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On the space of φ-nuclear operators on ℓ^2

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Abstract

We consider the generalization $\mathcal{S}_\varphi$ of the Schatten classes $\mathcal{S}_p$ obtained in correspondence with opportune continuous, strictly increasing, sub-additive functions φ such that $\varphi(0) = 0$ and $\varphi(1) = 1$. The purpose of this note is to study the spaces $\mathcal{S}_\varphi$ of the φ-nuclear operators and to compare their properties to those of the by now well-known space $\mathcal{S}_1$ of nuclear operators.

Let $\mathcal{L}(\ell^2)$ be the space of all bounded linear operators on ℓ^2. As well known, every compact operator T on ℓ^2 has a representation of the form

$$T = \sum_n \xi_n e_n \otimes f_n, \tag{1}$$

where (e_n) and (f_n) are orthonormal systems in ℓ^2 and the sequence (ξ_n) can always be taken to be non-increasing, non-negative and such that $\xi_n \longrightarrow 0$. For $p > 0$, it is customary to denote by $\mathcal{S}_p$ the space of all operators T as in (1) for which the quantity

$$\sigma_p(T) = \sum_n \xi_n^p$$

is finite [5, §15.5]. Thus, for $1 \leq p < \infty$, the $\mathcal{S}_p$ are the Schatten classes, while for $0 < p < 1$ the elements of $\mathcal{S}_p$ are the so-called p-nuclear operators [5, Theorem 18.5.2].

Now, following [3, §II.2], we consider the set Φ' of all continuous, strictly increasing, sub-additive functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$. For any function $\varphi \in \Phi'$ and any scalar sequence $\eta = (\eta_n)$ we put

$$\sigma_\varphi(\eta) = \sum_n \varphi(|\eta_n|)$$

and

$$\ell_\varphi = \{\eta : \ \sigma_\varphi(\eta) < \infty\}$$

and we observe that, because of sub-additivity, ℓ_φ is a linear space of sequences on which σ_φ is a metric generating a topology under which $(\ell_\varphi, \sigma_\varphi)$ becomes a complete, metrizable, topological vector space. Since each $\varphi \in \Phi'$ is equivalent to a concave function $\widetilde{\varphi} \in \Phi'$ and since $p\varphi \in \Phi'$ whenever $\varphi \in \Phi'$ and $p > 0$, we may always assume that φ is concave and satisfies

$$\varphi(1) = 1, \tag{2}$$

so that

$$\varphi(t) \geq t \qquad \text{for all} \qquad t \in [0,1]. \tag{3}$$

Then, we denote by Φ the set of all such functions and, from now on, we always assume that $\varphi \in \Phi$.

An operator $T \in \mathcal{L}(\ell^2)$ admitting the representation (1) with $(\xi_n) \in \ell_\varphi$ is called *φ-nuclear* and the set of all such operators is denoted by $\mathcal{S}_\varphi$. We observe that, when $\varphi(t) = t^p$ $(0 < p \leq 1)$, then $\ell_\varphi = l^p$ and hence $\mathcal{S}_\varphi = \mathcal{S}_p$, showing that the φ-nuclear operators are a generalization of the p-nuclear ones.

The purpose of this note is to study the spaces $\mathcal{S}_\varphi$ and to compare their properties to those of the by now well-known space $\mathcal{S}_1$ of nuclear operators. If $T \in \mathcal{S}_\varphi$, we put $\sigma_\varphi(T) = \sigma_\varphi(\xi)$ if $\xi = (\xi_n)$ is the sequence in the representation (1) of T.

Theorem 1

$\mathcal{S}_\varphi$ is an operator ideal and σ_φ is a translation-invariant metric on it generating a topology under which $\mathcal{S}_\varphi$ becomes a complete, metrizable, topological vector space in which the finite-rank operators are dense. Moreover, the inclusion map $(\mathcal{S}_\varphi, \sigma_\varphi) \to (\mathcal{S}_1, \sigma_1)$ is continuous.

