbb{N}}$. It follows without difficulty, that

$$\nu\big(p^{-1}(M)\big) = \mu(M),$$

for each μ-measurable set M.

Now choose disjoint non-measurable sets P and Q in $2^{\mathbb{N}}$ with

$$\mu^*(P) = \mu^*(Q) = \mu\big(2^{\mathbb{N}}\big) = 1.$$

Now $p^{-1}(P)$ and $p^{-1}(Q)$ are both unions of ρ-closed sets each separated by ρ-distance ε. Hence $p^{-1}(P)$ and $p^{-1}(Q)$ are disjoint ρ-closed sets in Z. Using the condition (i) we conclude that $p^{-1}(P)$ and $p^{-1}(Q)$ are measurable for the Radon measure ν. Hence

$$\nu\big(p^{-1}(P)\big) + \nu\big(p^{-1}(Q)\big) \le \nu(H) = \mu\big(2^{\mathbb{N}}\big) = 1,$$

and

$$\text{either} \qquad \nu\big(p^{-1}(P)\big) \le \frac{1}{2} \qquad \text{or} \qquad \nu\big(p^{-1}(Q)\big) \le \frac{1}{2}\,.$$

We suppose that $\nu(p^{-1}(P)) \le 1/2$. Then $\nu(H\backslash p^{-1}(P)) \ge 1/2$ and we can choose a compact set L in $H\backslash p^{-1}(P)$ with $\nu(L) > 1/4$. Since $p(L)$ is compact in $2^{\mathbb{N}}$, we have

$$\mu(p(L)) = \nu\big(p^{-1} \circ p(L)\big) \ge \nu(L) > \frac{1}{4}\,.$$

Since L is contained in $H\backslash p^{-1}(P)$, the set $p(L)$ is contained in $2^{\mathbb{N}}\backslash P$. Thus P is contained in the μ-measurable set $2^{\mathbb{N}}\backslash p(L)$ with μ-measure at most $3/4$. This contradicts the choice of P with $\mu^*(P) = 1$. The result follows. □

Note that we have now obtained the following implications contributing to the proof of Theorem 4.1.

$$\begin{array}{ccccccc}
(c) & & & & & & (j) \\
\Downarrow & & & & & & \Uparrow \\
(d) & & & & & & (i) \\
\Downarrow & & & & & & \Uparrow \\
(e) & \Rightarrow & (f) & \Rightarrow & (g) & \Rightarrow & (h)
\end{array}$$

At one stage we had hoped that, in proving some of these consequences of the fragmentation of the compact sets by a lower semi-continuous metric, we would have been able to work with non-negative inner regular Borel measures, that is, with Borel measures satisfying

$$\mu(E) = \sup \{\mu(F) : F \subset E \text{ and } F \text{ is closed}\},$$

for all measurable sets E, rather than with Radon measures. We now describe an example showing that this is not possible.

EXAMPLE 3.1. There is a compact Hausdorff space $\hat{Z}$ that is fragmented by a lower semi-continuous metric ρ and an open subset Z of $\hat{Z}$ and a non-negative inner regular Borel measure μ on Z with $\mu(Z) = 1$ but with $\mu(H) = 0$ for all compact subsets H of Z.

Construction. Let ω_1 be the first uncountable ordinal and take $\hat{Z}$ to be its successor $\omega_1 + 1$ taken with its order topology. Then $\hat{Z}$ is a compact Hausdorff space. Take $Z = \omega_1$. Then Z is an open set in $\hat{Z}$. Take ρ to be the trivial 0,1 metric on $\hat{Z}$ so that the ρ-topology on $\hat{Z}$ is the discrete topology. Then ρ is lower semi-continuous and $\hat{Z}$ is fragmented by ρ. Take μ to be the Dieudonné measure on ω_1 [15, §5.5]. Then μ is a non-negative inner regular Borel measure on Z. Further $\mu(Z) = 1$, but $\mu(H) = 0$ for all compact subsets H of Z.

4. Topological arguments

In this section we complete the proof of Theorem 4.1. Throughout we take ρ to be a lower semi-continuous metric on Z. We prove lemmas giving implications (see page 29) among the following conditions. In [19] we have proved that (a) $\iff$ (b) $\implies$ (c) $\implies$ (j) and that when Z is Čech-analytic (j) $\implies$ (a).

