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Cauchy-Riemann submanifolds of Kaehlerian Finsler spaces

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ABSTRACT

We study the geometry of the second fundamental form of a Cauchy-Riemann submanifold of a Kaehlerian Finsler space M^{2n}; any totally-real submanifold of M^{2n} with v-flat normal connection is shown to be a Berwald-Cartan space.

1. Introduction

Cauchy-Riemann (CR) submanifolds of Kaehlerian Finsler spaces were introduced by the author [9, p. 57], and have been taken recently under study by A. Bejancu [3]. He shows that v-integrable Finslerian almost complex structures correspond to Cauchy-Riemann structures (in the sense of A. Andreotti and C. D. Hill [1]) on the total space of the tangent bundle of the given (Finsler) manifold. We complete this result (cf. our theorem 1) in the following manner. If M^{2n} is a Kaehlerian Finsler space, then by a result of A. Farinola [11], its Finslerian almost complex structure is both h- and v-integrable; if moreover the nonlinear connection of the Cartan connection of M^{2n} is integrable, we find a new CR structure on $V(M^{2n}) = T(M^{2n})\setminus 0$ which is a direct summand (cf. our (2.1)) to the CR structure discovered by Bejancu [3] for the holomorphic tangent bundle of $V(M^{2n})$. Bejancu also considers CR submanifolds of Kaehlerian Finsler spaces and proves that any invariant submanifold is v-minimal; his notion of Kaehlerian Finsler space is slightly more general than that of the author [6], i.e., the ambient space is endowed with an arbitrary metrical and almost complex Finsler connection, while originally the author

[5, p. 95] has imposed the Kaehler condition $\nabla J = 0$ to the Cartan conection ∇ of the given Finsler space. As underlined by several authors (e.g., H. Rund [15]), Riemannian notions usually admit more than one significant counterpart. For instance, one may consider the problem of wether an invariant submanifold (of a Kaehlerian Finsler space) is h-minimal. We prove h-minimality under the additional assumption that the submanifold is totally-geodesic. This assumption is equivalent to the vanishing of the normal curvature vector of the submanifold, but does not yield the vanishing of the entire horizontal second fundamental form such that in general totally-geodesic submanifolds of a Finsler space fail to be h-minimal. We examine invariant submanifolds in complex v-space forms (see our theorem 2). Finally, we consider totally-real submanifolds (with either v-flat or h-flat normal connection) and generic submanifolds of a Kaehlerian Finsler space (cf. our theorems 4–6).

2. Statement of results

Let (M^{2n}, L, J) be a Kaehlerian Finsler space [6], and $(\pi^{-1}TM^{2n}, g)$ its induced Riemannian bundle, cf. our §3. By a recent result [11], the torsion tensor N_j^y (given by our (3.1)) vanishes. Therefore we may apply [3, Theorem 2.1, p. 160] such as to conclude that $V(M^{2n}) = T(M^{2n}) \setminus 0$ admits a (naturally induced) Cauchy-Riemann (CR) structure H^v, i.e., a complex subbundle H^v of the complexified tangent bundle

$$T^c(V(M^{2n})) = T(V(M^{2n})) \otimes \mathbf{C},$$

such that i) H^v is involutive, ii) $H^v \cap \overline{H^v} = (0)$, and iii) $\Re(H^v) = \ker(d\pi)$, where $\pi : V(M^{2n}) \to M^{2n}$ is the natural projection. On the other hand, let $\tilde{J}$ be the lift of J (with respect to the nonlinear connection of the Cartan connection of (M^{2n}, L)), cf. our (3.4). Then $\tilde{J}$ is naturally extended (by $\mathbf{C}$-linearity) to $T^c(V(M^{2n}))$; let $T^{1,0}(V(M^{2n}))$ be the bundle of all eigenvectors corresponding to the eigenvalue $i = \sqrt{-1}$ of $\tilde{J}$. Leaving definitions momentarily aside, we may formulate the following:

Theorem 1

Let (M^{2n}, L, J) be a Kaehlerian Finsler space whose Cartan connection has an integrable ($R^i_{jk} = 0$) associated nonlinear connection N on $V(M^{2n})$. Then $V(M^{2n})$ admits a Cauchy-Riemann structure H^h such that $\Re(H^h) = N$ and

