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The Mackey-Arens theorem for non-locally convex spaces

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ABSTRACT

Let $\mathcal{R}$ be a subcategory of the category of all topological vector spaces. Let $E \in \mathcal{R}$. The problem of the existence of the finest $\mathcal{R}$-topology on E with the same continuous linear functionals as the original one is discussed. Remarks concerning the Hahn-Banach Extension Property are included.

We use $E = (E, \tau)$ for a *Haussdorff topological vector space (tvs)*, E^*, E', E^+ for its algebraic, topological and sequential dual, respectively. Two vector topologies on E will be called *compatible* if they produce the same continuous linear functionals. A tvs E is *locally r-convex* (LrC), $0 < r \leq 1$ (*locally pseudoconvex*, LPC) if E has a 0-basis consisting of absolutely r-convex (balanced and pseudoconvex) sets; E is *almost convex* (AC) if E admits a fundamental system of bounded balanced and pseudoconvex sets [1, p. 76, 2, p. 108]. Clearly r-normed spaces have properties defined above.

Let $\mathcal{R}$ be a subcategory of the category TVS of all tvs. If $(E, \tau) \in \mathcal{R}$, we shall say that τ is a $\mathcal{R}$-topology. It is well known that for $\mathcal{R}$ = *locally convex spaces* every $(E, \tau) \in \mathcal{R}$ admits the finest $\mathcal{R}$-topology compatible with τ (the Mackey-Arens theorem, [2, p. 158]). Using the Hahn-Banach theorem one shows that for every locally convex tvs (E, τ) the supremum topology of all locally convex topologies compatible with τ is compatible with τ. We show that for some subcategories $\mathcal{R}$ of TVS (in particular TVS, LrC, $0 < r < 1$, LPC, AC) every tvs $(E, \tau) \in \mathcal{R}$ containing

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a dense infinite-codimensional subspace admits two vector topologies compatible with τ (and strictly finer than τ) under which E belongs to $\mathcal{R}$ but whose supremum topology is not compatible with τ.

We shall say that $\mathcal{R}$ has *property* $(*)$ if $\mathcal{R}$ is closed with respect to subspaces, finite products, isomorphic images and if moreover for every infinite-dimensional vector space E there exists a Hausdorff vector topology τ such that $(E,\tau)' = 0$ and $(E,\tau) \in \mathcal{R}$. Since every infinite-dimensional vector space E admits an r-normed topology τ (for any $0 < r < 1$) such that $(E,\tau)' = 0$ (cf. [5, Lemma 1] and its proof), then in particular TVS, LrC, $0 < r < 1$, LPC, AC have property $(*)$.

The proof of the following general result uses some ideas found in [5, 7, 8].

Proposition 1

Let $\mathcal{R}$ have property (*). If $(E,\tau) \in \mathcal{R}$ contains a dense infinite-codimensional subspace, then E does not admit the finest $\mathcal{R}$-topology compatible with τ.

Proof. Let G be a dense infinite-codimensional subspace in E. Choose $f \in E^*$, $f \neq 0$, such that $f(x) = 0$ for all $x \in G$. Set $X = E/G$. Let $h \in X^*$ be defined by $h(q(x)) = f(x)$, where $q : E \to X$ is the quotient map. Fix on X a Hausdorff $\mathcal{R}$-topology α_1 such that $(X,\alpha_1)' = 0$. For $y \in X$ with $h(y) = 2$ we define an injective linear map S of X onto X by $S(X) = x - h(x)y$, $x \in X$. Let α_2 be the image topology on X under the map S. Then h is continuous with respect to the topology $\alpha = \sup\{\alpha_1, \alpha_2\}$. In fact, if $x_\gamma \longrightarrow 0$ in α, then there exists a net y_γ in X such that $y_\gamma \longrightarrow 0$ in α_1 and

$$x_\gamma = y_\gamma - h(y_\gamma)y.$$

Therefore $h(x_\gamma) = -h(y_\gamma) \longrightarrow 0$. By β_i, $i = 1,2$, we denote on E the initial topology with respect to the canonical surjection $E \to (E/G, \alpha_i)$ and the injection $E \to (E,\tau)$. The sets of the form $U \cap q^{-1}(V)$ compose a 0-basis for β_i, where U, V run over 0-bases for τ and α_i, respectively. By property $(*)$ the topologies β_i are $\mathcal{R}$-topologies.

