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Kolmogorov diameters and orthogonality in non-archimedean normed spaces

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Abstract

The Kolmogorov n-diameter of a bounded set B in a non-archimedean normed space, as defined by the first author in a previous paper, is studied in terms of the norms of orthogonal subsets of B with $n+1$ points.

0. Introduction

The Kolmogorov diameters of a bounded set in a non-archimedean normed space have been recently introduced by the first author [2]. In that paper the relationships between Kolmogorov diameters and diametral dimension are also investigated and, as a result, non-archimedean power sequence spaces are characterized by means of their diametral dimension.

In this paper we explore the use of orthogonality in the study of n-diameters. Our main result is theorem 3.5 in which we obtain a very close relationship between

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the n-diameter $\delta_n^*(B)$ of an absolutely convex set B and a new constant, $P_n^t(B)$, defined in terms of the norms of those t-orthogonal subsets of B consisting of $n+1$ points. The inequalities which relate these two constants become equalities when every one-dimensional subspace is orthocomplemented, the valuation is dense and $t = 1$ (corollary 3.8).

Throughout this paper $\mathbb{K}$ will be a non-archimedean complete valued field endowed with a non-trivial valuation. If the valuation of $\mathbb{K}$ is discrete we will denote by π an element in $\mathbb{K}$ such that $|\pi| < 1$ is a generator of the value group $\{|\lambda| : \lambda \in \mathbb{K} - \{0\}\}$.

E will always denote a non-archimedean normed space over $\mathbb{K}$, $B(a,r)$ the closed ball with center $a \in E$ and radius $r \geq 0$, and B a nonempty bounded subset of E. By $\text{co}(B)$ we denote the convex hull of B, i.e.

$$\text{co}(B) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \in \mathbb{K},\ |\lambda_i| \leq 1,\ x_i \in B, n \in \mathbb{N} \right\}.$$

Also $[B]$ stands for the linear hull of B.

If t is a real number with $0 < t \leq 1$, a finite sequence $(x_1, \ldots, x_n)$ of elements of E is said to be *t-orthogonal* if

$$\|\lambda_1 x_1 + \cdots + \lambda_n x_n\| \geq t \max\left(\|\lambda_1 x_1\|, \ldots, \|\lambda_n x_n\|\right)$$

for all $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$. Notice that if $(x_1, \ldots, x_n)$ is a t-orthogonal sequence, so is $(x_1, \ldots, x_n, 0, \ldots, 0)$. A subset $X \subset E - \{0\}$ is *t-orthogonal* if the above inequality holds for all $n \in \mathbb{N}$, all $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ and all $x_1, \ldots, x_n \in X$ for which $x_i \neq x_j$ when $i \neq j$. Every t-orthogonal subset is a linearly independent set.

Now assume $\dim E < \infty$. A subset $X \subset E - \{0\}$ is said to be a *t-orthogonal basis* of E if it is a t-orthogonal subset and $[X] = E$ (where $[X]$ stands for the lineal hull of X). Every finite dimensional normed space has, for each $t \in (0,1)$, a t-orthogonal basis [5, Theorem 3.15].

A sequence (subset, basis) is said to be *orthogonal* if it is *1-orthogonal*. If $\mathbb{K}$ is spherically complete, every finite dimensional normed space has an orthogonal basis [5, Lemma 5.5].

Following [5], a normed space E is said to be *pseudoreflexive* if the natural map $j_E : E \to E''$ is an isometry. It is not difficult to see that a normed space E is pseudoreflexive if and only if for each $\epsilon > 0$ and each finite dimensional subspace F of E, there exists a projection $P \in L(E)$ of E onto F such that $\|P\| \leq 1 + \epsilon$. A projection $P \in L(E)$ is said to be an *orthoprojection* if $\|P\| = 1$. If $\mathbb{K}$ is spherically complete and F a finite dimensional subspace of a normed space E over $\mathbb{K}$, then there exists an orthoprojection of E onto F [5, Corollary 4.7].

1. The n-diameter of a bounded set

1.1. DEFINITION. Let B be a bounded subset of a normed space E. For each non-negative integer n, we define

a) $\delta_n(B)$ as the infimum of $|\lambda|$, for those $\lambda \in \mathbb{K} - \{0\}$ such that $B \subset F + B(0, |\lambda|)$ for some linear subspace F of E with $\dim F \leq n$.

b) $\delta_n^*(B)$ as the infimum of those $r > 0$, for which $B \subset F + B(0, r)$ for some linear subspace F of E with $\dim F \leq n$.

The definition of $\delta_n(B)$ was introduced by the first author [2] under the name of *n-diameter* of B. We shall write $\delta_{n,E}(B)$ (instead of $\delta_n(B)$) or $\delta_{n,E}^*(B)$ (instead of $\delta_n^*(B)$) if we want to emphasize the space E. In the case of real or complex ground field this definition goes back to A. N. Kolmogorov.

Remarks

1.2 It is obvious that $\delta_n(B) = \delta_n^*(B)$ if the valuation of $\mathbb{K}$ is dense and $|\pi|\delta_n(B) \leq \delta_n^*(B) \leq \delta_n(B)$ if the valuation is discrete.

From now on we are going to restrict ourselves to the study of the properties of δ_n^*, the properties of δ_n being completely similar to those of δ_n^*.

1.3. $\delta_n^*(B) = \delta_n^*(\text{co}(B))$ and $\delta_n^*(B) = \delta_n^*(\overline{B})$.

1.4. $\delta_0^*(B) \geq \delta_1^*(B) \geq \cdots \geq \delta_n^*(B) \geq \cdots$. By using 1.3 and [5, Theorem 4.37, $(\alpha) \iff (\gamma)$], one can easily prove that B is compactoid if and only if $\lim \delta_n^*(B) = 0$ (See also [2, Theorem 3.3] for a direct proof).