$$T^{1,0}(V(M^{2n})) = H^h \oplus H^v. \tag{2.1}$$

Let ∇ be the Cartan connection of the Finsler space (M^{2n}, L) and S its vertical curvature tensor. If $u \in V(M^{2n})$ and p is a 2-dimensional real subspace of the fibre $\pi_u^{-1}TM^{2n}$, let $s(p) = S_u(X,Y,X,Y)$, for some g_u-orthonormal basis $\{X,Y\}$ in p, be the *vertical sectional curvature* of (M^{2n}, L) [6, p. 97]. Let $\sigma : GF_2(M^{2n}) \to V(M^{2n})$ be the bundle of all 2-subspaces in fibres of the induced bundle $\pi^{-1}TM^{2n}$ of (M^{2n}, L). Its standard fibre is the Grassmann manifold $G_{2,2n}(\mathbf{R})$ of all 2-planes in $\mathbf{R}^{2n}$. Note that the vertical sectional curvature is a function $s : GF_2(M^{2n}) \to \mathbf{R}$ rather than a function on $V(M^{2n})$. Let now (M^{2n}, L, J) be a Kaehlerian Finsler space; a *Finslerian 2-plane* $p \in GF_2(M^{2n})$ is said to be *holomorphic* if $J(p) = p$. The restriction of s to the holomorphic 2-planes is refered to as the *holomorphic v-sectional curvature*. Then (M^{2n}, L, J) is said to be a *complex Finslerian v-space form* if there exists $c \in C^\infty(V(M^{2n}))$ such that the following equality $s = c \circ \sigma$ holds on all holomorphic 2-planes $p \in GF_2(M^{2n})$. Our §4 deals with the general notion of a Finsler connection (∇, N) on M^{2n}. If (∇, N) is a metrical Finsler connection on (M^{2n}, L, J) then its associated vertical curvature $S(X,Y,Z,W)$ is skew-symmetric in X, Y, respectively in Z, W, and thus the above procedure is easily generalized such as to yield a well defined concept of (holomorphic) v-sectional curvature. Moreover the main theorem in [6, p. 95] may be refined to the general case of a metrical Finsler conection (∇, N) on M^{2n}, $n \geq 2$, provided that J is parallel with respect to ∇. Moreover, if the holomorphic v-sectional curvature s (constructed with respect to (∇, N)), does not depend on the 2-plane $p \subset \pi_u^{-1}TM^{2n}$ but only on the direction $u \in V(M^{2n})$, then M^{2n} is also refered to as a *complex v-space form with respect to* (∇, N). We obtain the following:

Theorem 2

Let M^m be an invariant, i.e.

$$J_u(\pi_u^{-1}TM^m) = \pi_u^{-1}TM^m, \qquad u \in V(M^m),$$

submanifold of the complex v-space form $(M^{2n}(c), L, J)$, $n \geq 2$. Then M^m is a complex v-space form (with respect to the induced connection) of the same holomorphic v-sectional curvature c if and only if $Q = 0$, i.e. then vertical second fundamental form vanishes.

It has been recently shown [3, p. 165], that any invariant submanifold M^m of a Kaehlerian Finsler space is v-minimal, i.e. $\nu = 0$, cf. our §6. It is our purpose to study h-minimality (i.e. $\mu = 0$) of invariant submanifolds. We obtain the following:

Theorem 3

Let M^m be an invariant totally-geodesic submanifold of the Kaehlerian Finsler space (M^{2n}, L, J). Then M^m is h-minimal.

In contrast with the theory of submanifolds in Riemann spaces, in general a totally-geodesic submanifold of a Finsler space is neither h- nor v-minimal. Indeed, let H be te horizontal second fundamental form of M^m in M^{2n}, cf. our §6. By a theorem of M. G. Brown [13], M^m is totally-geodesic if and only if $N_0 = 0$, $N_0 = H(v,v)$, where v is the Liouville vector of M^m. As proved by O. Varga, see [13], the vanishing of the normal curvature ($N_0 = 0$) yields the vanishing of the normal curvature vector ($N = 0$), too. Here $N(X) = H(X,v)$. Nevertheless, in general $N = 0$ does not imply the vanishing of the entire horizontal second fundamental form.

A *Berwald-Cartan* space is a Finsler space whose Cartan third curvature tensor (vertical curvature tensor) vanishes. Typical examples of Berwald-Cartan spaces are real Finsler surfaces (2-dimensional Finsler spaces) [14]. We slightly generalize this concept by calling Berwald-Cartan all submanifolds whose vertical curvature (associated with the induced connection) vanishes; clearly, such a submanifold is Berwald-Cartan in the original sense provided that the induced and intrinsic connections coincide.

Theorem 4

Let M^n be a totally-real, i.e.

$$J_u(\pi_u^{-1} TM^n) = E(\psi)_u, \qquad u \in V(M^n),$$

submanifold of the Kaehlerian Finsler space (M^{2n}, L, J). If M^n has a v-flat normal connection (i.e. $S^{\perp} = 0$) then M^n is a Berwald-Cartan space.

Theorem 5

Let M^n be a totally-real submanifold of the Kaehlerian Finsler space (M^{2n}, L, J), having an h-flat normal connection. Then the induced connection of M^n has a vanishing horizontal curvature tensor, i.e. $R = 0$.

Let (M^{2n}, J, L) be an almost Hermitian Finsler space and $(\pi^{-1}TM^{2n}, g_0)$ its induced bundle. Let us lift J, g_0 (by using the nonlinear connection N^0 of the Cartan connection), to an almost Hermitian structure $\tilde{g}_0$, $\tilde{J}$ on $V(M^{2n})$, cf. our §3. By taking the vertical, respectively the horizontal lifts, of the Finslerian distributions of a CR submanifold M^m of M^{2n} one obtains:

Theorem 6

Let $(M^m, \mathcal{D}, \mathcal{D}^\perp)$ be a CR submanifold of the almost Hermitian Finsler space (M^{2n}, L, J). Then each integral manifold $V_x(M^m) = \pi^{-1}(x)$, $x \in M^m$, of the vertical distribution $\ker(d\pi)$ on $V(M^m)$ is a CR submanifold of the almost Hermitian manifold $(V(M^{2n}), \tilde{g}_0, \tilde{J})$ whose holomorphic and totally-real distributions are $\gamma\mathcal{D}$ and $\gamma\mathcal{D}^\perp$. If additionally M^m is generic ($\dim_{\mathbf{R}} \mathcal{D}^\perp = 2n - m$) and the nonlinear connection N of its induced connection is integrable, then each maximal integral manifold S of N carries a pair of distributions $\beta\mathcal{D}$ and $\beta\mathcal{D}^\perp$ such that $\beta\mathcal{D}^\perp$ is totally-real, i.e. anti-invariant under $\tilde{J}$. Moreover $\beta\mathcal{D}$ and $\beta\mathcal{D}^\perp$ are orthogonal if and only if

$$g_0(N(X), N(Y)) = 0 \tag{2.2}$$

for any $X \in \mathcal{D}$, $Y \in \mathcal{D}^\perp$. Also $\beta\mathcal{D}$ is holomorphic (with respect to $\tilde{J}$) if and only if

$$N \circ J = J \circ N, \tag{2.3}$$

i.e. the normal curvature vector and the Finslerian almost complex structure commute. Consequently if (2.2) and (2.3) hold, then $(S, \beta\mathcal{D}, \beta\mathcal{D}^\perp)$ is a CR submanifold of $(V(M^{2n}), \tilde{g}_0, \tilde{J})$.

3. Complex Finsler structures

Let (M^{2n}, L), $n \geq 1$; be a real $2n$-dimensional Finsler space with the Lagrangian function $L : T(M^{2n}) \to [0, +\infty)$. Here $T(M^{2n}) \to M^{2n}$ denotes the tangent bundle over M^{2n}. Let $j : M^{2n} \to T(M^{2n})$ be the natural imbedding, i.e.

$$j(x) = 0_x \in T_x(M^{2n}), \qquad x \in M^{2n}.$$

We put $V(M^{2n}) = T(M^{2n}) \setminus j(M^{2n})$.

Let $\pi : V(M^{2n}) \to M^{2n}$ be the natural projection and $\pi^{-1}TM^{2n} \to V(M^{2n})$ the pullback bundle of $T(M^{2n})$ by π. A bundle morphism $J : \pi^{-1}TM^{2n} \to \pi^{-1}TM^{2n}$, $J^2 = -I$, is said to be a *Finslerian almost complex structure* on M^{2n}.

Let $u \in V(M^{2n})$; then $\pi_u^{-1}TM^{2n} = \{u\} \times T_x(M^{2n})$, $x = \pi(u)$, denotes the fibre over u in $\pi^{-1}TM^{2n}$. Any ordinary almost complex structure $J : T(M^{2n}) \to T(M^{2n})$, $J^2 = -I$, admits a *natural lift* to a Finslerian almost complex structure $\bar{J}$ given by

$$J_u X = (u, J_x \tilde{\pi} X), \qquad x = \pi(u),\ X \in \pi_u^{-1}TM^{2n},\ u \in V(M^{2n}).$$

Here $\tilde{\pi}$ denotes the projection onto the second factor of the product manifold $V(M^{2n}) \times T(M^{2n})$.

Let (U, x^i) be a system of local coordinates on M^{2n} and let $(\pi^{-1}(U), x^i, y^i)$ be the naturally induced local coordinates on $V(M^{2n})$. The *vertical lift* is the bundle isomorphism $\gamma : \pi^{-1}TM^{2n} \to \ker(d\pi)$ defined by

$$\gamma X_i = \dot{\partial}_i, \qquad \dot{\partial}_i = \frac{\partial}{\partial y^i}.$$

Clearly, the definition of γ does not depend upon the choice of local coordinates. Here

$$X_i(u) = \left(u, \frac{\partial}{\partial x^i}\Big|_{\pi(u)}\right), \qquad u \in \pi^{-1}(U),\ 1 \leq i \leq 2n.$$

To make this construction precise, let us note that any tangent vector field X on M^{2n} admits a *natural lift* $\dot{X}$ to a cross-section of $\pi^{-1}TM^{2n}$ defined by

$$\dot{X}(u) = \big(u, X(\pi(u))\big), \qquad u \in V\big(M^{2n}\big).$$

Note that X_i are the natural lifts of the (locally defined) tangent vector fierlds $\partial/\partial x^i$. Clearly $\{X_i\}_{1 \leq i \leq 2n}$ form a (local) frame of $\pi^{-1}TM^{2n}$.

Let

$$g_{ij} = \frac{1}{2}\,\dot{\partial}_i\dot{\partial}_j L^2.$$

We put

$$g_u(X, Y) = g_{ij}(u)X^iY^j, \qquad X = X^iX_i,\ Y = Y^iX_i,\ u \in \pi^{-1}(U),$$

for any cross-sections X, Y in $\pi^{-1}TM^{2n}$. The definition of g_u does not depend upon the choice of local coordinates around $x = \pi(u)$, $u \in M^{2n}$. Since (M^{2n}, L) is a Finsler space, for each $u \in \pi^{-1}(U)$ the quadratic form $g_{ij}(u)\xi^i\xi^j$ is positive definite. Therefore $\pi^{-1}TM^{2n}$ turns into a Riemannian vector bundle with the Riemann (bundle) metric $g : u \to g_u$. Then $(\pi^{-1}TM^{2n}, g)$ is called *induced bundle* of the Finsler manifold (M^{2n}, L). Cross-sections in the induced bundle are refered to as *Finsler vector fields* on M^{2n}.

