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Extended Chebyshev systems for the expansions of exp(At)

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ABSTRACT

Consider a matrix A of order $n \times n$ having real entries (i.e., $A \in \mathbb{R}^{n\times n}$). The degree of its minimal polynomial is μ. It is proved that the identity $\exp(At) = \sum_{k=0}^{\rho-1} \alpha_k(t)A^k$ stands for sets $\{\alpha_u(t) : u = 0, 1, \dots, \rho - 1\}$ of functions of real variable defined in any real interval I. These sets are unique for each integer $\rho \geq \mu$ and can be determined from a system of linear equations. In addition, these sets are always Chebyshev systems on a real interval $(\gamma, \gamma + \pi/\omega)$, with $\omega = \max_{1\leq k\leq\sigma}(\Im(\lambda_k))$, λ_k $(k = 1, 2, \dots, \sigma)$ being the eigenvalues of A, and any $\gamma \in \mathbb{R}$. These results generalize a weaker parallel known result which stands for the set of minimum cardinal (i.e., for $\rho = \mu$). The generalizations obtained lead to important consequences when solving some algebraic problems in control theory.

1. Introduction

It is well known [1] that the matrix function $\exp(At)$ with $A \in \mathbb{R}^{n\times n}$ (or $A \in \mathbb{C}^{n\times n}$) can be expressed as

$$\exp(At) = \sum_{u=0}^{\mu-1} \alpha_u(t)A^u \qquad \forall t \in I,$$

with I being a real interval, μ being the degree of the minimal polynomial of A, and $\alpha_0(t), \dots, \alpha_{\mu-1}(t)$ being real functions which are uniquely determined by solving the set of linear equations

$$t^j \exp(\lambda_k t) = \begin{bmatrix} 1 & \lambda & \cdots & \lambda^{\mu-1} \end{bmatrix} \begin{bmatrix} \alpha_0(t) & \alpha_1(t) & \cdots & \alpha_{\mu-1}(t) \end{bmatrix}^T \Big|_{\lambda=\lambda_k}^{(j)} \tag{1}$$

$$(k = 1, 2, , \dots, \sigma;\ j = 0, 1, \dots, \mu_k - 1)$$

where the symbols $f^{(j)}$ and T denote, respectively, jth derivative and transpose, and $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_\sigma\}$ is the spectrum of A in $\mathbb{C}$, with each λ_k being of multiplicity μ_k $(k = 1, 2, \dots, \sigma)$ in the minimal polynomial of A.

The paper is organized as follows. In section 2, the identity

$$\exp(At) = \sum_{u=0}^{\rho-1} \alpha_u(t)A^u$$

is proved for any integer $\rho \geq \mu$ and conditions for unicity of the set

$$\{\alpha_u(t) : u = 0, 1, \dots, \rho - 1\}$$

are stated. In section 3, it is proved that this set is a Chebyshev system. This implies that

$$\det\big(\alpha_i(t_j)\big)_{\substack{i=0,1,\dots,\rho-1 \\ j=0,1,\dots,\rho-1}} \neq 0$$

for some $t_j \in \mathbb{R}$ and $t_i \neq t_j$ for $i \neq j$. Some implications of this fact in problems of Control Theory are presented in section 4, and, finally, conclusions end the paper.

The major mathematical contributions of the paper are:

a) Generalizations of the expansion of $\exp(At)$ for any integer $\rho \geq \mu$ and for certain kinds of complex matrices A are given.

b) The maintenance of the Chebyshev system structure of the set

$$\{\alpha_u(t) : u = 0, 1, \dots, \rho - 1\}$$

even if some of the $\alpha_u(t)$ are complex is proved. This is a sufficient condition to ensure that the matrix

$$\left(\alpha_i(t_j)\right)_{\substack{i=0,1,\dots,\rho-1\\ j=0,1,\dots,\rho-1}} \qquad t_j \in \mathbf{R},\ t_i \neq t_j \quad (i,j=1,2,\dots,\rho;\ i\neq j)$$

has a unique inverse within certain real intervals.