This characterization of compactoid subsets is the non-archimedean counterpart to the folllowing well known result: *A bounded subset B of a real or complex normed space E is precompact if and only if* $\lim \delta_n(B) = 0$ [3, Proposition 9.1.4].

1.5 Theorem

Let λ be an element of $\mathbb{K}$ such that $\lambda = 1$ if the valuation is discrete and $|\lambda| > 1$ if the valuation is dense. Let B be a bounded subset of E, $n \in \mathbb{N}$ and $t \in (0, 1)$. Let also $\beta_n^t(B)$, $\gamma_n^\lambda(B)$, $\theta_n(B)$ and $\omega_n(B)$ be defined in the following way:

a) *$\omega_n(B)$ as the infimum of those $r > 0$, for which $B \subset F + B(0, r)$ for some linear subspace F of E with $\dim F \leq n$ and $F \subset [B]$.*

b) *$\theta_n(B)$ as the infimum of those $r > 0$, for which $B \subset \text{co}\{a_1, \ldots, a_n\} + B(0, r)$, for some sequence $(a_1, \ldots, a_n)$ in E.*

c) *$\beta_n^t(B)$ as the infimum of those $r > 0$, for which $B \subset \text{co}\{a_1, \ldots, a_n\} + B(0, r)$, for some t-orthogonal sequence $(a_1, \ldots, a_n)$ in E.*

d) $\gamma_n^{\lambda}(B)$ *as the infimum of those* $r > 0$, *for which* $B \subset \operatorname{co}\{a_1, \ldots, a_n\} + B(0,r)$, *for some sequence* $(a_1, \ldots, a_n)$ *in* $\lambda \operatorname{co}(B)$.

e) $\psi_n^{t,\lambda}(B)$ *as the infimum of those* $r > 0$, *for which* $B \subset \operatorname{co}\{a_1, \ldots, a_n\} + B(0,r)$, *for some t-orthogonal sequence* $(a_1, \ldots, a_n)$ *in* $\lambda \operatorname{co}(B)$.

Then, $\delta_n^*(B) = \omega_n(B) = \theta_n(B) = \beta_n^t(B) = \gamma_n^{\lambda}(B) = \psi_n^{t,\lambda}(B)$.

Proof. It is obvious that

$$\psi_n^{t,\lambda}(B) \geq \beta_n^t(B) \geq \theta_n(B) \geq \delta_n^*(B)$$

and that

$$\psi_n^{t,\lambda}(B) \geq \gamma_n^{\lambda}(B) \geq \omega_n(B) \geq \delta_n^*(B).$$

Let us split the rest of the proof into three steps:

(1) $\beta_n^t(B) \geq \gamma_n^{\lambda}(B)$.

Let $r > 0$ for which there exists a t-orthogonal sequence $(a_1, \ldots, a_n)$ of elements of E such that $B \subset \operatorname{co}\{a_1, a_2, \ldots, a_n\} + B(0,r)$.

First assume that the valuation on $\mathbb{K}$ is discrete. Then by [1, Lemma 1.2], there exist $b_1, \ldots, b_n \in \operatorname{co}(B)$ such that

$$\operatorname{co}(B) \subset \operatorname{co}\{b_1, \ldots, b_n\} + B(0,r),$$

and hence $\beta_n^t(B \geq \gamma_n^{\lambda}(B)$.

Now we assume that the valuation on $\mathbb{K}$ is dense and let $\xi \in \mathbb{K}$ such that $1 < |\xi| \leq |\lambda|$. By using again [1, Lemma 1.2] there exist $b_1, \ldots, b_n \in \operatorname{co}(B)$ such that

$$\xi^{-1}\operatorname{co}(B) \subset \operatorname{co}\{b_1, \ldots, b_n\} + B(0,r),$$

which implies

$$B \subset \operatorname{co}\{c_1, \ldots, c_n\} + \xi\, B(0,r),$$

where $c_k = \xi\, b_k \in \lambda\operatorname{co}(B)$ $(k = 1, \ldots, n)$. Then, $\gamma_n^{\lambda}(B) \leq |\xi\,|\beta_n^t(B$ for all $\xi \in \mathbb{K}$ such that $|\xi| \in \big(1, |\lambda|\big]$. Hence, $\beta_n^t(B \geq \gamma_n^{\lambda}(B)$.

(2) $\delta_n^*(B) \geq \beta_n^t(B)$.

Since B is bounded there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in B$. Let $r > 0$ for which there exists a linear subspace F with $\dim F \leq n$ such that $B \subset F + B(0,r)$. First assume $m = \dim F > 0$. We claim that

$$B \subset \big(F \cap B(0,s)\big) + B(0,r) \qquad \text{where } s = \max\{M, r\}.$$

Indeed, let $x \in B$ and let f be an element of F such that $\|x - f\| \leq r$. Then, $\|f\| = \|x - f - x\| \leq \max\{M, r\} = s$ and we are done.

Now, let $\{f_1, \dots, f_m\}$ be a t-orthogonal basis of F such that $\|f_i\| \geq s\,t^{-1}$ $(i = 1, \dots, m)$. If $x \in F \cap B(0, s)$, then x can be written as

$$x = \sum_{i=1}^{m} \lambda_i f_i$$

with

$$s \geq \|x\| \geq t \max_{1 \leq i \leq n} |\lambda_i|\, \|f_i\| \geq s \max_{1 \leq i \leq n} |\lambda_i|$$

and so $x \in \operatorname{co}\{f_1, \dots, f_m\}$. Thus, $F \cap B(0, s) \subset \operatorname{co}\{f_1, \dots, f_m\}$ and hence

$$B \subset \operatorname{co}\{f_1, \dots, f_m\} + B(0, r).$$

If $\dim F = 0$, this last inclusion also holds for $f_1 = \dots = f_m = 0$. This allows us to conclude that $\delta_n^*(B) \geq \beta_n^t(B)$.