Proof. The ideal properties of $\mathcal{S}_\varphi$ are evident from the hypoteses on φ and so is the fact that σ_φ is a translation-invariant metric on $\mathcal{S}_\varphi$. Thus $(\mathcal{S}_\varphi, \sigma_\varphi)$ is a metrizable topological vector space and it is also clear that the finite-rank operators are dense in it. Suppose now that $T \in \mathcal{S}_\varphi$; then $\sigma_\varphi(T) = \sum_n \varphi(\xi_n) < \infty$ and hence there exists a k such that $\varphi(\xi_n) \leq 1$ for all $n \geq k$. Because of (3) and the fact that φ is increasing, we must have $0 \leq \xi_n \leq 1$ for $n \geq k$ and hence $\xi_n \leq \varphi(\xi_n)$ again by (3). But then

$$\sum_{n \geq k} \xi_n \leq \sum_{n \geq k} \varphi(\xi_n) < \infty$$

and $T \in \mathcal{S}_1$. This argument shows at the same time that $\mathcal{S}_\varphi \subset \mathcal{S}_1$ and that the inclusion map $(\mathcal{S}_\varphi, \sigma_\varphi) \to (\mathcal{S}_1, \sigma_1)$ is continuous. Finally the completeness of $(\mathcal{S}_\varphi, \sigma_\varphi)$ follows from that of (S_1, σ_1) and of $(\ell_\varphi, \sigma_\varphi)$ by a standard argument. □

Now we put

$$B_\varphi = \{T \in \mathcal{S}_\varphi : \sigma_\varphi(T) \leq 1\}$$

and

$$B_1 = \{T \in \mathcal{S}_1 : \sigma_1(T) \leq 1\}.$$

Then we have the following

Lemma

B_1 is the closure in $(\mathcal{S}_1, \sigma_1)$ of the absolutely convex hull of B_φ.

Proof. Let $T = \sum_n \xi_n e_n \otimes f_n \in \mathcal{S}_1$ be such that $\sigma_1(T) \leq 1$. Then $\sum_n \xi_n \leq 1$. Moreover, by (2),

$$\sigma_1(e_n \otimes f_n) = 1 = \varphi(1) = \sigma_\varphi(e_n \otimes f_n),$$

hence $e_n \otimes f_n \in B_\varphi$ for all n and the lemma follows. □

Denote by $\mathcal{S}'_\varphi$ the topological dual of $(\mathcal{S}_\varphi, \sigma_\varphi)$ and put

$$\|A\|_\varphi = \sup \{|\langle T, A\rangle| : T \in B_\varphi\}$$

for $A \in \mathcal{S}'_\varphi$. Then we have

Theorem 2

$(\mathcal{S}'_\varphi, \|\cdot\|_\varphi)$ is a Banach space isometric to $\mathcal{L}(\ell^2)$.

Proof. By the lemma, $\mathcal{S}'_\varphi = (\mathcal{S}_\varphi, \sigma_1)' = \mathcal{S}'_1$ and

$$\|A\|_\varphi = \sup\{|\langle T, A\rangle| : T \in B_1\}.$$

Hence $(\mathcal{S}'_\varphi, \|\cdot\|_\varphi) = (\mathcal{S}'_1, \|\cdot\|_1) = \mathcal{L}(\ell^2)$ by [6]. □

Turning now our attention to the extreme points of B_φ we find

Theorem 3

Let $T \in B_\varphi$. Then the following assertions are equivalent:
(i) T is an extreme point;
(ii) $T = e \otimes f$, with $\|e\| = \|f\| = 1$.

Proof. (ii) $\Longrightarrow$ (i): Any $T = e \otimes f$, with $\|e\| = \|f\| = 1$, belongs to B_φ and hence to B_1. By [2, Theorem 3.1], T is then an extreme point of B_1 and hence of B_φ, since $B_\varphi \subset B_1$.

(i) $\Longrightarrow$ (ii): Let $T \in B_\varphi$ be an extreme point and write, as in (1),

$$T = \sum_n \xi_n e_n \otimes f_n,$$

with

$$\sum_n \varphi(\xi_n) \leq 1. \tag{4}$$

Suppose that there are two integers j and k, with $j \neq k$, for which $\xi_j \neq 0$ and $\xi_k \neq 0$. Then, by (4),

$$0 < \varphi(\xi_j) + \varphi(\xi_k) = \rho \leq 1.$$

Thus, in the two dimensional xy-plane the point (ξ_j, ξ_k) belongs to the set

$$C_\rho = \{(x, y) : \varphi(|x|) + \varphi(|y|) \leq \rho\}$$

and is not an extreme point for such a set, since φ is concave and both ξ_j and ξ_k are non-zero. It follows that there are scalars $\alpha > 0$, $\beta > 0$ and t, with $0 < t < 1$, such that

$$(\xi_j, \xi_k) = t(\alpha, 0) + (1-t)(0, \beta) = (t\alpha, (1-t)\beta)$$

and that the segment

$$\{s(\alpha, 0) + (1-s)(0, \beta) : 0 \leq s \leq 1\}$$

is contained in C_ρ. In particular,

$$\max\{\varphi(\alpha), \varphi(\beta)\} \le \rho = \varphi(\xi_j) + \varphi(\xi_k). \tag{5}$$

