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On certain multidimensional generalized Kober operators

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ABSTRACT

In this paper, we introduce certain multidimensional generalized Kober operators associated with the Gauss's hypergeometric function, which provide an elegant multivariate analogue of the operators introduced by Saxena and Kumbhat [16].

1. Introduction

The object of this paper is to establish some theorems for the multidimensional generalized Kober operators introduced in this paper.

These operators are the extensions of the operators of fractional integration defined and studied earlier by many authors, notably by Saxena and Kumbhat [16–19], Kober [6], Saxena [13], Kalla [2–5], Kiryakova [7], Saigo [11] and several others.

In what follows, the symbol $f[(x)]$ will be used to represent $f(x_1,\ldots,x_n)$ and $\phi[(t/x)]$ to represent $\phi(t_1/x_1,\ldots,t_n/x_n)$.

2. The multidimensional generalized Kober operators

The multidimensional generalized Kober operators are defined in the following manner:

$$R\{f[(x)]\} = R\{\alpha_j,\beta_j,\eta_j,\delta_j,a_j : f[(x)]\}$$

$$= \prod_{j=1}^{n} \left[\frac{x_j^{-\eta_j-\delta_j}}{\Gamma(\delta_j)} \int_0^{x_1} \cdots \int_0^{x_n} t_j^{\eta_j} (x_j - t_j)^{\delta_j - 1} \right.$$

$$\left. \times {}_2F_1\left(\alpha_j, \beta_j; \delta_j; a_j\left(1 - \frac{t_j}{x_j}\right)\right) f[(t)]\, \phi[(t/x)]\, dt_1 \cdots dt_n \right], \quad (2.1)$$

$$K\{f[(x)]\} = K\left\{\alpha_j, \beta_j, \zeta_j, \gamma_j, a_j : f[(x)]\right\}$$

$$= \prod_{j=1}^{n} \left[\frac{x_j^{\zeta_j}}{\Gamma(\gamma_j)} \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} t_j^{-\zeta_j-\gamma_j} (t_j - x_j)^{\gamma_j - 1} \right.$$

$$\left. \times {}_2F_1\left(\alpha_j, \beta_j; \gamma_j; a_j\left(1 - \frac{x_j}{t_j}\right)\right) f[(t)]\, \phi[(x/t)]\, dt_1 \cdots dt_n \right], \quad (2.2)$$

where the kernel $\phi[(t)]$ is such that the integrals make sense. ${}_2F_1(\alpha_j, \beta_j; \gamma_j; x)$ is the Gauss' hypergeometric function; α_j, β_j, γ_j, δ_j, η_j and ζ_j are complex numbers.

In what follows j varies from 1 to n.

The operators exist under the following sets of conditions:

(i) $1 \le p_j;\ q_j < \infty;\ p_j^{-1} + q_j^{-1} = 1$

(ii) $\Re(\eta_j) > -q_j^{-1}$, $\Re(\delta_j) > 0$, $\Re(\beta_j) > 0$, $\Re(\delta_j - \alpha_j - \beta_j) > 0$

(iii) $\Re(\zeta_j) > -p_j^{-1}$, $\Re(\gamma_j) > 0$, $\Re(\gamma_j - \alpha_j - \beta_j) > 0$

(iv) $f[(x)] \in [L_{p_j}(0, \infty)]$,

where $[L_{p_j}(0, \infty)]$ denotes the space of Lebesgue integrable functions of n variables, or more exactly, with p_j–th power integrable with respect to each of the variables x_j.

This is a space with 'mixed norm'. This norm for two variables is pointed out in [12].

3. The multidimensional Mellin transform

The multidimensional Mellin transform pair, required in the analysis that follows, is given by

$$F[(s)] = M\{f[(x)]\} = \int_0^{\infty} \cdots \int_0^{\infty} f[(t)] \prod_{j=1}^{n} \left(t_j^{s_j - 1}\right) dt_1 \cdots dt_n \qquad (3.1)$$

and

$$f[(t)] = \frac{1}{(2\pi i)^n} \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_n - i\infty}^{c_n + i\infty} F[(s)] \prod_{j=1}^{n} \left(t_j^{-s_j}\right) ds_1 \cdots ds_n, \qquad (3.2)$$

under suitable conditions on the parameters and the variables.

4. Special cases

The following special cases of the operators are worth mentioning:

(i) For $a_j = 0$, $n = 1$ and $\phi(x) \equiv 1$, we obtain the classical Erdélyi-Kober operator:

$$R[f(x)] = I^{\eta,\delta} f(x) = \frac{x^{-\eta-\delta}}{\Gamma(\delta)} \int_0^x t^\eta (x-t)^{\delta-1} f(t)\, dt. \tag{4.1}$$

It is well known [1] that a sufficient condition for $I^{\eta,\delta} f \in L_p$ is that $f \in L_p$, $\Re(\eta) > -1/q$ and $R(\delta) > 0$.