2. The expansion of exp(At)

Consider any matrix $A \in \mathbf{R}^{n\times n}$ (or $A \in \mathbf{C}^{n\times n}$). Consider also the following generalization of (1)

$$t^j \exp(\lambda_k t) = \begin{bmatrix} 1 & \lambda & \cdots & \lambda^{\rho-1} \end{bmatrix} \begin{bmatrix} \alpha_0(t) & \alpha_1(t) & \cdots & \alpha_{\rho-1}(t) \end{bmatrix}^T \Big|_{\lambda=\lambda_k}^{(j)} \tag{1}$$

$$(k = 1,2,,\dots,\sigma;\ j = 0,1,\dots,\rho_k - 1)$$

where ρ and ρ_k $(k = 1,2,\dots,\sigma)$ are positive and nonnegative integers satisfying the two following constraints

(C.1) $\qquad \rho \geq \mu; \qquad \rho_k \geq \mu_k \qquad (k = 1,2,\dots,\sigma)$

(C.2) $\qquad \rho = \sum_{k=1}^{\sigma} \rho_k$

The following result generalizes the expansion of $\exp(At)$ given in [1].

Theorem 1

For any $A \in \mathbf{R}^{nxn}$ (or $\mathbf{C}^{nxn}$), the following holds.

(i) For each possible choice of the set $(\rho_1, \rho_2, \dots, \rho_\sigma)$ and ρ satisfying (C.1)–(C.2) the system (2) has a unique solution defined by a set of functions of real variable $\{\alpha_u(t) : u = 0,1,\dots,\rho-1\}$ on any real interval I.

(ii) For this set, the identity $\exp(At) = \sum_{u=0}^{\rho-1} \alpha_u(t) A^u$ stands on I.

Proof. Using a compact matrix notation, system (2) is denoted by:

$$P\alpha(t) = Q \tag{3}$$

where P is a matrix of order $\rho \times \rho$ of entries in $\mathbb{C}$, and $\alpha(t)$, $Q \in \mathbb{C}^{\rho\times 1}$, P, Q and $\alpha(t)$ are defined in partitioned forms by system (4):

$$P = \left[P_1^T \quad P_2^T \quad \dots \quad P_\sigma^T\right]^T$$
$$Q = \left[Q_1 \quad Q_2 \quad \dots \quad Q_\sigma\right]^T$$
$$\alpha(t) = \left[\alpha_0(t) \quad \alpha_1(t) \quad \dots \quad \alpha_{\rho-1}(t)\right]^T$$
$$Q_k = \left[\exp(\lambda_k t) \quad t\exp(\lambda_k t) \quad \dots \quad t^{\rho_k-1}\exp(\lambda_k t)\right]$$

$$P_k = \begin{bmatrix} 1 & \lambda_k & \lambda_k^2 & \lambda_k^3 & \cdot & \cdot & \cdot & \cdot & \lambda_k^{\rho-1} \\ 0 & 1 & 2\lambda_k & 3\lambda_k^2 & \cdot & \cdot & \cdot & \cdot & (\rho-1)\lambda_k^{\rho-2} \\ 0 & 0 & 2 & 6\lambda_k & \cdot & \cdot & \cdot & \cdot & \prod_{e=1}^{2}(\rho-e)\lambda_k^{\rho-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & (\rho_k-1)! & \rho_k!\lambda_k & \cdots & \prod_{e=1}^{\rho_k-1}(\rho-e)\lambda_k^{\rho-\rho_k} \end{bmatrix}$$

$$(k = 1, 2, \dots, \sigma)$$

System (2) has a unique solution $\alpha(t)$ if and only if $\operatorname{rank}(P) = \rho$. This can be proved by reordering the rows of P by constructing a new matrix $\hat{P}$, being equivalent to P, which is partitioned by

$$\hat{P} = [\hat{P}_0^T \quad \hat{P}_1^T \quad \dots \quad \hat{P}_s^T]^T; \qquad s = \max_{1\le k\le\sigma}(\rho_k - 1) \tag{5}$$

with

$$\hat{P}_0 = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{\rho-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{\rho-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \lambda_\sigma & \lambda_\sigma^2 & \dots & \lambda_\sigma^{\rho-1} \end{bmatrix}$$