(a) *Z is σ-fragmented by ρ.*
(b) *Z is σ-separable with respect to ρ.*
(c) *Each compact subset of Z is fragmented by ρ.*
(j) *For no $\varepsilon > 0$ does Z contain a compact set H that admits a continuous map p onto the Cantor set $2^{\mathbb{N}}$ with the inverse images of distinct points of $2^{\mathbb{N}}$ separated by ρ-distance ε.*
(k) *For no $\varepsilon > 0$ does Z contain a relatively compact ε-tree for ρ.*
(l) *For each compact set K in Z, the points of continuity of the identity map from K to K with its ρ-topology are dense in K.*
(m) *For each compact set K in Z, the topology of K coincides with the ρ-topology of K on some dense $\mathcal{G}_\delta$-subset of K.*
(n) *For each compact set K of Z, each ρ-Borel set of K has the Baire property in K.*

Lema 4.1

(k) $\implies$ (j).

Proof. Suppose that (j) fails. Then we can choose $\varepsilon > 0$, a compact set H contained in Z and a continuous map p of H onto $2^{\mathbb{N}}$ with the inverse images of distinct points of $2^{\mathbb{N}}$ separated by ρ-distance ε. For each finite sequence

$$s = s_1, s_2, \ldots, s_n$$

of 0's and 1's let

$$\sigma(s) = s_1, s_2, \ldots, s_n, 0, 0, \ldots$$

be formed by adjoining an infinite sequence of zeros to s. Then $\sigma(s) \in 2^{\mathbb{N}}$ and we can choose a point $x(s)$ in $p^{-1}(\sigma(s))$. It is easy to verify that $\{x(s)\}$ is a ε-tree contained in the compact set H. Thus (k) fails. This proves the lemma. □

Lemma 4.2

(j) $\implies$ (k).

Proof. We suppose that (k) fails. Then, for some $\varepsilon > 0$, Z contains a relatively compact ε-tree. This means that we can choose $\varepsilon > 0$ and a countable set $\{x(s)\}$ of points of Z, indexed by the finite sequences s of 0's and 1's, and for each $l \geq 0$ and each sequence s of length l the sets

$$\mathrm{cl}\{x(t) : t|l+1 = s, 0 \text{ and } t \text{ has length } \geq l+1\}$$

and

$$\mathrm{cl}\{x(t) : t|l+1 = s, 1 \text{ and } t \text{ has length } \geq l+1\}$$

are compact sets separated by ρ-distance ε. Let $l(s)$ denote the length of the sequence s. Write

$$H(s) = \mathrm{cl}\{x(t) : l(t) \geq l(s) \text{ and } t|l(s) = s\},$$

for each s. Note that $x(s) \in H(s)$ so that, for each s, $H(s)$ is non-empty and compact. Further, for each s, perhaps of zero length, the sets $H(s, 0)$ and $H(s, 1)$ are separated by ρ-distance ε. Hence

$$H = \bigcap \left\{ \bigcup \{H(s) : l(s) = l\} : l \geq 1 \right\}$$

is a compact subset of Z constructed by a generalized Cantor set construction.

Define a map p from H to $2^{\mathbb{N}}$ by taking $p(h)$, for $h \in H$, to be the unique sequence in $2^{\mathbb{N}}$ for which

$$h \in \bigcap_{r=1}^{\infty} H\left(p(h)|r\right).$$

Then, for each σ in $2^{\mathbb{N}}$, the set

$$p^{-1}(\sigma) = \bigcap_{r=1}^{\infty} H(\sigma|r)$$

is compact and non-empty. It follows, in particular, that p maps H onto $2^{\mathbb{N}}$. To prove that p is continuous, it suffices to prove that

$$p^{-1}\left(I(\sigma|r)\right)$$

is a relatively open set in H, whenever $\sigma|r \in 2^r$ and

$$I(\sigma|r) = \left\{\tau \in 2^{\mathbb{N}} : \tau|r = \sigma|r\right\}.$$

This set $p^{-1}(I(\sigma|r))$ coincides with the compact set

$$\bigcap_{s=r}^{\infty} \bigcup \{H(\tau|s) : \tau|r = \sigma|r\}.$$

Hence $p^{-1}(I(\sigma|r))$ is the complement of H of the $2^r - 1$ compact subsets