(3) $\psi_n^{t,\lambda}(B) \leq \gamma_n^\lambda(B)$.

Let $r > 0$ for which there exists a sequence $(a_1, \dots, a_n)$ in $\lambda \operatorname{co}(B)$ such that $B \subset \operatorname{co}\{a_1, \dots, a_n\} + B(0, r)$. Set $H = [a_1, \dots, a_n]$, and let $m = \dim H > 0$ (the case $m = 0$ is trivial). By [6, Lemma 2.2] there are m linearly independent elements $b_1, \dots, b_m \in \{a_1, \dots, a_n\}$ such that $\operatorname{co}\{a_1, \dots, a_n\} = \operatorname{co}\{b_1, \dots, b_m\}$. Next, we consider two norms in H as follows

(a) $\|\cdot\|_1$ is such that $(H, \|\cdot\|_1)$ has an orthogonal basis and

$$t\|x\| \leq \|x\|_1 \leq \|x\|$$

for all $x \in H$, where $\|\cdot\|$ is the original norm on H. One such norm exists by [5, Theorem 3.15 (iv)].

(b) $\|\cdot\|_2$ is defined by

$$\|x\|_2 = \|\sum_{i=1}^{m} \lambda_i b_i\|_2 = \max_{1 \leq i \leq m} |\lambda_i|.$$

Both $(H, \|\cdot\|_1)$ and $(H, \|\cdot\|_2)$ have an orthogonal basis and so there exists an orthogonal basis $\{c_1, \dots, c_m\}$ which is orthogonal in both spaces [4, Theorem 1.11]. Moreover, since $\{\|x\|_2 : x \in H\} \subset \{|\lambda| : \lambda \in \mathbb{K}\}$, there is no loss of generality if we suppose that $\|c_i\|_2 = 1$ for $i = 1, \dots, m$.

We claim that $\operatorname{co}\{b_1, \dots, b_m\} = \operatorname{co}\{c_1, \dots, c_m\}$.

Indeed,

$$\operatorname{co}\{b_1, \dots, b_m\} = \{x \in H : \|x\|_2 \leq 1\}$$

and hence

$$\operatorname{co}\{c_1, \dots, c_m\} \subset \operatorname{co}\{b_1, \dots, b_m\}.$$

On the other hand, b_i can be written as

$$b_i = \sum_{j=1}^{m} \lambda_i^j c_j \qquad (i = 1, \dots, m)$$

and hence

$$1 = \|b_i\|_2 = \max_{1 \leq j \leq m} |\lambda_i^j| \|c_j\|_2 = \max_{1 \leq j \leq m} |\lambda_i^j|.$$

Then, $b_i \in \operatorname{co}\{c_1, \dots, c_m\}$ for all $i \in \{1, \dots, m\}$ and $\operatorname{co}\{b_1, \dots, b_m\} \subset \operatorname{co}\{c_1, \dots, c_m\}$.

We have that $B \subset \operatorname{co}\{c_1, \dots, c_m\} + B(0, r)$ with $c_1, \dots, c_m \in \lambda \operatorname{co}(B)$. Also,

$$\begin{aligned} \|\lambda_1 c_1 + \cdots + \lambda_m c_m\| &\geq \|\lambda_1 c_1 + \cdots + \lambda_m c_m\|_1 \\ &= \max_{1 \leq i \leq m} |\lambda_i| \, \|c_i\|_1 \\ &\geq t \max_{1 \leq i \leq m} |\lambda_i| \, \|c_i\| \end{aligned}$$

which shows that $\{c_1, \dots, c_m\}$ is t-orthogonal. Thus, $\psi_n^{t,\lambda}(B) \leq \gamma_n^\lambda(B)$. □

Remarks

1.6. The constant $\beta_n^t(B)$ introduced in the above theorem does not depend on the choice of t. In the same way, $\gamma_n^\lambda(B)$ does not depend on λ and $\psi_n^{t,\lambda}(B)$ does depend neither on λ nor on t.

1.7. Also, if $\mathbb{K}$ is spherically complete, every finite dimensional normed space has an orthogonal basis and hence theorem 1.4 is also valid for $t = 1$.

The same holds for any $\mathbb{K}$ if every one-dimensional subspace of E is orthocomplemented [5, 4.35]. Indeed, every finite-dimensional subspace of E is orthocomplemented. Then every linear subspace of a finite-dimensional subspace E_n of E is orthocomplemented in E and hence in E_n. By [5, 5.15] this implies that E_n has an orthogonal basis.

1.8. In view of 1.5 it is reasonable to compare $\delta_n^*(B)$ with the following new constant:

$$\sigma_n(B) = \inf\{r > 0 : B \subset \text{co}\{a_1, \ldots, a_n\} + B(0, r),\ a_1, \ldots, a_n \in \text{co}(B)\}.$$

It is obvious that $\sigma_n(B) \geq \delta_n^*(B)$ and that they are equal if the valuation on $\mathbb{K}$ is discrete. So we will assume that the valuation on $\mathbb{K}$ is dense.

Since $\sigma_n(B) = \sigma_n(\text{co}(B))$, we are also going to restrict ourselves to absolutely convex subsets B of E. Recall that B is said to be *c'-compactoid* or *pure compactoid* [7, Definition 3.1] if for each $r > 0$ there exists a finite set $M \subset B$ such that $B \subset \text{co}(M) + B(0, r)$. Then, it is obvious that B is c'-compactoid if and only if $\lim \sigma_n(B) = 0$.

Let now B be a compactoid subset of a normed space E such that B is not c'-compact (e.g., if $\mathbb{K}$ is spherically complete take $E = \mathbb{K}$ and $B = \{\lambda \in \mathbb{K} : |\lambda| < 1\}$). Then, $\lim \sigma_n(B) > 0$ but $\lim \delta_n^*(B) = 0$.