Moreover, by [9, Lemma 2.9] we have

$$\beta_i|G = \tau|G, \qquad \beta_i/G = \alpha_i, \qquad \tau < \beta_i.$$

Hence $\beta|G = \tau|G$, $\beta/G = \alpha$, where $\beta = \sup\{\beta_1, \beta_2\}$. Since G is τ-dense and $(X,\alpha_i)' = 0$, then τ and β_i are compatible. This completes the proof since f is τ-discontinuous but β-continuous. □

In [6, Lemma 2.1] we proved that a tvs E with $E^+ \neq E^*$ contains a dense infinite-codimensional subspace. Hence *Mazur* (in particular *ultrabornological or bornological*) spaces E with $E' \neq E^*$ contain dense infinite-codimensional subspaces.

For some subcategories of TVS we note even the following

Proposition 2

Let (E,τ) be a locally r-convex, $0 < r < 1$, (resp. locally pseudoconvex) tvs. Then E admits the finest locally r-convex (resp. locally pseudoconvex) vector topology compatible with τ if and only if $E^* = E'$.

Proof. If $E^* = E'$, then the finest locally r-convex (resp. locally pseudoconvex) vector topology on E is compatible with τ. Now assume that $E^* \neq E'$ and choose $f \in E^* \setminus E'$. Fix on E two r-normed topologies α_1 and α_2 such that $(E,\alpha_i)' = 0$, $i = 1,2$, but f is $\sup\{\alpha_1,\alpha_2\}$-continuous, see the proof of Proposition 1. Let $\sigma(E,E')$ be the weak topology of E. Since the topology $\inf\{\sigma(E,E'),\alpha_i\}$ is indiscrete, there exists an isomorphism Q from $(E,\sup\{\sigma(E,E'),\alpha_i\})$ onto a dense subspace

$$W = \{(x,x) : x \in E\}$$

of the product

$$Y_i = \big(E,\sigma(E,E')\big) \times (E,\alpha_i).$$

Let $\beta_i = \sup\{\sigma(E,E'),\alpha_i\}$. Choose $g \in (E,\beta_i)'$. Then $g \circ Q^{-1}$ is continuous on W and hence there exists $h \in Y_i'$ such that $h|W = g \circ Q^{-1}$. Since $h(x,y) = h(x,0)$, $x,y \in E$, then g is τ-continuous. Therefore τ and β_i are compatible. Since f is τ-discontinuous but $\sup\{\beta_1,\beta_2\}$-continuous, then E does not admit the finest locally r-convex (resp. locally pseudoconvex) vector topology compatible with τ. □

A tvs (E,τ) is said to have the *Hahn-Banach Extension Property* (HBEP) if every τ-closed subspace of E is $\sigma(E,E')$-closed. In [3] Kalton showed that every metrizable and complete tvs with the HBEP must be locally convex. It seems to be unknown whether the product of two tvs with the HBEP has the HBEP. It turns out that this question is strictly connected with the problem of the existence of infinite-dimensional atomic spaces. Applying the argument used in the proof of Proposition 2 we show the following

Corollary

If there exists an infinite-dimensional atomic space, then there exists two tvs with the HBEP whose product has not the HBEP.

Recall that E is atomic [4] if every proper closed subspace of E is finite-dimensional.

Proof. Assume that (E,τ_1) is an infinite dimensional atomic space. Then $M' = 0$ for any infinite-dimensional subspace M of E. Let β be a Haussdorff locally convex topology on E with $E^* \neq (E,\beta)'$. Fix $f \in E^* \setminus (E,\beta)'$. Let τ_2 be a vector topology on E such that (E,τ_1) and (E,τ_2) are isomorphic and f is $\sup\{\tau_1,\tau_2\}$-continuous (cf. the proof of Proposition 1). Since (E,τ_i) are atomic, then for any subspace M of E the topologies $\sup\{\tau_i|M,\sigma(M,(M,\beta)')\}$ are compatible with $\beta|M$. Hence $(E,\sup\{\tau_i|M,\sigma(M,(M,\beta)')\}$ have the HBEP, $i = 1,2$. Let

$$\gamma_i = \sup\{\tau_i,\sigma(M,(M,\beta)')\}.$$

If Q is the isomorphism from $(E,\sup\{\gamma_1,\gamma_2\})$ onto

$$W = \{(x,x) : x \in E\} \subset (E,\gamma_1) \times (E,\gamma_2), \qquad Q : x \longmapsto (x,x),$$

then $f \circ Q^{-1}$ can not be extended to a continuous linear functional over $(E,\gamma_1) \times (E,\gamma_2)$. $\square$

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