$$\hat{P}_v = \begin{bmatrix} \underbrace{0\dots0}_{v} & v! & (v+1)!\lambda_{v1} & \dots & \prod_{e=1}^{v}(\rho-e)\lambda_{v1}^{\rho-v-1} \\ \underbrace{0\dots0}_{v} & v! & (v+1)!\lambda_{v2} & \dots & \prod_{e=1}^{v}(\rho-e)\lambda_{v2}^{\rho-v-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \underbrace{0\dots0}_{v} & v! & (v+1)!\lambda_{vc_v} & \dots & \prod_{e=1}^{v}(\rho-e)\lambda_{vc_v}^{\rho-v-1} \end{bmatrix} \tag{6}$$

$$(v = 1, 2, \ldots, s)$$

where $c_v = \text{card}(\Lambda_v)$, and

$$\Lambda_v = \{\lambda_k : \rho_k - 1 \geq v,\ k = 1, 2, \ldots, \sigma\}$$

is a subset $\{\lambda_{v1}, \lambda_{v2}, \ldots, \lambda_{vc_v}\}$ of the spectrum Λ of A in $\mathbb{C}$. Matrices $\hat{P}_v$ ($v = 0, 1, \ldots, s$) have entries in $\mathbb{C}$ and are of orders $c_v \times \rho$ (for $v \geq 1$) and $\sigma \times \rho$ (for $v = 0$). Since P and $\hat{P}$ are both of order $\rho \times \rho$, it is obvious from (5)–(6) that

$$\sum_{v=1}^{s} c_v = \rho - \sigma.$$

Let $W_0, W_1, \ldots, W_s$ be the vector subspaces of $\mathbb{C}^{1\times\rho}$ generated by the rows of $\hat{P}_0, \hat{P}_1, \ldots, \hat{P}_s$. It is obvious from (5)-(6) that

(a) The respective rows of $\hat{P}_v$ for $v = 0, 1, \ldots, s$ are linearly independent.

(b)

$$W_i \cap \left(\sum_{p \in I_s - \{i\}} w_p \right) = \left\{ (\overbrace{0\ \ 0\ \ \ldots\ \ 0}^{p}) \right\},$$

for all $i \in I_s = \{0, 1, \ldots, s\}$.

Thus, the vector subspace

$$W \equiv \sum_{i=0}^{s} W_i = \bigoplus_{i=0}^{s} W_i$$

($\oplus$ denotes direct sum of vector subspaces).

From (a)–(b), it follows that the ρ rows of $\hat{P}$ and P are linearly independent and then P is non-singular. This proves proposition (i). To prove (ii), consider the analytic function $f(\lambda) = \exp(\lambda t)$ in $\mathbb{C}$ and the polynomial

$$p(\lambda) = \sum_{u=0}^{\rho-1} \alpha_u(\lambda)\lambda^u$$

of coefficients in $\mathbb{C}$ which are defined in any real interval I. Denoting by superscripts (q) the derivatives of order q with respect to λ, one gets [1, 2]:

$$\left\{\begin{array}{l} f(\lambda_k) = \exp(\lambda_k t) \\ f^{(q)}(\lambda_k) = t^q \exp(\lambda_k t) \\ p(\lambda_k) = \sum_{n=0}^{\rho-1} \alpha_n(t)\lambda_k^n \\ \quad = \left[1 \quad \lambda \quad \ldots \quad \lambda^{\rho-1}\right]\left[\alpha_0(t) \quad \alpha_1(t) \quad \ldots \quad \alpha_{\rho-1}(t)\right]^T\Big|_{\lambda=\lambda_k} \\ p^{(q)}(\lambda_k) = \left[1 \quad \lambda \quad \ldots \quad \lambda^{\rho-1}\right]\left[\alpha_0(t) \quad \alpha_1(t) \quad \ldots \quad \alpha_{\rho-1}(t)\right]^T\Big|_{\lambda=\lambda_k}^{(q)} \\ (k = 1,2,\ldots,\sigma; \qquad q = 1,2,\ldots,\mu_k - 1) \end{array}\right\} \tag{7}$$

From (2) and (7), it follows that $f(\lambda_k) = q(\lambda_k)$ and

$$f^{(q)}(\lambda_k) = p^{(q)}(\lambda_k); \qquad k = 1,2,\ldots,\sigma; \quad q = 1,2,\ldots,\mu_k - 1.$$

Since eqns. (7) are valid for any arbitrary $t \in I$, proposition (ii) follows directly. □

It must be noticed that the proof of part i) of this theorem can be also focused by thinking that the system has a unique solution if and only if the homogeneous system has a unique solution. This is equivalent to prove that the polynomial, of degree less or equal than $\rho - 1$ in λ,

$$\left[1 \quad \lambda \quad \ldots \quad \lambda^{\rho-1}\right]\left[\alpha_0(t) \quad \alpha_1(t) \quad \ldots \quad \alpha_{\rho-1}(t)\right]^T$$

is the zero polynomial. This is true since the homogeneous system implies that such a polynomial must have the σ zeros λ_k with multiplicities greater or equal than ρ_k. Since the sum of those multiplicities is greater or equal than ρ, one deduces that such a polynomial has to be zero. This lies in the context of a polynomial interpolation of generalized Hermite type.