If we now put

$$T_1 = \alpha e_j \otimes f_j + \sum_{n \neq j,k} \xi_n e_n \otimes f_n,$$
$$T_2 = \beta e_k \otimes f_k + \sum_{n \neq j,k} \xi_n e_n \otimes f_n,$$

then $T = tT_1 + (1-t)T_2$. Moreover,

$$\sigma_\varphi(T_1) = \varphi(\alpha) + \sum_{n \neq j,k} \varphi(\xi_n) \le 1$$

by (4) and (5), showing that $T_1 \in B_\varphi$. Since the same is true for T_2, we conclude that such a T cannot be an extreme point. □

Finally, we investigate the isometries of $(\mathcal{S}_\varphi, \sigma_\varphi)$, i.e. the linear bijections $J : \mathcal{S}_\varphi \to \mathcal{S}_\varphi$ such that $\sigma_\varphi[J(T)] = \sigma_\varphi(T)$. We find that the results of [1] can be extended to the following

Theorem 4

Let $J : \mathcal{S}_\varphi \to \mathcal{S}_\varphi$ be linear and onto. The following assertions are equivalent:
(i) J is an isometry;
(ii) There exist two unitary maps U, V on ℓ^2 such that $J = U \otimes V$.

Proof. (ii) $\Longrightarrow$ (i): If $T \in \mathcal{S}_\varphi$ has the representation (1), then

$$J(T) = \sum_n \xi_n U(e_n) \otimes V(f_n). \tag{6}$$

Because U and V are unitary on ℓ^2, the sequences $(U(e_n))$ and $(V(f_n))$ are orthonormal systems in ℓ^2 and hence, from (6),

$$\sigma_\varphi[J(T)] = \sum_n \varphi(\xi_n) = \sigma_\varphi(T),$$

i.e. J is an isometry.

(i) $\Longrightarrow$ (ii): If J is an isometry, then so is $J' : \mathcal{S}'_\varphi \to \mathcal{S}'_\varphi$ and hence also $J'' : \mathcal{S}''_\varphi \to \mathcal{S}''_\varphi$. But, by Theorem 2 and [6], $\mathcal{S}''_\varphi = \mathcal{L}(\ell^2)' = \mathcal{S}''_1$, hence the restriction J_0 of J'' to $\mathcal{S}_1 \subset \mathcal{S}''_1$ is an isometry of $\mathcal{S}_1$ onto $\mathcal{S}_1$ (because J maps $\mathcal{S}_\varphi$ onto $\mathcal{S}_\varphi$) and (ii) follows by the theorem in [1]. □

Remark. One may also define an operator ideal η_φ as follows: $T \in \eta_\varphi$ if T has a representation of the form

$$T = \sum_n \xi_n u_n \otimes v_n, \tag{7}$$

where $(\xi_n) \in \ell_\varphi$ and $(u_n), (v_n) \subset \ell^2$ with $\|u_n\| = \|v_n\| = 1$ for all n. Recalling [4, §3], we see that η_φ is the ideal of *pseudo-φ-nuclear operators*, which may be defined between arbitrary Banach spaces E, F and not just on ℓ^2. Endowed with the metric

$$\nu_\varphi(T) = \inf \sum_n \varphi(\xi_n),$$

where the infimum is taken over all representations of the form (7), η_φ becomes a complete, metrizable, topological vector space for which all the results proved above for $\mathcal{S}_\varphi$ continue to hold. In particular, if φ belongs to the class Φ_ω of [3, § II.2], then ℓ_φ is idempotent, hence $\ell_\varphi = \ell^1 \cdot \ell_\varphi$ and, therefore, $\eta_\varphi = \mathcal{S}_\varphi$ by [4, Theorem 3]. In this case the metrics ν_φ and σ_φ are equivalent, in the sense that they generate the same topology on $\eta_\varphi = \mathcal{S}_\varphi$ (use the open mapping theorem and the fact that $\nu_\varphi(T) \le \sigma_\varphi(T)$ always).

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