(ii) For $a_j = 1$, $n = 1$, $\phi(x) = 1$, we obtain the operators defined by Saxena and Kumbhat [16]:

$$R[f(x)] = \frac{x^{-\eta-\delta}}{\Gamma(\delta)} \int_0^x t^\eta (x-t)^{\delta-1} {}_2F_1\left(\alpha,\beta;\delta;1-\frac{t}{x}\right) f(t)\, dt. \tag{4.2}$$

This case is considered first by E. R. Love [8]. In the special case $\alpha = \eta+\delta$, (4.2) reduces to the operators considered by M. Saigo [11]. They may also be obtained as a special case of the operators of general fractional integration of V. Kiryakova [7] (for $m = 2$).

(iii) Putting $a_j = 1/\alpha_j$ and taking the limit as α_j tends to ∞ we obtain the following operators:

$$\begin{aligned} R\{f[(x)]\} &= R\left\{\alpha_j,\beta_j,\eta_j,\delta_j,\frac{1}{\alpha_j} : f[(x)]\right\} \\ &= \prod_{j=1}^n \left[\frac{x_j^{-\eta_j-\delta_j}}{\Gamma(\delta_j)} \int_0^{x_1} \cdots \int_0^{x_n} t_j^{\eta_j} (x_j - t_j)^{\delta_j - 1}\right. \\ &\qquad \left. \times {}_1F_1\left(\beta_j;\delta_j;a_j\left(1-\frac{t_j}{x_j}\right)\right) f[(t)]\, \phi[(t/x)]\, dt_1 \cdots dt_n\right] \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} K\{f[(x)]\} &= K\left\{\alpha_j,\beta_j,\zeta_j,\gamma_j,\frac{1}{\alpha_j} : f[(x)]\right\} \\ &= \prod_{j=1}^n \left[\frac{x_j^{\zeta_j}}{\Gamma(\gamma_j)} \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} t_j^{-\zeta_j-\gamma_j} (t_j - x_j)^{\gamma_j - 1}\right. \\ &\qquad \left. \times {}_1F_1\left(\beta_j;\gamma_j;a_j\left(1-\frac{x_j}{t_j}\right)\right) f[(t)]\, \phi[(x/t)]\, dt_1 \ldots dt_n\right] \end{aligned} \tag{4.4}$$

where the kernel $\phi[(t)]$ is such that the integrals make sense and the following conditions hold:

(i) $1 \leq p_j;\ q_j < \infty;\ p_j^{-1} + q_j^{-1} = 1$
(ii) $\Re(\eta_j) > -q_j^{-1},\ \Re(\delta_j) > 0$
(iii) $\Re(\gamma_j) > 0,\ Re(\zeta_j) > -p_j^{-1}$.

5. Theorems

Theorem 1

If $f[(x)] \in [L_{p_j}(0,\infty)]$, $1 \leq p_j \leq 2$ (or $f[(x)] \in [M_{p_j}(0,\infty)]$ with $p_j > 2$), $p_j^{-1} + q_j^{-1} = 1$, $\Re(\eta_j) > -q_j^{-1}$, $R(\gamma_j) > 0$, $R(\delta_j) > 0$, $\Re[\gamma_j - \alpha_j - \beta_j] > 0$, $\Re[\delta_j - \alpha_j - \beta_j] > 0$, for $j \in (1,2,\ldots,n)$, and the integrals involved are absolutely convergent, then the following result holds:

$$M\left[R\{\alpha_j,\beta_j,\eta_j,\delta_j,a_j : f[(x)]\}\right] = Mf[(x)]\,K[\alpha_j,\beta_j,\eta_j - s_j + 1,\delta_j,a_j : 1], \tag{5.1}$$

where $[M_{p_j}(0,\infty)]$ denotes the class of all functions $f[(x)]$ of $[L_{p_j}(0,\infty)]$ with $p_j > 2$ which are inverse Mellin transforms of functions of $[L_{q_j}(-\infty,\infty)]$.