Also, with the same proof as in 1.5 one can easily see that $\sigma_n(B)$ equals to the infimum of all $r > 0$ such that $B \subset \text{co}\{a_1, \ldots, a_n\} + B(0, r)$ for some t-orthogonal sequence $(a_1, \ldots, a_n)$ in $\text{co}(B)$.

1.9. Assume that $\dim[B] \geq n$ and let λ and t as in theorem 1.5. Then, reasoning as in theorem 1.5, $\delta_n^*(B)$ equals to each of the following numbers:

a) The infimum of all $r > 0$ such that $B \subset F + B(0, r)$ for some linear subspace F of E with $\dim F = n$.
b) The infimum of all $r > 0$ such that $B \subset F + B(0, r)$ for some linear subspace F of E with $\dim F = n$ and $F \subset [B]$.
c) The infimum of all $r > 0$ such that $B \subset \text{co}\{a_1, \ldots, a_n\} + B(0, r)$ for some linearly independent subset $\{a_1, \ldots, a_n\}$ of E.
d) The infimum of all $r > 0$ such that $B \subset \text{co}\{a_1, \ldots, a_n\} + B(0, r)$ for some t-orthogonal subset $\{a_1, \ldots, a_n\}$ of E.
e) The infimum of all $r > 0$ such that $B \subset \text{co}\{a_1, \ldots, a_n\} + B(0, r)$ for some linearly independent subset $\{a_1, \ldots, a_n\}$ of $\lambda\, \text{co}(B)$.
f) The infimum of all $r > 0$ such that $B \subset \text{co}\{a_1, \ldots, a_n\} + B(0, r)$ for some t-orthogonal subset $\{a_1, \ldots, a_n\}$ of $\lambda\, \text{co}(B)$.

1.10 Corollary

Let F be a normed space such that $E \subset F$ and let B be a bounded subset of E. Then, $\delta_{n,E}^*(B) = \delta_{n,F}^*(B)$.

2. A related constant

2.1. DEFINITION. Let B be a bounded subset of a normed space E and let $t \in (0,1]$. For each non-negative integer n, we define $P_n^t(B)$ as the infimum of all $r \geq 0$ such that if

$$Y \subset B \cap \{x \in E : \|x\| > t^{-1}r\},$$

and Y is t-orthogonal, then $\#Y \leq n$.

When $t = 1$, we shall write $P_n(B)$ instead of $P_n^1(B)$.

Remarks

2.2. In the definition of $P_n^t(B)$ the infimum is attained; that is, if

$$Y \subset B \cap \{x \in E : \|x\| > t^{-1} P_n^t(B)\},$$

and Y is t-orthogonal, then $\#Y \leq n$.

2.3.

$$P_0^t(B) = t \sup \{\|x\| : x \in B\} = t\, \delta_0^*(B).$$

$$P_0^t(B) \geq P_1^t(B) \geq \cdots \geq P_n^t(B) \geq \cdots.$$

Also if $t_1 \leq t_2$, then $t_2^{-1} P_n^{t_2}(B) \leq t_1^{-1} P_n^{t_1}(B)$ for each n.

2.4 Theorem

Let B be a bounded subset of a normed space E, $t \in (0,1]$ and n a non-negative integer. Then

a) $P_n^t(B) = 0$ if there are no t-orthogonal subsets of B consisting of $n+1$ points.

b) Otherwise

$$P_n^t(B) = t \sup_Y \{\inf\{\|y\| : y \in Y\}\}$$

where the supremum is taken over the t-orthogonal subsets Y of B such that $\#Y = n+1$.

Proof. The first part of the proof is trivial. Now assume $P_n^t(B) > 0$ and take $s \in (0, P_n^t(B))$. Then by 2.1 there exists a t-orthogonal subset Y_0 contained in $B \cap \{x \in E : \|x\| > t^{-1}s\}$ with $n+1$ elements. Then, $\inf\{\|y\| : y \in Y_0\} > t^{-1}s$ and hence

$$P_n^t(B) \leq t \sup_Y \{\inf\{\|y\| : y \in Y\}\},$$

with Y as above. This inequality also holds if $P_n^t(B) = 0$.

In order to prove the equality let us suppose that

$$\sup_Y \{ \inf\{\|y\| : y \in Y\} \} > t^{-1} P_n^t(B).$$

In this case there exists a t-orthogonal subset Y of B with $n+1$ elements such that

$$\inf \{\|y\| : y \in Y\} > t^{-1} P_n^t(B)$$

which contradicts 2.2. □

2.5. Corollary

Let B be a bounded subset of a normed space E, $t \in (0,1]$ and n a non negative integer. Then,

$$P_n^t(B) = \sup_M \{P_n^t(B \cap M)\},$$

where the supremum is taken over the $n+1$-dimensional subspaces M of E.

Proof. Let m be this supremum. It is obvious that $m \leq P_n^t(B)$. The equality holds if $P_n^t(B) = 0$. Hence, suppose $P_n^t(B) > 0$ and let $r \in (0, P_n^t(B))$. Then, there is a t-orthogonal set $Y \subset B$ such that $\#Y = n+1$ and $\|y\| > t^{-1}r$ for all $y \in Y$. Let $M = [Y]$. Then, $\dim M = n+1$, $Y \subset B \cap M$, and hence $P_n^t(B \cap M) > r$. This proves that $m \geq P_n^t(B)$ and we are done. □

2.6 Proposition

For each $t \in (0,1]$, $P_n^t(B) \leq \delta_n^*(B)$.