$$p^{-1}\big(I(\tau|r)\big), \qquad \tau|r \in 2^r, \qquad \tau|r \neq \sigma|r,$$

and so is relatively open in H. Thus p is a continuous map of H onto $2^{\mathbb{N}}$. Further, if σ and τ are distinct points of $2^{\mathbb{N}}$, we can take r to be the least integer with $\sigma|r \neq \tau|r$. Then $p^{-1}(\sigma)$ and $p^{-1}(\tau)$ are subsets of $H(\sigma|r)$ and of $H(\tau|r)$, which are separated b ρ-distance ε. This proves the lemma. □

Lemma 4.3

If K is a compact subset of Z that is fragmented by ρ, then the points of continuity of the identity map from K to K with its ρ-topology are dense in K. Further, (c) $\Longrightarrow$ (l).

Proof. Let G_0 be any non-empty open subset of K. We construct a sequence $G_1, G_2, \ldots$ of non-empty open sets with

$$\operatorname{cl} G_n \subset G_{n-1} \qquad \text{and} \qquad \rho\text{-}\operatorname{diam} G_n < 1/n,$$

for $n \geq 1$. When $n \geq 1$ and G_{n-1} has been chosen, we use the assumption that K is fragmented by ρ to choose an open set U_n in K with

$$G_{n-1} \cap U_n \neq \emptyset \qquad \text{and} \qquad \rho\text{-}\operatorname{diam} G_{n-1} \cap U_n < 1/n.$$

We then choose a point x_n in $G_{n-1} \cap U_n$ and an open set G_n with

$$x_n \in G_n \subset \operatorname{cl} G_n \subset G_{n-1} \cap U_n.$$

That ensures that

$$\emptyset \neq G_n, \quad \operatorname{cl} G_n \subset G_{n-1} \quad \text{and} \quad \rho\text{-}\operatorname{diam} \operatorname{cl} G_n < 1/n.$$

The construction now follows by induction.

The sets

$$\operatorname{cl} G_n, \quad n = 1, 2, \ldots$$

form a decreasing sequence of non-empty compact sets. Take x to be any point in the intersection of these sets. Then, for each $n \geq 1$, the point x has a neighbourhood G_n in K of ρ-diameter less than $1/n$. Thus the identity map from K to K with its ρ-topology is continuous at x. Since $x \in G_0$ and G_0 may be any open set, these points of continuity are dense in K.

Now (l) follows from (c). □

Lemma 4.4

Suppose that K is a compact set in Z and that the points of continuity of the identity map from K to K with its ρ-topology are dense in K. Then the topology of K coincides with the ρ-topology of K on some dense $\mathcal{G}_\delta$-set of K. Further (l) $\Longrightarrow$ (m).

Proof. Let H be the set of all points of continuity of the identity map from K to K with its ρ-topology. Then H is a $\mathcal{G}_\delta$-set in K and it is dense in K by our assumption. The identity map from H to H with its ρ-topology is clearly continuous, and its inverse map is also continuous since the ρ-topology on K is stronger than the original topology, by Lemma 2.1. Hence the ρ-topology on H coincides with the topology induced on H by the original topology on K.

It is now clear that $(l) \Longrightarrow (m)$. □

Lemma 4.5

Let K be a compact set in Z and suppose that the topology of K coincides with the ρ-topology of K on some dense $\mathcal{G}_\delta$-set of K. Then each ρ-Borel set in K has the Baire property in K. Further (m) $\Longrightarrow$ (n).

Proof. Let H be a dense $\mathcal{G}_\delta$-set in K chosen so that the topology of H inherited from K coincides with the ρ-topology on H. Then $K \backslash H$ is of the first category in K.

Consider any ρ-Borel set B in K. Then $B \cap H$ is a ρ-Borel set in H. By the coincidence of the topologies on H, the set $B \cap H$ is a Borel set in H and so also in K. Hence $B \cap H$ has the Baire property in K. Since $B \backslash H$ is of the first category in K, it follows that

$$B = (B \cap H) \cup (B \backslash H)$$

also has the Baire property in K.