Let J be a Finslerian almost complex structure on M^{2n}. Since the induced bundle and the *vertical bundle* $\ker(d\pi)$ (over $V(M^{2n})$) are isomorphic, $\ker(d\pi)$ turns into a Riemannian bundle in a natural way, with the metric $g^v(Z, W) = g(\gamma^{-1}Z, \gamma^{-1}W)$ for any vertical tangent vector fields Z, W on $V(M^{2n})$. Moreover $\ker(d\pi)$ is a complex vector bundle since each fibre $\ker(d_u\pi)$ carries the complex structure J_u^v defined by

$$J_u^v = \gamma_u \circ J_u \circ \gamma_u^{-1}, \qquad u \in V\big(M^{2n}\big).$$

Let $[J^v, J^v]$ be the torsion of the (1,1)-tensor field J^v. This is $\ker(d\pi)$-valued, since $\ker(d\pi)$ is involutive. Let $JX_i = J_i^j(x,y)X_j$. Then $J_j^i J_k^j = -\delta_k^i$. We define $N_J^v(X_i, X_j) = N_{ij}^k X_k$ where:

$$N_{jk}^i = J_j^m \frac{\partial J_k^i}{\partial y^m} - J_k^m \frac{\partial J_j^i}{\partial y^m} + J_i^m \frac{\partial J_k^m}{\partial y^j} - J_m^i \frac{\partial J_j^m}{\partial y^k}. \tag{3.1}$$

Clearly the definition of N_J^v does not depend upon the choice of local coordinates. Actually one has:

$$\gamma N_J^v(X,Y) = [J^v, J^v](\gamma X, \gamma Y) \tag{3.2}$$

for any Finsler vector fields X, Y on M^{2n}. Following [3, p. 159], if $N_J^v = 0$ then J may be called a *complex Finsler structure*. Nevertheless, this concept is not entirely satisfactory since, in this sense, the natural lift of any (not necessarily integrable) almost complex structure on M^{2n} is a complex Finsler structure, by (3.1).

A differentiable $2n$-distribution $N : u \mapsto N_u$ on $V(M^{2n})$ is called a *nonlinear connection* on $V(M^{2n})$ if each N_u is a direct summand to the vertical space $\ker(d_u\pi)$ in $T_u(V(M^{2n}))$, $u \in V(M^{2n})$. In classical tensor notation N might be represented by the Pfaffian system:

$$dy^i + N_j^i\, dx^j = 0. \tag{3.3}$$

Here $N_j^i \in C^\infty(\pi^{-1}(U))$ are the coefficients of the nonlinear connection; that is, if $u \in \pi^{-1}(U)$, then N_u is spanned by the tangent vectors

$$\frac{\delta}{\delta x^i} = \partial_i - N_i^j\, \dot\partial_j, \qquad \partial_i = \frac{\partial}{\partial x^i}.$$

Let N be a nonlinear connection on $V(M^{2n})$. We shall need the bundle epimorphism $L : T(V(M^{2n})) \to \pi^{-1}TM^{2n}$, defined by

$$L_u Z = (u, (d_u\pi)Z), \qquad Z \in T_u(V(M^{2n})),\ u \in V(M^{2n}).$$

Clearly, the restriction of L_u to N_u is a $\mathbb{R}$-linear isomorphism $N_u \approx \pi_u^{-1}TM^{2n}$. Let β_u denote its inverse. The resulting bundle isomorphism $\beta : \pi^{-1}TM^{2n} \to N$ is refered to as the *horizontal lift* (with respect to the nonlinear connection N). If J is a Finslerian almost complex structure on M^{2n} then we may put $J^h = \beta \circ J \circ L$ such as to define a complex structure on the bundle $N \to V(M^{2n})$.

We shall need the *Dombrowski mapping*, i.e. the bundle morphism

$$G : T(V(M^{2n})) \longrightarrow \pi^{-1}TM^{2n}, \qquad GZ = \gamma^{-1}Z_v,$$

where Z_v denotes the vertical part of Z with respect to the direct sum decomposition

$$T_u(V(M^{2n})) = N_u \oplus \ker(d_u\pi), \qquad u \in V(M^{2n}).$$

With any Finslerian almost complex structure J one may associate an almost complex structure $\tilde{J}$ on $V(M^{2n})$ defined by:

$$\tilde{J} = \beta \circ J \circ L + \gamma \circ J \circ G. \tag{3.4}$$

Note that the restriction of J_u to N_u, respectively to $\ker(d_u\pi)$, coincides with J_u^h, respectively with J_u^v, for any $u \in V(M^{2n})$.

Let (M^{2n}, L) be a Finsler space and $(\pi^{-1}TM^{2n}, g)$ its induced bundle. Let N be a fixed nonlinear connection on $V(M^{2n})$. The *Sasaki metric* $\tilde{g}$ is given by:

$$\tilde{g}(Z, W) = g(LZ, LW) + g(GZ, GW). \tag{3.5}$$

Thus $V(M^{2n})$ turns naturally into a (noncompact) Riemannian manifold. Moreover, $(V(M^{2n}), \tilde{g}, \tilde{J})$ is well known to be almost Hermitian.

We shall need the torsion $N_J^h(X_i, X_j) = A_{ij}^k X_k$ where:

$$A_{jk}^i = J_j^m \frac{\delta J_k^i}{\delta x^m} - J_k^m \frac{\delta J_j^i}{\delta x^m} + J_m^i \frac{\delta J_k^m}{\delta x^j} - J_i^m \frac{\delta J_j^m}{\delta x^k}. \tag{3.6}$$

Clearly the definition of N_J^h does not depend upon the choice of local coordinates. Note that:

$$N_J^h(X, Y) = L[J^h, J^h](\beta X, \beta Y) \tag{3.7}$$

for any Finsler vector fields X, Y on M^{2n}.