Proof. Let $r > \delta_n^*(B)$. Then there exists a vector subspace F of E, $\dim F \leq n$, such that $B \subset F + B(0,r)$. Now assume there exists a t-orthogonal subset Y of $B \cap \{x \in E : \|x\| > t^{-1}r\}$ with more than n points. Let $\{y_1, \ldots, y_{n+1}\}$ be a t-orthogonal subset of elements of Y. For each i ($i = 1, \ldots, n+1$), let $f_i \in F$ be such that $\|y_i - f_i\| \leq r < t\|y_i\|$. By using [4, Lemma 6.d], we deduce that $\{f_1, \ldots, f_{n+1}\}$ is a t-orthogonal subset of elements of F. But this is impossible because $\dim F \leq n$. Hence $P_n^t(B) \leq r$ and so $\delta_n^*(B) \geq P_n^t(B)$. □

2.7 Corollary

For a bounded absolutely convex subset B of a normed space E, the following properties are equivalent:

a) $\delta_n^*(B) = 0$

b) *There exists $t \in (0,1]$ such that* $P_n^t(B) = 0$

c) $\dim[B] \leq n$.

Proof. (c) $\Longrightarrow$ (a) is obvious and (a) $\Longrightarrow$ (b) follows from 2.6.

(b) $\Longrightarrow$ (c). Since $P_n^t(B) = 0$, every t-orthogonal subset $Y \subset B - \{0\}$ has no more than n elements.

Let us prove that $\dim[B] \leq n$. If not, we could find $n+1$ nonzero elements in $[B]$ which are t-orthogonal. Since $[B] = \bigcup_{\lambda \in \mathbb{K}} \lambda B$, this would imply the existence of a t-orthogonal subset of $B - \{0\}$ with $n+1$ elements. This contradiction completes the proof. □

The above corollary 2.7 does not hold if we drop the hypotesis of convexity as the example 2.9 shows. In this case we must replace the property (b) by a stronger one.

2.8 Corollary

For a bounded subset B of a normed space E, the following properties are equivalent:

a) $\delta_n^*(B) = 0$
b) *There exists a sequence (t_m) in $(0,1]$ with $t_m \longrightarrow 0$ such that $P_n^{t_m}(B) = 0$ for all m*
c) $\dim[B] \leq n$.

Proof. Since the other implications are obvious, we only need to show that (b) $\Longrightarrow$ (c). So assume (b) and suppose that $\dim[B] > n$. Then, there are $n+1$ linearly independent elements $x_1, \ldots, x_{n+1}$ of B. There exists $t \in (0,1]$ such that $\{x_1, \ldots, x_{n+1}\}$ is t-orthogonal. Then $\{x_1, \ldots, x_{n+1}\}$ is t_m-orthogonal, if $t_m \leq t$, which contradicts theorem 2.4. Hence the result follows. □

2.9 EXAMPLE. One cannot expect an equality in the formula of proposition 2.6. Let, for instance, $\mathbb{K}$ be not spherically complete and let $\mathbb{K}_\nu^2$ be the space considered in [5, p. 68]. Since the space $\mathbb{K}_\nu^2$ does not contain any two nonzero elements that are orthogonal, we deduce that $P_n(B) = 0$ for any B and $n \geq 1$ whereas $\delta_1^*(B) \neq 0$ if $\dim[B] > 1$.

2.10 EXAMPLE. Unlike δ_n^*, $P_n^t(B)$ can be different from $P_n^t(\mathrm{co}(B))$. Indeed, let $\mathbb{K}$ be any field and let $E = \mathbb{K}^2$ with the usual (product) norm. Let x_1, x_2 be two linearly independent elements of E which are not t-orthogonal. Then, if $B = \{x_1, x_2\}$, $P_1^t(B) = 0$ whereas $P_1^t(\mathrm{co}(B)) > 0$.

3. The main result

The aim of this paragraph is to get a better relationship between $P_n^t(B)$ and $\delta_n^*(B)$ than the one obtained in 2.6. Because of 2.10 we are going to restrict ourselves to absolutely convex bounded sets B. Also, when $t = 1$, we will need some additional properties on the space E because of 2.9 (see corollary 3.8 below).

First we need the following lemmas which are quite similar to the ones given in [5, p. 137] (within the proof of lemma 4.36).

3.1 Lemma

Let B be a bounded absolutely convex subset of a pseudoreflexive normed space E over $\mathbb{K}$ and let $0 < t < 1$. Let $d_0 = 1 < d_1 < \cdots < d_{n+1}$ be such that $d_1 d_2 \cdots d_{n+1} < d$, where $d = t^{-1}$ if the valuation of $\mathbb{K}$ is dense, and $d = \min\{|\pi|^{-1}, t^{-1}\}$ if the valuation is discrete. Choose β, $\alpha_1, \ldots, \alpha_{n+1}$ in $\mathbb{K}$ and $v_1, \ldots, v_{n+1}$ in $\mathbb{R}$ as follows:

(i) if the valuation is dense, we take $|\beta| > 1$ and then choose $v_k = |\alpha_k| > 1$, where $\alpha_1, \ldots, \alpha_{n+1}$ in $\mathbb{K}$ are such that $|\alpha_1 \cdots \alpha_{n+1}| \le |\beta|$.

(ii) if the valuation is discrete, we take

$$\beta = \alpha_1 = \cdots = \alpha_{n+1} = 1 \qquad \text{and} \qquad v_1 = \cdots = v_{n+1} = v,$$

where $v > 1$ is such that $v d_{n+1} < |\pi|^{-1}$.

Then, there is a t-orthogonal sequence $z_1, \ldots, z_{n+1}$ (eventually some z_i can be zero) in E and projections $Q_0 = I_E$, Q_1, $Q_2, \ldots, Q_{n+1}$ in $L(E)$ such that:

(1) $\|Q_k\| \le d_k$ $(k = 0, 1, \ldots, n+1)$

(2) $z_k \in Q_{k-1}(B)$ and

$$\|z_k\| \ge v_k^{-1} \sup\{\|x\| : x \in Q_{k-1}(B)\} \qquad (k = 1, \ldots, n+1)$$

(3) $R_k = Q_{k-1} - Q_k$ is a projection of E onto $[z_k]$ $(k = 1, \ldots, n+1)$

(4) $Q_m Q_k = Q_m$ if $m \ge k$ $(m, k = 0, 1, \ldots, n+1)$.

