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# Dominated operators on C[0,1] and the (CRP)(*)

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ABSTRACT

We show that a B-space $E$ has the (CRP) if and only if any dominated operator $T$ from $C[0,1]$ into $E$ is compact. Hence we apply this result to prove that $c_0$ embeds isomorphically into the B-space of all compact operators from $C[0,1]$ into an arbitrary B-space $E$ without the (CRP).
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Let $E$ be a B-space. We say that $E$ has the Compact Range Property (in symbols (CRP)) if any $E$-valued countably additive measure with finite variation has relatively compact range [9]. In the paper [7], we showed that "*$E$ has the* (CRP) *if and only if, for any compact Hausdorff space $K$, any dominated operator $T : C(K) \to E$ is compact*" (see [4] for the definition of dominated operator). The purpose of this note is to show that, in order to prove that a B-space has the (CRP), it is enough to check the compactness of dominated operators on $C[0,1]$.

Once we have this result, we are able to construct a copy of $c_0$ inside of $\mathcal{K}(C[0,1], E)$, for any B-space $E$ failing the (CRP) (here $\mathcal{K}(C[0,1], E)$ denotes the B-space of all compact operators from $C[0,1]$ into $E$).

In the book [10] Talagrand stated, without any proof, the equivalence of the following two facts:

(i) *$E$ has the* (CRP),

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(ii) *any operator from* $L^1[0,1]$ *into* $E$ *is a Dunford-Pettis operator.*

We present a proof of the implication (ii) $\Longrightarrow$ (i), because we need it in the proof of our main result. Let $S$ be an arbitrary set, $\Sigma$ a $\sigma$-field of subsets of $S$ and $\nu : \Sigma \to E$ a countably additive measure with finite variation $|\nu|$. Consider a sequence $(A_n) \subset \Sigma$ and the characteristic functions $\chi_{A_n}$. It is well known that there exist $S_1 \in \Sigma$, a sub-$\sigma$-field $\Sigma_1$ of $\Sigma$ and $\mu_1$ $(= |\nu|_{|\Sigma_1})$ in such a way that $(\chi_{A_n}) \subset L^1(S_1, \Sigma_1, \mu_1)$, that in turn is a separable B-space. Hence $L^1(S_1, \Sigma_1, \mu_1)$ is isometrically isomorphic to one of the following three spaces:

(a) $\ell^1(\Gamma)$, for some set $\Gamma$ with $\mathrm{card}(\Gamma) \leq \mathrm{card}(\mathbb{N})$,

(b) $L^1[0,1]$,

(c) $\left(\ell^1(\Gamma) \oplus L^1[0,1]\right)_1$, for some set $\Gamma$ with $\mathrm{card}(\Gamma) \leq \mathrm{card}(\mathbb{N})$.

Now, we define an operator $T : L^1(S_1, \Sigma_1, \mu_1) \to E$ by

$$T(f) = \int_{S_1} f(s_1)\, d\nu, \qquad f \in L^1(S_1, \Sigma_1, \mu_1).$$

It is clear that $T(\chi_{A_n}) = \nu(A_n)$, for all $n \in N$. Since $(\chi_{A_n})$ is a relatively weakly compact subset of $L^1(S_1, \Sigma_1, \mu_1)$, if (a) is true we get that $((\chi_{A_n})$ and hence) $(\nu(A_n))$ is relatively compact, because $\ell^1(\Gamma)$ enjoys the Schur property. If (b) is true, our assumption (ii) enters into play to prove that, still, $(\nu(A_n))$ is relatively compact. If we show that the same is true under (c), we are done, thanks to the arbitrariness of $(A_n)$. Let $j$ be the existing isometric isomorphism from $L^1(S_1, \Sigma_1, \mu_1)$ onto $\left(\ell^1(\Gamma) \oplus L^1[0,1]\right)_1$. Of course, $\left(j^{-1}(\chi_{A_n})\right)$ is relatively weakly compact in $\left(\ell^1(\Gamma) \oplus L^1[0,1]\right)_1$. If