4. Kaehlerian Finsler spaces

Let (M^{2n}, L) be a Finsler space endowed with the Finslerian almost complex structure J. Then (M^{2n}, L, J) is said to be an *almost Hermitian Finsler space* if $g(J,X,J,Y) = g(X,Y)$, for any Finsler vector fields X, Y on $V(M^{2n})$. Let ∇ be a connection in the induced bundle $(\pi^{-1}TM^{2n}, g)$. It is said to be *metrical* if $\nabla g = 0$, respectively *almost complex* if $\nabla J = 0$. A tangent vector field Z on $V(M^{2n})$ is *horizontal* (with respect to ∇) if $\nabla_Z v = 0$. Here v denotes the *Liouville vector field*, i.e. the cross-section in the induced bundle defined by $v(u) = (u,u)$, $u \in V(M^{2n})$. Let N be the distribution of all horizontal tangent vectors on $V(M^{2n})$; it is refered to as the *horizontal distribution* of ∇. Then ∇ is *regular* if its horizontal distribution N is a nonlinear connection on $V(M^{2n})$. A pair (∇, N) consisting of a connection in $\pi^{-1}TM^{2n}$ and a nonlinear connection on $V(M^{2n})$ is called a *Finsler connection* on M^{2n}. Note that any regular connection in $\pi^{-1}TM^{2n}$ gives raise to a Finsler connection on M^{2n}.

Let (∇, N) be a Finsler connection; two concepts of torsion tensor fields are usually associated with (∇, N), namely

$$\tilde{T}(Z,W) = \nabla_Z LW - \nabla_W LZ - L[Z,W],$$

$$\tilde{T}_1(Z,W) = \nabla_Z GW - \nabla_W GZ - G[Z,W],$$

for any tangent vector fields Z, W on $V(M^{2n})$. Let also $\tilde{R}$ denote the curvature 2-form of ∇. Several fragments of $\tilde{T}$, $\tilde{T}_1$, and $\tilde{R}$ are usually derived by means of the bundle morphisms β, γ, i.e.

$$T(X,Y) = \tilde{T}(\beta X, \beta Y), \qquad C(X,Y) = \tilde{T}(\gamma X, \beta Y),$$

$$R^1(X,Y) = \tilde{T}_1(\beta X, \beta Y), \qquad P^1(X,Y) = \tilde{T}_1(\gamma X, \beta Y),$$

$$S^1(X,Y) = \tilde{T}_1(\gamma X, \gamma Y), \qquad R(X,Y)Z = \tilde{R}(\beta X, \beta Y)Z,$$

$$P(X,Y)Z = \tilde{R}(\gamma X, \beta Y)Z, \qquad S(X,Y)Z = \tilde{R}(\gamma X, \gamma Y)Z.$$

We may define no 'vertical' component of $\tilde{T}$ since clearly $\tilde{T}(\gamma X, \gamma Y) = 0$ for any Finsler vector fields X, Y on M^{2n}. Note that:

$$\gamma R^1(X,Y) = [\beta X, \beta Y], \tag{4.1}$$

i.e. R^1 is the obstruction towards the integrability on N. In spite of being defined in terms of ∇ the torsion R^1 depends essentially on N only, as easily seen in local coordinates, i.e.

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j},$$

where $R^1(X_i, X_j) = R^k_{ij} X_k$.

The fundamental theorem of Finsler geometry asserts that there exists a unique regular connection ∇ in the induced bundle $(\pi^{-1}TM^{2n}, g)$ of the given Finsler manifold (M^{2n}, L) such that i) ∇ is metrical, ii) $T = 0$, $S^1 = 0$. It is refered to as the *Cartan connection* of (M^{2n}, L). Then (M^{2n}, L, J) is called a *Kaehlerian Finsler space* if its Cartan connection is almost complex [6].

5. Cauchy-Riemann structures on the tangent bundle

If N is an arbitrary C^∞-manifold and $T^{\mathbf{C}}(N) = T(N) \otimes \mathbf{C}$ denotes the complexification of its tangent bundle, then a Cauchy-Riemann (CR) structure on N is a complex subbundle H of $T^{\mathbf{C}}(N)$ such that i) $H \cap \bar{H} = (0)$, ii) H is involutive. Here a bar denotes complex conjugation. At this point we may prove our theorem 1. Let J be a Finslerian almost complex structure on M^{2n}. Let N be a nonlinear connection on $V(M^{2n})$. If $N^h_J = 0$, we consider the $\mathbf{C}$-vector subbundle H^h of $T^{\mathbf{C}}(V(M^{2n}))$ defined by $u \mapsto H^h_u$, where H^h_u consists of all complex tangent vectors $X \otimes 1 - J^h X \otimes i$, $i = \sqrt{-1}$, $X \in N_u$, $u \in V(M^{2n})$. Suppose $R^1 = 0$. By (3.7) and (4.1), it follows that $[J^h, J^h] = 0$. Consequently H^h is involutive. Also, by the very definition of H^h, one has $\Re(H_u) = N_u$, $u \in V(M^{2n})$. That is, $V(M^{2n})$ turns into a CR manifold. □

Note that theorem 1 holds for nonlinear connections whose R^1 torsion field vanishes identically. The following example shows that such nonlinear connections actually exist. Let ∇ be a flat connection on M^{2n} and $\Gamma^i_{jk}(x)$ the connection coefficients. Then $N^i_j = \Gamma^i_{jk} y^k$ defines (by our (3.3)) a nonlinear connection on $V(M^{2n})$ with $R^i_{jk} = 0$.