Proof. Take $Q_0 = I_E$ and suppose that $z_1, \ldots, z_{m-1}, Q_0, \ldots, Q_{m-1}$ $(1 \le m \le n+1)$ satisfy (1)–(4). Choose z_m satisfying (2). Since E is pseudoreflexive, there exists a projection Q of E onto $[z_m]$ with $\|Q\| \le d_m / d_{m-1}$. Taking $Q_m = (I_E - Q)Q_{m-1}$, we have that Q_m is a projection with $\|Q_m\| \le d_m$ and $R_m = Q_{m-1} - Q_m = Q Q_{m-1}$ is a projection onto $[z_m]$. Thus, by induction we can choose $z_1, \ldots, z_{n+1}$ and Q_0, $Q_1, \ldots, Q_{n+1}$ satisfying (1)–(4). It remains to show that $z_1, \ldots, z_{n+1}$ are t-orthogonal. To this end we first notice that $R_k(z_m) = 0$ if $m \ne k$ while $R_k(z_k) = z_k$. Since $\|R_k\| \le d_k < t^{-1}$, we have

$$\left\| \sum_{i=1}^{n+1} \lambda_i z_i \right\| \ge t \max_k \left\| R_k \left(\sum_{i=1}^{n+1} \lambda_i z_i \right) \right\| = t \max_k |\lambda_k| \, \|z_k\|$$

which completes the proof. □

3.2. Remark. Assume also that E is a normed space for which every subspace of dimension 1 is orthocomplemented. Then the conclusions of lemma 3.1 hold for $t = 1$ and $d_k = 1$ $(k = 0, 1, \ldots, n+1)$.

3.3 Lemma

With the notations of the preceding lemma, the following hold:

(a) *If the valuation of $\mathbb{K}$ is dense and if $\gamma \in \mathbb{K}$, $|\gamma| > |\beta| t^{-1}$, then*
  (i) $Q_m(B) \subset \gamma B$ $(m = 0, 1, \ldots, n+1)$
  (ii) *for $m \geq k$ we have* $\|z_m\| \leq |\beta\gamma| \|z_k\|$ $(m, k = 1, \ldots, n+1)$.

(b) *If the valuation of $\mathbb{K}$ is discrete, then*
  (iii) $Q_m(B) \subset Q_{m-1}(B) \subset B$ $(m = 1, \ldots, n+1)$
  (iv) *For $m \geq k$ we have* $\|z_m\| \leq |\pi|^{-1} \|z_k\|$ $(m, k = 1, \ldots, n+1)$.

Proof. We first observe that each $Q_m(B)$ is absolutely convex. Let now $x \in B$ and $1 \leq m \leq n+1$. There exists $\lambda \in \mathbb{K}$ (we take $\lambda = 0$ if $z_m = 0$) such that $R_m(x) = \lambda x_m$. Thus

$$|\lambda| \, \|z_m\| = \|R_m(x)\| = \|(R_m Q_{m-1})(x)\| \leq d_m \, \|Q_{m-1}(x)\| \leq d_m v_m \, \|z_m\|$$

and so $|\lambda| \leq d_m v_m$.

If the valuation of $\mathbb{K}$ is discrete, then $|\lambda| \leq d_m v < |\pi|^{-1}$ and so $|\lambda| \leq 1$, which implies that $R_m(x) \in Q_{m-1}(B)$ and therefore

$$Q_m(x) = Q_{m-1}(x) - R_m(x) \in Q_{m-1}(B).$$

Hence, for discrete valuation, we have $Q_m(B) \subset Q_{m-1}(B)$, which implies that $Q_m(B) \subset Q_0(B) = B$. Also, if $m \geq k$, then $z_m \in Q_{m-1}(B) \subset Q_{k-1}(B)$ and so

$$\|z_m\| \leq v \, \|z_k\| \leq |\pi|^{-1} \|z_k\|.$$

Assume next that the valuation is dense and let $|\gamma| > |\beta| \, t^{-1}$. Choose $\delta \in \mathbb{K}$, $|\beta| < |\delta| < t \, |\gamma|$. For each m, $1 \leq m \leq n+1$, choose $\gamma_m \in \mathbb{K}$ such that

$$d_m \, |\alpha_m| \leq |\gamma_m| \leq d_m \, |\alpha_m| \, |\delta\beta^{-1}|^{1/(n+1)}.$$

Now, for $x \in B$, we have $R_m(x) \in \gamma_m Q_{m-1}(B)$ and so

$$Q_m(x) = Q_{m-1}(x) - R_m(x) \in \gamma_m Q_{m-1}(B).$$

Therefore $Q_m(B) \subset \gamma_1 \cdots \gamma_m B$.

But

$$|\gamma_1 \cdots \gamma_m| \leq |\gamma_1 \cdots \gamma_{n+1}| \leq d_1 \cdots d_{n+1}\,|\beta|\,|\delta\beta^{-1}| \leq t^{-1}|\delta| < |\gamma|$$

and thus

$$Q_m(B) \subset \gamma B.$$

If $m \geq k$, then

$$z_m \in Q_{m-1}(B) \subset \gamma_k\gamma_{k+1}\cdots\gamma_{m-1}Q_{k-1}(B)$$

and hence

$$\|z_m\| \leq |\gamma|\,|\alpha_k|\,\|z_k\| \leq |\beta\gamma|\,\|z_k\|.$$

This completes the proof. □

3.4. Remark. Assume also that E is a normed space for which every subspace of dimension 1 is orthocomplemented. Then, if the valuation of $\mathbb{K}$ is dense the following improvement of the statement (a) of 3.3 holds:

(i) $Q_m(B) \subset \beta B$ $(m = 0, 1, \ldots, n+1)$ and
(ii) For $m \geq k$, $(m.k = 1, \ldots, n+1)$, $\|z_m\| \leq |\beta|^2\|z_k\|$.