Proof.

$$\begin{aligned} M\Big[R\{\alpha_j,\beta_j,\eta_j,\delta_j,a_j : f[(x)]\}\Big] &= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \Bigg[x_j^{s_j-1}\Bigg\{\frac{x_j^{-\eta_j-\delta_j}}{\Gamma(\delta_j)} \\ &\times \int_0^{x_1}\cdots\int_0^{x_n} t_j^{\eta_j}(x_j - t_j)^{\delta_j-1} \\ &\times {}_2F_1\left(\alpha_j,\beta_j;\delta_j;a_j\left(1 - \frac{t_j}{x_j}\right)\right) \\ &\times f[(t)]\,\phi[(t/x)]\,dt_1\cdots dt_n\Bigg\}dx_1\cdots dx_n\Bigg] \\ &= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \Bigg[t_j^{s_j-1} f[(t)]\,dt_1\cdots dt_n \\ &\times \Bigg\{\frac{t_j^{\eta_j-s_j+1}}{\Gamma(\delta_j)}\int_{t_1}^\infty\cdots\int_{t_n}^\infty x_j^{s_j-\eta_j-\delta_j-1} \\ &\times (x_j - t_j)^{\delta_j-1}{}_2F_1\left(\alpha_j,\beta_j;\delta_j;a_j\left(1 - \frac{t_j}{x_j}\right)\right) \\ &\times \phi[(t/x)]\,dx_1\cdots dx_n\Bigg\}\Bigg], \end{aligned}$$

Interchanging the order of integration, which is permissible under the conditions stated with the theorem, (5.1) readily follows from the equations (2.2) and (3.1). □

The following two theorems can be proved in the same way.

Theorem 2

If $f[(x)] \in [L_{p_j}(0,\infty)]$, $1 \le p_j \le 2$ (or $f[(x)] \in [M_{p_j}(0,\infty)]$ with $p_j > 2$), $p_j^{-1} + q_j^{-1} = 1$, $\Re(\gamma_j) > 0$, $\Re(\delta_j) > 0$, $\Re(\zeta_j) > -p_j^{-1}$, $\Re[\gamma_j - \alpha_j - \beta_j] > 0$, $\Re[\delta_j - \alpha_j - \beta_j] > 0$, for $j \in (1,2,\ldots,n)$, and the integrals involved are absolutely convergent, then the following result holds:

$$M\big[K\{\alpha_j,\beta_j,\zeta_j,\gamma_j,a_j : f[(x)]\}\big] = Mf[(x)]\,R\big[\alpha_j,\beta_j,\zeta_j+s_j-1,\gamma_j,a_j : 1\big]. \quad (5.2)$$

Theorem 3

If $f[(x)] \in [L_{p_j}(0,\infty)]$, $p_j^{-1} + q_j^{-1} = 1$, $g[(x)] \in [L_{p_j}(0,\infty)]$, $\Re(\gamma_j) > 0$, $\Re(\delta_j) > 0$, $\Re[\gamma_j - \alpha_j - \beta_j] > 0$, $\Re[\delta_j - \alpha_j - \beta_j] > 0$, for $j \in (1,2,\ldots,n)$ and the integrals involved are absolutely convergent, then the following result holds:

$$\int_0^\infty \cdots \int_0^\infty g[(x)]\,R\{\alpha_j,\beta_j,\eta_j,\delta_j,a_j : f[(x)]\}\,dx_1\cdots dx_n$$
$$= \int_0^\infty \cdots \int_0^\infty f[(x)]\,K\{\alpha_j,\beta_j;\eta_j,\delta_j,a_j : g[(x)]\}\,dx_1\cdots dx_n. \quad (5.3)$$

Theorem 4

If $f[(x)] \in [L_{p_j}(0,\infty)]$, $1 \le p_j < 2$, (or $f[(x)] \in [M_{p_j}(0,\infty)]$ with $p_j > 2$), $p_j^{-1} + q_j^{-1} = 1$, $\Re(\eta_j) > -q_j^{-1}$, $\Re(\gamma_j) > 0$, $\Re(\delta_j) > 0$, $\Re[\gamma_j - \alpha_j - \beta_j] > 0$, $\Re[\delta_j - \alpha_j - \beta_j] > 0$, for $j \in (1,2,\ldots,n)$, the integrals involved are absolutely convergent, and

$$R\{\alpha_j,\beta_j,\eta_j,\delta_j,a_j : f[(x)]\} = g[(x)], \quad (5.4)$$

then the following result holds:

$$f[(x)] = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n t_j^{-1} g[(t)]\,h[(x/t)]\,dt_1\cdots dt_n, \quad (5.5)$$

where

$$h[(x)] = \frac{1}{(2\pi i)^n}\int_{c_1'-i\infty}^{c_1'+i\infty} \cdots \int_{c_n'-i\infty}^{c_n'+i\infty} \frac{\prod_{j=1}^n x_j^{-s_j}}{K[(s)]}\,ds_1\cdots ds_n \quad (5.6)$$

and

$$K[(s)] = K\big[\alpha_j,\beta_j,\eta_j - s_j + 1,\delta_j,a_j : 1\big]. \quad (5.7)$$

Proof. Multiplying both sides of equation (5.4) by $\prod_{j=1}^{n} x_j^{s_j-1}$, integrating from 0 to ∞ with respect to x_j and applying theorem 1, we find that

$$Mf[(x)] = \frac{Mg[(x)]}{K[(s)]}.$$

Now using the multidimensional inverse Mellin transform, we get the desired result. □

Remark. Theorem 4 is only a formal inversion formula. It is well known that such types of fractional integration operators could be inverted by 'fractional derivatives', i.e. by means of integro-differential operators [7]. One may expect to write down such an inversion formula only for special choices of the kernel function $\phi(x)$ by specializing the parameters, e.g., as an H-function of one or more variables, etc.