If J is a Finslerian almost complex structure with $N^h_J = 0$, (with respect to a fixed nonlinear connection), then J is said to be *h-integrable*. To unify terminology, complex Finsler structures (cf. our §3) may be refered to as being *v-integrable*. It should be underlined that if J is obtained by natural lifting from an ordinary almost complex structure J then N^h_J still contains information on the integrability of J, while N^v_J is vanishing identically.

6. Invariant submanifolds of Kaehlerian Finsler spaces

Let $\psi : M^m \to M^{2n}$ be an isometric immersion (i.e. $L(u) = L_0(\psi_* u)$, $u \in T(M^m)$) of a real m-dimensional Finsler space (M^m, L) in a Kaehlerian Finsler space (M^{2n}, L_0, J). Throuhout a knot indicates objects (metrics, connections, etc.) associated with the ambient space M^{2n}.

Let then $(\pi^{-1}TM^m, g)$, $(\pi^{-1}TM^{2n}, g_0)$ be the corresponding induced bundles. Here π denotes both the projections $V(M^m) \to M^m$ and $V(M^{2n}) \to M^{2n}$. Let $D\psi$ be the restriction of $\psi_* \times \psi_*$ to $\pi^{-1}TM^m$. We denote by $E(\psi)_u$ the orthogonal complement (with respect to $g_{0,u}$) of $\pi_u^{-1}TM^m$ (thought of as the subspace $(D\psi)_u \pi_u^{-1}TM^m$ in $\pi_u^{-1}TM^{2n}$). Let $E(\psi)$ be the disjoint union of all $E(\psi)_u$, $u \in V(M^m)$. Then $E(\psi) \to V(M^m)$ is a rank $2n - m$ real differentiable vector bundle, i.e. the *normal bundle* of ψ.

A Finslerian k-distribution $\mathcal{D}$ on M^m is an assignement $\mathcal{D} : u \mapsto \mathcal{D}_u$ such that each $\mathcal{D}_u$ is a k-dimensional real subspace of $\pi_u^{-1}TM^m$, $u \in V(M^m)$, $k \geq 1$. With each Finslerian distribution $\mathcal{D}$ on M^m we may associate a distribution on $V(M^m)$, namely its vertical lift

$$\gamma\mathcal{D} : u \longmapsto \gamma_u \mathcal{D}_u \subset \ker(d_u \pi).$$

All Finslerian distributions considered are assumed to be smooth, i.e. $\mathcal{D}$ is smooth if for any $u_0 \in V(M^m)$ there exists an open neighbourhood U of $x_0 = \pi(u_0)$ in M^m and a family $\{X_i\}_{1 \leq i \leq k}$ of smooth Finsler vector fields defined on $\pi^{-1}(U)$, such that for each $u \in \pi^{-1}(U)$, $\{X_i(u)\}_{1 \leq i \leq k}$ is a linear basis of $\mathcal{D}_u$.

Cauchy-Riemann (CR) submanifolds of almost Hermitian (Riemannian) manifolds were considered firstly by A. Bejancu [16]. The author [9, p. 57] has generalized this notion to the Finsler geometry in the following manner. A submanifold M^m of the almost Hermitian Finsler space M^{2n} is said to be a *CR submanifold* if it carries a pair of complementary (with respect to g) Finslerian distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ such that $\mathcal{D}$ is invariant (i.e. $J_u \mathcal{D}_u = \mathcal{D}_u$, $u \in V(M^m)$), while $\mathcal{D}^\perp$ is anti-invariant (i.e. $J_u \mathcal{D}_u^\perp \subset E(\psi)_u$, $u \in V(M^m)$). Moreover, together with [3, p. 162], we call M^m an *invariant* submanifold if $\mathcal{D}^\perp = (0)$, respectively an *anti-invariant* submanifold if $\mathcal{D} = (0)$. If M^m is anti-invariant and $m = n$ then M^n is said to be a *totally-real* submanifold of M^{2n}. Finally, if $\dim_{\mathbb{R}} \mathcal{D}^\perp = 2n - m$ then M^m is said to be a *generic* CR submanifold.

We recall, e.g. [10, p. 3], the Gauss and Weingarten equations of ψ, i.e.

$$\begin{aligned} \nabla^0_Z Y &= \nabla_Z Y + \tilde{H}(Z, Y) \\ \nabla^0_Z \xi &= -\tilde{A}_\xi Z + \nabla^\perp_Z \xi. \end{aligned} \tag{6.1}$$

Here ∇^0 denotes the Cartan connection of (M^{2n}, L_0) while ∇, $\tilde{H}$, $\tilde{A}_\xi$, $\nabla^\perp$ denote respectively the *induced connection*, the *second fundamental form* (of ψ), the *Weingarten operator* (associated with the cross-section ξ in $E(\psi)$), and the *normal connection* (in $E(\psi)$). The induced connection ∇ (although generally different from the Cartan connection of (M^m, L)) is known to be regular [10, p. 4]. Let then N denote its nonlinear connection (on $V(M^m)$) and β the corresponding horizontal lift. We consider the following fragments of $\tilde{H}$, $\tilde{A}$, i.e.

$$H(X,Y) = \tilde{H}(\beta X, Y), \qquad Q(X,Y) = \tilde{H}(\gamma X, Y),$$

$$A_\xi X = \tilde{A}_\xi \beta X, \qquad W_\xi X = \tilde{A}_\xi \gamma X,$$

for any Finsler vector fields X and Y on M^m, respectively any cross-section ξ in $E(\psi)$. Then H and Q are respectively called the *horizontal and vertical second fundamental forms* of ψ.