3.5 Theorem

Let B be a bounded absolutely convex subset of a pseudoreflexive normed space E over $\mathbb{K}$ and let $0 < t < 1$.

(1) If the valuation of $\mathbb{K}$ is dense, then

$$P_n^t(B) \leq \delta_n^*(B) \leq t^{-1}P_n^t(B).$$

(2) If the valuation of $\mathbb{K}$ is discrete, then

$$P_n^t(B) \leq \delta_n^*(B) \leq |\pi|^{-1}t^{-1}P_n^t(B).$$

Proof. (1) Let $t < s < 1$ and let β, γ, α_k, z_k, Q_k, R_k, $k = 1, \ldots, n+1$, be as in 3.1 and 3.3 for this choice of s. Let $b_k = \gamma^{-1}z_k$, $k = 1, \ldots, n$ and $b_{n+1} = \beta^{-1}\gamma^{-2}z_{n+1}$. Then $b_1, \ldots, b_{n+1}$ are elements of B which are s-orthogonal and hence t-orthogonal. Since $\|b_{n+1}\| \leq \|b_k\|$, for $k < n+1$, we must have

$$\|b_{n+1}\| \leq rt^{-1} \qquad \text{where } r = P_n^t(B).$$

For $x \in B$, we have

$$\left\| x - \sum_{i=1}^{n} R_i(x) \right\| = \|Q_n(x)\| \le |\alpha_{n+1}| \, \|z_{n+1}\| \le |\beta\gamma|^2 \|b_{n+1}\| \le |\beta\gamma|^2 r t^{-1} = m.$$

If $F = [z_1, \ldots, z_n]$, then $B \subset F + B(0, m)$ and so

$$\delta_n^*(B) \le m = |\beta\gamma|^2 r t^{-1}$$

Let $\epsilon > 0$. If we had chosen β, γ, s such that

$$|\beta| < 1 + \epsilon, \qquad s > \frac{1}{1+\epsilon}, \qquad |\gamma| < \epsilon + |\beta| \, s^{-1} < \epsilon + (1+\epsilon)^2,$$

then we would have

$$\delta_n^*(B) \le (1+\epsilon)^2 \left[(1+\epsilon)^2 + \epsilon\right]^2 r t^{-1}.$$

Taking $\epsilon \longrightarrow 0$, we get

$$\delta_n^*(B) \le r t^{-1} = t^{-1} P_n^t(B).$$

(2) Let $s = t$ and let z_k, Q_k, R_k be as in 2.10. Take $b_k = z_k$, $k = 1, \ldots, n$ and $b_{n+1} = \pi z_{n+1}$. Then $b_1, \ldots, b_{n+1}$ are t-orthogonal elements of B and $\|b_{n+1}\| \le \|b_k\|$ if $k \le n+1$. If $r = P_n^t(B)$, then we must have $\|b_{n+1}\| \le r t^{-1}$. Now for $x \in B$, we have

$$\left\| x - \sum_{i=1}^{n} R_i(x) \right\| = \|Q_n(x)\| \le v \, \|z_{n+1}\| \le v \, |\pi|^{-1} r t^{-1}$$

and so

$$\delta_n^*(B) \le v \, |\pi|^{-1} r t^{-1}.$$

Taking $v \longrightarrow 1^+$, we get $\delta_n^*(B) \le |\pi|^{-1} r t^{-1}$, which completes the proof. □

Remarks

3.6. Since the values of δ_n^* and P_n^t do not depend on the space in which B is embeded, we can replace in the above theorem the hypothesis "E is pseudoreflexive" by "$[B]$ is pseudoreflexive".

3.7. Theorem 2.5 does not hold for $t = 1$ (see 2.8). However, taking into account this theorem as well as 3.2 and 3.4 we can easily prove the following result.

3.8 Corollary

Let B be a bounded absolutely convex subset of a normed space E over $\mathbb{K}$ in which every subspace of dimension 1 is orthocomplemented. Then:

(1) if the valuation of $\mathbb{K}$ is dense, then:

$$\delta_n^*(B) = P_n(B)$$

(2) if the valuation of $\mathbb{K}$ is discrete, then

$$P_n(B) \leq \delta_n^*(B) \leq |\pi|^{-1} P_n(B).$$

3.9 Corollary

If B is a bounded absolutely convex subset of a pseudoreflexive normed space E over $\mathbb{K}$ and if the valuation of $\mathbb{K}$ is dense, then

$$\delta_n^*(B) = \lim_{t \to 1^-} P_n^t(B).$$

3.10 Corollary

Let B be a bounded absolutely convex subset of a pseudoreflexive normed space E over $\mathbb{K}$. Then:

(a) if the valuation of $\mathbb{K}$ is dense, then

$$\delta_n^*(B) = \sup_M \delta_n^*(B \cap M),$$

where M can be any $n+1$-dimensional linear subspace of E

(b) if the valuation of $\mathbb{K}$ is discrete, then

$$\sup_M \delta_n^*(B \cap M) \leq \delta_n^*(B) \leq |\pi|^{-1} \sup_M \delta_n^*(B \cap M),$$

with M as in (a).

Proof. Let d be this supremum. It is obvious that $d \leq \delta_n^*(B)$.

(a) Assume that the valuation on $\mathbb{K}$ is dense. If $d < \delta_n^*(B)$, then by 3.5 and 3.9 there would exist $t \in (0,1)$ such that $d < P_n^t(B) \leq \delta_n^*(B)$. By 2.5 there exists a vector subspace M with dimension $n+1$ such that $d < P_n^t(B \cap M) \leq P_n^t(B)$. And this last inequality contradicts 3.5.