$$P_1 : \left(\ell^1(\Gamma) \oplus L^1[0,1]\right)_1 \longrightarrow \ell^1(\Gamma) \qquad \text{and} \qquad P_2 : \left(\ell^1(\Gamma) \oplus L^1[0,1]\right)_1 \to L^1[0,1]$$

are the existing projections we have

$$T(\chi_{A_n}) = (T \circ j)\left[P_1\left(j^{-1}(\chi_{A_n})\right) + P_2\left(j^{-1}(\chi_{A_n})\right)\right].$$

By virtue of the Schur property, $\left(P_1\left(j^{-1}(\chi_{A_n})\right)\right)$ is relatively compact; hence $(T \circ j)\left[P_1\left(j^{-1}(\chi_{A_n})\right)\right]$ is. On the other hand, $\left(P_2\left(j^{-1}(\chi_{A_n})\right)\right)$ is relatively weakly compact in $L^1[0,1]$, hence $(T \circ j)\left[P_2\left(j^{-1}(\chi_{A_n})\right)\right]$ is relatively compact in $E$, because of our hypothesis (ii). The proof is complete.

Now, we are ready to show the main result of the paper. We refer to [4] for the definition of dominated operators: an operator $T : C[0,1] \to E$ is a dominated operator if there exists an $E$-valued regular Borel measure $\mu$ on $K$ with finite variation such that

$$\left\|T(f)\right\|_E \leq \int_K |f(s)|\, d\mu = \|f\|_{L^1(K, Bo(K), \mu)}, \qquad f \in C[0,1].$$

**Theorem 1**

*$E$ has the (CRP) if and only if any dominated operator from $C[0,1]$ into $E$ is compact.*

*Proof.* If $E$ has the (CRP), the result follows from [7, Theorem 1]. Conversely, let us assume the compactness of any dominated operator from $C[0,1]$ into $E$. We show that any operator $T : L^1[0,1] \to E$ is Dunford-Pettis. To do this, it is enough to prove that $T \circ I : L^\infty[0,1] \to E$ is compact (here $I$ denotes the embedding of $L^\infty[0,1]$ into $L^1[0,1]$). If we consider

$$T_1 = T \circ I_{|C[0,1]} : C[0,1] \longrightarrow E,$$

it is very easy to see that $T_1$ is dominated and hence compact by our hypothesis. Now, let $f$ be an element of the unit ball $B_\infty$ of $L^\infty[0,1]$. Using Lusin Theorem and Tiezte Extension Theorem it is quite easy to construct a sequence $(f_n)$ in the unit ball $B$ of $C[0,1]$ such that $\|f_n - f\|_1 \longrightarrow 0$. Hence $(T \circ I)(B_\infty) \subseteq \overline{T_1(B)}$ thanks to the definition of dominated operator. So we obtain the compactness of $T \circ I$. The proof is over. □

Now, we apply the above result to the construction of an isomorphic copy of $c_0$ inside of the B-space of all compact operators from $C[0,1]$ into a B-space $E$ without the (CRP). In the following result we need some properties of 1-absolutely summing and 2-absolutely summing operators (for the definitions the reader can look at [3]). We make use of the following facts: ($\alpha$) any dominated operator on $C[0,1]$ is 1-absolutely summing [2, pp. 183–184], ($\beta$) any 1-absolutely summing operator is 2-absolutely summing [3].