A similar remark also holds for Theorem 5 given below, which can be proved in the same way.

Theorem 5

If $f[(x)] \in [L_{p_j}(0,\infty)]$, $1 \leq p_j \leq 2$, (or $f[(x)] \in [M_{p_j}(0,\infty)]$ with $p_j > 2$), $p_j^{-1} + q_j^{-1} = 1$, $\Re(\gamma_j) > 0$, $\Re(\delta_j) > 0$, $\Re[\gamma_j - \alpha_j - \beta_j] > 0$, $\Re[\delta_j - \alpha_j - \beta_j] > 0$, for $j \in (1,2,\ldots,n)$, the integrals involved are absolutely convergent, and

$$K\{\alpha_j, \beta_j, \zeta_j, \gamma_j, a_j : f[(x)]\} = n^*[(x)], \tag{5.8}$$

then the following result holds:

$$f[(x)] = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{n} t_j^{-1} n^*[(t)]\, g[(x/t)]\, dt_1 \cdots dt_n \tag{5.9}$$

where

$$g[(x)] = \frac{1}{(2\pi i)^n} \int_{c_1'-i\infty}^{c_1'+i\infty} \cdots \int_{c_n'-i\infty}^{c_n'+i\infty} \frac{\prod_{j=1}^{n} x_j^{-s_j}}{\tau[(s)]}\, ds_1 \cdots ds_n \tag{5.10}$$

and

$$\tau[(s)] = R\big[\alpha_j, \beta_j, \zeta_j + s_j - 1, \gamma_j, a_j : 1\big]. \tag{5.11}$$

6. Some general properties of generalized Kober operators

The results presented in this section can be readily established with the help of the definitions (2.1) and (2.2).

$$\prod_{j=1}^{n} x_j^{-1} R\big[\alpha_j, \beta_j, \eta_j, \delta_j, a_j : f(x_j^{-1})\big] = \prod_{j=1}^{n} K\big[\alpha_j, \beta_j, \eta_j, \delta_j, a_j : f(x_j)\big], \tag{6.1}$$

$$\prod_{j=1}^{n} x_j^{-1} K\big[\alpha_j, \beta_j, \zeta_j, \gamma_j, a_j : f(x_j^{-1})\big] = \prod_{j=1}^{n} R\big[\alpha_j, \beta_j, \zeta_j, \gamma_j, a_j : f(x_j)\big]. \tag{6.2}$$

(6.3) and (6.4) given below exhibit the rules for a 'generalized' commuting of the operators R and K with power functions of x.

$$\prod_{j=1}^{n} x_j^{\lambda_j} R\big[\alpha_j, \beta_j, \eta_j, \delta_j, a_j : f(x_j)\big] = \prod_{j=1}^{n} R\big[\alpha_j, \beta_j, \eta_j - \lambda_j, \delta_j, a_j : x_j^{\lambda_j} f(x_j)\big], \tag{6.3}$$

$$\prod_{j=1}^{n} x_j^{\lambda_j} K\big[\alpha_j, \beta_j, \zeta_j, \gamma_j, a_j : f(x_j)\big] = \prod_{j=1}^{n} K\big[\alpha_j, \beta_j, \zeta_j + \lambda_j, \gamma_j, a_j : x_j^{\lambda_j} f(x_j)\big]. \tag{6.4}$$

The following two properties express the homogeneity of the operators R and K.

If

$$\prod_{j=1}^{n} R\{\alpha_j, \beta_j, \eta_j, \delta_j, a_j : f[(x)]\} = \prod_{j=1}^{n} g[(x_j)], \tag{6.5}$$

then

$$\prod_{j=1}^{n} R\{\alpha_j, \beta_j, \eta_j, \delta_j, a_j : f\big[(c_j\, x_j)\big]\} = \prod_{j=1}^{n} g\big[(c_j\, x_j)\big]. \tag{6.6}$$

If

$$\prod_{j=1}^{n} K\{\alpha_j, \beta_j, \zeta_j, \gamma_j, a_j : f\big[(x_j)\big]\} = \prod_{j=1}^{n} \phi[(x_j)], \tag{6.7}$$

then

$$\prod_{j=1}^{n} K\big\{\alpha_j, \beta_j, \zeta_j, \gamma_j, a_j : f\big[(c_j\, x_j)\big]\big\} = \prod_{j=1}^{n} \phi\big[(c_j\, x_j)\big]. \tag{6.8}$$

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