It is now clear that (m) $\Longrightarrow$ (n). □

Lemma 4.6

Let K and H be compact Hausdorff spaces and let f be a continuous map of K onto H. Suppose that K contains no compact subset L with

$$L \neq K \qquad \text{but} \qquad f(L) = f(K).$$

Then f maps each nowhere dense subset of K to a nowhere dense subset of H.

Proof. It suffices to suppose that N is a subset of K and that $f(N)$ fails to be nowhere dense in H and to prove that N cannot be nowhere dense in K. Then $\mathrm{cl}\, f(N) = f(\mathrm{cl}\, N)$ contains a nonempty open subset, say U, of H. Write $L = \mathrm{cl}\, N \cup f^{-1}(H \backslash U)$. Then L is closed and

$$f(L) \supset f(\mathrm{cl}\, N) \cup (H \backslash U) = H.$$

Hence, by hypothesis, $L = K$. Thus $\mathrm{cl}\, N$ contains the non-empty open set

$$K \backslash f^{-1}(H \backslash U) = f^{-1}(U),$$

and N is dense in this set. □

Lemma 4.7

(n) $\Longrightarrow$ (j).

Proof. We suppose that (n) holds but that, for some $\varepsilon > 0$, there is a compact subset H of K that admits a continuous map f onto $2^{\mathbb{N}}$ with the inverse images of distinct points of $2^{\mathbb{N}}$ separated by ρ-distance ε.

By a Zorn's lemma argument, there will be a minimal compact subset L of H subject to the condition that $f(L) = 2^{\mathbb{N}}$. We suppose, as we may without loss of generality, that H is chosen to coincide with such a minimal set L.

Take V to be any subset of $2^{\mathbb{N}}$ that does not have the Baire property in $2^{\mathbb{N}}$ (see, for example [23, p. 91]). Write $W = f^{-1}(V)$. For each v in V, the set $f^{-1}(v)$ is compact in H and so is ρ-closed in H. As each pair of such sets $f^{-1}(v)$ are separated by ρ-distance ε, it follows that W is ρ-closed in H. By condition (n) this set W has the Baire property in H. This enables us to choose a closed set F in H with

$$W \backslash F \qquad \text{and} \qquad F \backslash W$$

both of the first category in H. Now $f(F)$ is closed in $2^{\mathbb{N}}$, and, by Lemma 4.6, the sets

$$f(W \backslash F) \qquad \text{and} \qquad f(F \backslash W)$$

are of the the first category in $2^{\mathbb{N}}$. Since $V = f(W)$ we see that the sets

$$V \backslash f(F) = f(W) \backslash f(F) \subset f(W \backslash F),$$

$$f(F) \backslash V = f(F) \backslash f(W) \subset f(F \backslash W)$$

are of the first category in $2^{\mathbb{N}}$, and V has the Baire property in $2^{\mathbb{N}}$.

This contradiction shows that K can contain no such set H. It follows that (j) holds. □

Lemma 4.8

(j) $\Longrightarrow$ (c).

Proof. We suppose that (c) fails. Then Z contains a compact set H that is not fragmented by ρ. Since H, being compact, is Čech-analytic, it follows [19, Theorem 4.1] that for some $\varepsilon > 0$, H contains a compact set L that admits a continuous map onto the Cantor set $2^{\mathbb{N}}$ with the inverse images of distinct points of $2^{\mathbb{N}}$ separated by ρ-distance at least ε. Thus (j) fails. This prove the lemma. □

Proof of Theorem 4.1. In Lemmas 3.2 to 3.10 and Lemmas 4.1 to 4.8 we have proved all the implications in the following diagram.

$$
\begin{array}{ccccccc}
 & & & & (k) & & \\
 & & & & \Updownarrow & & \\
(d) & \Longleftarrow & (c) & \Longleftarrow & (j) & \Longleftarrow & (i) \\
 & & \Downarrow & & \Uparrow & & \Uparrow \\
\Downarrow & & (l) & \Longrightarrow (m) \Longrightarrow & (n) & & (h) \\
 & & & & & & \Uparrow \\
(e) & \Longrightarrow & & (f) & & \Longrightarrow & (g)
\end{array}
$$

As we remarked at the beginning of this section the remaining implications in Theorem 4.1 are proved in [19]. □

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