At this point we may prove our theorem 2. To this end, let (M^{2n}, L_0, J) be a Kaehlerian Finsler space, $n \geq 2$, and suppose that its holomorphic v-sectional curvature s_0 (with respect to the Cartan connection) is given by $s_0 = c \circ \sigma$ for some $c \in C^\infty(V(M^{2n}))$. By a result in [6], c actually falls into a point function only, i.e. $c \in C^\infty(M^{2n})$, and the vertical curvature tensor $S_0(X,Y,Z,W) = g_0(S_0(Z,W)Y,X)$ is given by:

$$\begin{aligned}(6.2)\qquad S_0(X,Y,Z,W) = \frac{c}{4}\,[&g_0(X,Z)g_0(Y,W) - g_0(X,W)g_0(Y,Z)\\ &+ g_0(X,JZ)g_0(Y,JW) - g_0(X,JW)g_0(Y,JZ)\\ &+ 2g_0(X,JY)g_0(Z,JW)]\end{aligned}$$

for any Finsler vector fields X, Y, Z, and W on M^{2n} [6, p. 97, (3.4)]. We need to recall [10, p. 6] the following Gauss-Codazzi equation:

$$S_0(X,Y)Z = S(X,Y)Z + W_{Q(X,Z)}Y - W_{Q(Y,Z)}X + (\nabla_{\gamma X}Q)(Y,Z) - (\nabla_{\gamma Y}Q)(X,Z)$$

for any Finsler vector fields X, Y and Z on M^m. Here S stands for the vertical curvature of the induced connection ∇ on M^m.

It is well known [10, p. 5] that Q is symmetric. Since M^m is invariant and $\nabla^0_{\gamma X}Y = \nabla_{\gamma X}Y + Q(X,Y)$, it follows that J is v-parallel with respect to ∇ while Q is subject to

$$(6.4)\qquad Q(X,JY) = Q(JX,Y) = JQ(X,Y).$$

Let $p \in GF_2(M^m)$ be a holomorphic 2-plane and $\{X, JX\}$ an orthonormal basis in p. Taking the inner product of (6.3) with W and using (6.2), one derives the expression of S. This yields the holomorphic v-sectional curvature s on M^m, that is:

$$s(p) = c(x) - 2\left\|Q_u(X,X)\right\|^2 \tag{6.5}$$

where

$$x = \pi(u), \qquad u = \sigma(p), \qquad u \in V(M^m),\ x \in M^m,$$

and the proof of theorem 2 is complete. □

Let M^m be a submanifold of the Finsler space (M^{2n}, L). Let $\{E_1, \ldots, E_m\}$ be an orthonormal frame of the induced bundle $(\pi^{-1}TM^m, g)$. The following (well defined) normal sections, i.e.

$$\mu = \frac{1}{m}\delta^{ij}H(E_i, E_j), \qquad \nu = \frac{1}{m}\delta^{ij}Q(E_i, E_j),$$

are refered to as the *h-mean curvature vector*, and *v-mean curvature vector* of M^m in M^{2n}, respectively.

At this point, we may prove our theorem 3. To this end, let M^m be an invariant submanifold of the Kaehlerian Finsler space (M^{2n}, L, J). We recall [10, p. 4, (1.2)], i.e.

$$T(X,Y) + H(X,Y) - H(Y,X) = C_0((N(X),Y) - C_0((N(Y),X) \tag{6.7}$$

for any Finsler vector fields X, Y on M^m. Here T denotes the horizontal component of the torsion $\tilde{T}$ of the induced connection, while C_0 stands for the mixt component of the torsion $\tilde{T}_0$ (of the Cartan connection of M^{2n}). Also $N(X) = H(X, v)$. Generally $T \neq 0$ and therefore H is not symmetric. From our formula

$$\nabla^0_{\beta X}Y = \nabla_{\beta X}Y + H(X,Y)$$

it follows that J is h-parallel (with respect to ∇) but one may only prove that $H(X, JY) = JH(X,Y)$. Nevertheless, if M^m is assumed to be totally-geodesic ($N = 0$) then (6.7) yields the symmetry of H and a standard argument leads to $\mu = 0$. □

7. Totally-real submanifolds of Kaehlerian Finsler spaces

Let M^m be an anti-invariant submanifold of the Kaehlerian Finsler space (M^{2n}, J, L). We suppose for the rest of this paragraph that $m = n$, i.e. M^n is totally-real. Let $\tilde{R}^{\perp}$ be the curvature 2-form of the normal connection $\nabla^{\perp}$. Three fragments of $\tilde{R}^{\perp}$ may de defined in a natural way, i.e.

$$R^{\perp}(X,Y) = \tilde{R}^{\perp}(\beta X, \beta Y),$$
$$P^{\perp}(X,Y) = \tilde{R}^{\perp}(\gamma X, \beta Y),$$
$$S^{\perp}(X,Y) = \tilde{R}^{\perp}(\gamma X, \gamma Y).$$