(b) Assume that the valuation on $\mathbb{K}$ is discrete. Then

$$\delta_n^*(B) \leq |\pi|^{-1}t^{-1}P_n^t(B) = |\pi|^{-1}t^{-1}\sup_M P_n^t(B \cap M) \leq |\pi|^{-1}t^{-1}d. \ \square$$

From now on, we are going to assume that the space E is not necessarily pseudoreflexive. In this case we are able to give some partial counterparts to theorem 3.5 and to corollary 3.8.

3.11 Theorem

Let B be a bounded absolutely convex subset of any normed space E over $\mathbb{K}$, $t \in (0,1)$ and n a non-negative integer. If for some $m > n$, $\delta_m^*(B) < \delta_n^*(B)$, then

$$\delta_n^*(B) \geq P_n^t(B) \geq t\delta_n^*(B).$$

Moreover, if every subspace of dimension 1 is orthocomplemented, then $\delta_n^*(B) = P_n(B)$.

Proof. Set

$$p := \min\{m \in \mathbb{N} : \delta_m^*(B) < \delta_n^*(B)\}.$$

Then

$$\delta_p^*(B) < \delta_{p-1}^*(B) = \cdots = \delta_n^*(B)$$

Take $r \in (\delta_p^*(B), \delta_n^*(B))$ and $\lambda \in \mathbb{K}$ as in the statement of the theorem 1.5. Then, there exists a t-orthogonal sequence $(a_1, \ldots, a_p)$ in λB such that

$$B \subset \operatorname{co}\{a_1, \ldots, a_p\} + B(0,r).$$

Also, $\|a_1\|, \ldots, \|a_p\| > r$ because $r < \delta_{p-1}^*(B)$. So, $Y_p = \{\lambda^{-1}a_1, \ldots, \lambda^{-1}a_p\}$ is a t-orthogonal subset of B and $\#Y_p = p > n$. If we choose a subset Y of Y_p with $n+1$ elements,

$$\inf\{\|y\| : y \in Y\} > |\lambda|^{-1}r$$

and hence $P_n^t(B) > t|\lambda|^{-1}r$ by 2.4. Since the last inequality holds for all $r \in (\delta_p^*(B), \delta_n^*(B))$ and for all λ as in 1.5, we deduce $P_n^t(B) \geq t\delta_n^*(B)$. The second part of the theorem follows from 1.7. $\square$

3.12 Theorem

Let B be a bounded absolutely convex subset of any normed space E over $\mathbb{K}$, $t \in (0,1)$ and $n \in \mathbb{N}$. If $\delta_n^*(B) < \delta_{n-1}^*(B)$, then

$$\delta_n^*(B) \geq P_n^t(B) \geq t\delta_n^*(B).$$

Proof. Take $r' \in (\delta_n^*(B), \delta_{n-1}^*(B))$. By 1.5 there exist $a_1, \ldots, a_n \in \lambda B$ such that

$$B \subset \text{co}\{a_1, \ldots, a_n\} + B(0, r'),$$

$\|a_1\|, \ldots, \|a_n\| > r'$, and $\{a_1, \ldots, a_n\}$ is $t^{1/2}$-orthogonal.

Set $F := [a_1, \ldots, a_m]$ and assume whithout loss of generality that $\delta_n^*(B) \neq 0$. Then, if $0 < r'' < \delta_n^*(B)$, B is not contained in $F + B(0, r'')$. Hence, there exists $b \in B$ such that $\|b - f\| > r''$ for all $f \in F$. Let

$$b = \sum_{i=1}^{n} \mu_i a_i + a_{n+1}, \qquad |\mu_i| \leq 1 \ (i = 1, \ldots, n), \quad \|a_{n+1}\| \leq r'.$$

It is obvious that $\|a_{n+1}\| > r''$. Also

$$a_{n+1} = b - \sum_{i=1}^{n} \mu_i a_i \in \lambda B.$$

Moreover, for any $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$,

$$\left\| a_{n+1} - \sum_{i=1}^{n} \lambda_i a_i \right\| = \left\| b - \sum_{i=1}^{n} \mu_i a_i - \sum_{i=1}^{n} \lambda_i a_i \right\| > r'' \geq \|a_{n+1}\| \frac{r''}{r'}$$

which shows that for $r''/r' \geq t^{1/2}$, $\{a_1, \ldots, a_n, a_{n+1}\}$ is t-orthogonal [5, Lemma 3.2]. Also

$$Y = \left\{\lambda^{-1} a_1, \ldots, \lambda^{-1} a_n, \lambda^{-1} a_{n+1}\right\} \subset B$$

and

$$\inf\left\{\|y\| : y \in Y\right\} > |\lambda|^{-1} r''.$$

This implies $P_n^t(B) > t|\lambda|^{-1} r''$ and hence $P_n^t(B) \geq t\delta_n^*(B)$. □

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References

1. S. Caeneepel and W.H. Schikhof, Two elementary proofs of Katsaras' theorem on p-adic compactoids, in *Proceedings of the Conference on p-adic Analysis*, ed. by N. De Grande-De Kimpe and L. Van Hamme, pp. 41–44, Hengelhoef, 1986.
2. A. K. Katsaras, On non-archimedean locally convex spaces, *Bull. Greek Math. Soc.*, **29** (1990), 61–83.
3. A. Pietsch, *Nuclear Locally Convex Spaces*, Springer, 1972.
4. A. C. M. van Rooij, *Notes on p-Adic Banach Spaces I–VI*, Math. Inst., Katholieke Universiteit, Reports 7633 and 7725, Nijmegen, 1976–77.
5. A. C. M. van Rooij, *Non-archimedean Functional Analysis*, Marcel Dekker, New York, 1978.
6. W. H. Schikhof, *Finite Dimensional Ultrametric Convexity*, Math. Inst., Katholieke Universiteit, Report 8538, Nijmegen, 1985.
7. W. H. Schikhof, *A Complementary Variant of c-Compactness in p-Adic Funtional Analysis*, Math. Inst., Katholieke Universiteit, Report 8647, Nijmegen, 1985.