**Theorem 2**

*Let $E$ be a B-space without (CRP). Then $c_0$ embeds into $\mathcal{K}(C[0,1], E)$.*

*Proof.* Since $E$ doesn't possess the (CRP), there is a dominated operator $T$ from $C[0,1]$ into $E$ that is not compact. From ($\alpha$) and ($\beta$) it follows that $T$ is a 2-absolutely summing operator. From the Grothendieck-Pietsch Domination Theorem [3, p. 60] there exists a regular Borel probability measure $\mu$ defined on $B_{C^*[0,1]}$ (in its w* topology) for which $T$ factorizes through a closed subspace $X$ of $L^2(\mu)$ in the following way

$$\begin{array}{ccc} C[0,1] & \xrightarrow{T} & E \\ {}_G\searrow & & \nearrow_R \\ & X & \end{array}$$

where $G$ and $R$ are suitable (non compact) operators. We note that $X$ is the closure in the $L^2$-norm of $C[0,1]$ and hence it is a separable Hilbert space. Hence $X$ has an unconditional basis $(e_n)$. If $P_k : X \to \overline{\text{span}}(e_k)$, $k \in N$, is the existing projection, then the series $\sum_{i=1}^{\infty}(P_iG)(x)$ converges unconditionally to $G(x)$; further, any operator $P_iG$ is compact, because it admits a finite dimensional range. Similarly, the series $\sum_{i=1}^{\infty}(RP_iG)(x)$ converges unconditionally to $T(x)$ and $RP_iG$ is compact, for each $i \in N$. However, $\sum_{i=1}^{\infty} RP_iG$ doesn't converge in norm to $T$, because $T$ is not compact. Hence, we can proceed as in [8] to get the copy of $c_0$ inside of $\mathcal{K}(C[0,1],E)$; since the paper [8] is still unpublished we hint at the construction of the $c_0$-copy. Since $\sum_{i=1}^{\infty} RP_iG$ doesn't converge to $T$, there is a $\eta > 0$ and two subsequences $(n_k)$ and $(m_k)$ of $\mathbb{N}$, with $m_k < n_k < m_{k+1}$ for all $k \in N$, so that

$$\left\| \sum_{i=m_k}^{n_k} RP_iG \right\| > \eta \qquad \text{for all } k \in N.$$

Let $\mathcal{F}$ be the field of finite subsets of $\mathbb{N}$ and their complements.

Hence we can define a vector measure $\psi : \mathcal{F} \to \mathcal{K}(C[0,1],E)$ by putting

$$\psi(\Delta) = \sum_{k\in\Delta} \left[ \sum_{i=m_k}^{n_k} RP_iG \right]$$

if $\Delta$ is finite, and

$$\psi(\Delta) = \sum_{k\in\Delta^c} \left[ -\sum_{i=m_k}^{n_k} RP_iG \right]$$

if $\Delta^c$ is finite. It is quite easy to see that $\psi$ is a well-defined vector measure, that is not strongly additive. Hence a theorem due to Diestel and Faires [1] allows us to conclude the proof. □

**Corollary**

*Assume that $Y$ is a Banach space containing a copy of $\ell^1$. Then $c_0$ embeds into $\mathcal{K}(C[0,1],Y^*)$.*

*Proof.* If $Y$ contains a copy of $\ell^1$, $Y^*$ fails to possess the weak Radon Nikodym property and hence the (CRP) (as remarked in [7]). □

*Remark 1.* Since it is known that for $K$ an uncountable compact metric space $C(K)$ is isomorphic to $C[0,1]$, then both Theorem 1 and Theorem 2 are true in this new setting.

*Remark 2.* If $E$ is a Gelfand-Phillips space (see [5] for a definition), $\mathcal{K}(C[0,1], E)$ is [5], and so $c_0$ embeds complementably into it, by virtue of a result in [6].

*Remark 3.* In general we can say that if $X$ and $E$ are two Banach spaces such that there exists a noncompact 2-absolutely summing operator $T$ from $X$ into $E$, then $c_0$ embeds isomorphically into $\mathcal{K}(X, E)$. For instance, it is known that if $X$ is an $\mathcal{L}_\infty$-space and $E$ is an $\mathcal{L}_1$-space, then any operator from $X$ into $E$ is 2-absolutely summing. Hence, as soon as $\mathcal{L}(X, E) \neq \mathcal{K}(X, E)$, then $c_0$ embeds into $\mathcal{K}(X, E)$.

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