Then $\nabla^{\perp}$ is said to be *v-flat* (respectively *h-flat*) if $S^{\perp} = 0$, respectively if $R^{\perp} = 0$). We may prove now our theorem 4. To this end we recall [10, p. 8, (1.12)]:

$$g_0(S_0(X,Y)\xi, \eta) = g_0(S^{\perp}(X,Y)\xi, \eta) - g([W_\xi, W_\eta]X, Y). \tag{7.1}$$

Since the form of (7.1) is similar to that of the Ricci equation of a submanifold in a Riemann space, we may apply the arguments in [16, p. 82] such as to generalize [16, p. 82, proposition 1.3]. Indeed, for $\xi = JZ$, $\eta = JU$, equation (7.1) followed by (6.3) lead to:

$$S(X,Y)Z = W_{Q(Y,Z)}X - W_{Q(X,Z)}Y + JQ(X, W_{JZ}Y) - JQ(Y, W_{JZ}X). \tag{7.2}$$

On the other hand, as M^n is totally-real and $\nabla^0 J = 0$ the Gauss and Weingarten formulae (6.1) lead to:

$$W_{JY}X = -JQ(X,Y). \tag{7.3}$$

Substitution from (7.3) into (7.2) finally yields $S = 0$. □

To prove theorem 5, we recall (1.4) and [10, pp. 6–7, (1.9)], i.e.

$$\begin{aligned} R_0(X,Y)Z + P_0(N(X),Y)Z - P_0(N(Y),X)Z + S_0(N(X),N(Y))Z \\ = R(X,Y)Z + A_{H(X,Z)}Y - A_{H(Y,Z)}X + (\nabla_{\beta X}H)(Y,Z) \\ - (\nabla_{\beta Y}H)(X,Z) + H(T(X,Y),Z) + Q(R^1(X,Y),Z), \end{aligned} \tag{7.4}$$

$$\begin{aligned} g_0(R_0(X,Y)\xi,\eta) + g_0(P_0(N(X),Y)\xi,\eta) \\ - g_0(P_0(N(Y),X)\xi,\eta) + g_0(S_0(N(X),N(Y))\xi,\eta) \\ = g_0(R^{\perp}(X,Y)\xi,\eta) + g(A_\eta Y, A_\xi X) - g(A_\eta X, A_\xi Y). \end{aligned} \tag{7.5}$$

Note that A_ξ is not self-adjoint, since H is not symmetric. As $\nabla^{\perp}$ is h-flat and:

$$A_{JY}X = -JH(X,Y) \tag{7.6}$$

we may combine (7.4) and (7.5) such as to yield $R = 0$. □

8. Generic submanifolds

Let $(M^m, \mathcal{D}, \mathcal{D}^\perp)$ be a CR submanifold of the almost Hermitian Finsler space (M^{2n}, L, J). Clearly, the vertical distribution $\ker(d\pi)$ on $V(M^m)$ is integrable, and its maximal integral manifolds are the fibres of π, i.e.

$$\ker(d_u\pi) = T_u\big(V_x(M^m)\big), \qquad V_x(M^m) = \pi^{-1}(x),$$

for all $u \in \pi^{-1}(x)$, $x \in M^m$. Let $\gamma\mathcal{D}$ and $\gamma\mathcal{D}^\perp$ be the vertical lifts of $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. It is a simple matter to verify that $(V_x(M^m), \gamma\mathcal{D}, \gamma\mathcal{D}^\perp)$ turns to be a CR submanifold of $(V(M^{2n}), J, g_0)$, in the sense of [16]. To prove the second part of theorem 6, let $\beta\mathcal{D}$ and $\beta\mathcal{D}^\perp$ be the horizontal lifts of $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively, with respect to the nonlinear connection of the induced connection of M^m, say N. Let S be a leaf of N, $T_u(S) = N_u$, $u \in S$. Let $i : S \to V(M^m)$ be the natural inclusion and ψ the given immersion of M^m in M^{2n}. We regard S as a submanifold of $V(M^{2n})$ by considering the immersion $\psi_* \circ i$. If $X \in \mathcal{D}$, $Y \in \mathcal{D}^\perp$, then

$$\tilde{g}_0(\beta X, \beta Y) = g_0\big(N(X), N(Y)\big),$$

due to:

$$\beta Z = \beta_0 Z + \gamma H(Z, v), \tag{8.1}$$

for any Finsler vector field Z on M^m [10, p. 3]. Here $\beta_0 : \pi^{-1}TM^{2n} \to N^0$ denotes the horizontal lift with respect to the Cartan connection of (M^{2n}, L). Thus $\beta\mathcal{D}$ and $\beta\mathcal{D}^\perp$ are mutually orthogonal (with respect to $\tilde{g}_0$) if and only if (2.2) holds. Let $X \in \mathcal{D}$. Then

$$\tilde{J}\beta X = \beta JX + \gamma\{JH(X, v) - H(JX, v)\},$$

and consequently $\beta\mathcal{D}$ is $\tilde{J}$-invariant if and only if (2.3) holds. Finally, for $Y \in \mathcal{D}^\perp$ and any Finsler vector field Z on M^m, one has:

$$g_0(\tilde{J}\beta Y, \beta Z) = g_0\big(JN(Y), N(X)\big) = 0,$$

by (8.1) and since M^m is generic, i.e. $JN(Y)$ is tangential. □

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