From (1) and the steps in the proof of Theorem 1, the following result stands.

Corollary 1.1

Let $\mathcal{A}$ be the set of the square matrices A of any order having the same spectrum $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\sigma)$ in $\mathbb{C}$ and the same degree of the minimal polynomial μ. Then, for any integer $\rho \geq \mu$, the set $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$ computed from (2) is invariant for all the square matrices $A \in \mathcal{A}$ having the same multiplicities for each λ_k $(k = 1, 2, \ldots, \sigma)$. This result does not depend on the order of A.

We have essentially proved that (2) can be used to compute $\exp(At)$, even if A has complex entries, for any $\rho \geq \mu$. The set $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$ can have complex elements. This can occur even if A has real entries for some choices $(\rho_1, \rho_2, \ldots, \rho_\sigma)$ as it is shown in the following illustrating example.

EXAMPLE. Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $(0, -i, i)$. Their multiplicities in the minimal polynomial are $(1, 1, 1)$. Taking $\rho = 4$ with $\rho_1 = 1$, $\rho_2 = 2$ and $\rho_3 = 1$, it follows that

$$\exp(At) = \sum_{u=0}^{3} \alpha_u(t) A^u$$

with

$$\alpha_0(t) = 1,$$

$$\alpha_1(t) = 1/2\,\big(\sin t - t\cos t\big) - 1/2\,\big(2 - t\sin t - 2\cos t\big)i,$$

$$\alpha_2(t) = 1 - \cos t,$$

$$\alpha_3(t) = 1/2\,\big(-\sin t - t\cos t\big) - 1/2\,\big(2 - t\sin t - 2\cos t\big)i. \ \square$$

The next result gives sufficient conditions to determine the set $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$ in $\mathbb{R}$ under constraints on ρ. It will be used as an intermediate result to derive new results in section 3.

Theorem 2

Consider $A \in \mathbb{R}^{n\times n}$ having μ as degree of its minimal polynomial and ρ any integer number satisfying $\rho \geq \mu$. If $\lambda_{k'} = \bar{\lambda}_{k''}$ with $\rho_{k'} = \rho_{k''}$, then the unique set $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$ obtained from (2) contains uniquely real functions of real variable for each possible choice $(\rho_1, \rho_2, \ldots, \rho_\sigma)$.

Proof. From (3) (4), it follows that

$$P_k\alpha(t) = Q_k \qquad (k = 1, 2, \ldots, \sigma).$$

P_k and Q_k are, respectively, of orders $\rho_k \times \rho$ and $\rho_k \times 1$ and have real (complex) entries if $\lambda_k \in \mathbb{R}$ ($\mathbb{C}$). For each pair of complex eigenvalues of A, $\lambda_{k'} = \beta_{k'} + \gamma_{k'} i$; $\lambda_{k''} = \bar{\lambda}_{k'} = \beta_{k'} - \gamma_{k'} i$, it is possible to define the following two subsystems of linear equations

$$P_{k'}\alpha(t) = Q_{k'}; \qquad P_{k''}\alpha(t) = Q_{k''} \tag{8}$$

with $P_{k'}$ and $P_{k''}$ being matrices of order $\rho_k \times \rho$, and $Q_{k'}$ and $Q_{k''}$ matrices of order $\rho_k \times 1$, having all their entries in $\mathbb{C}$. Denoting $\rho_{k'} = \rho_{k''} = \rho_k$, eqns. (8) can be equivalently rewritten as

$$(P_{k'} + P_{k''})\alpha(t) = Q_{k'} + Q_{k''}; \qquad (P_{k'} - P_{k''})\alpha(t) = Q_{k'} - Q_{k''}. \tag{9}$$

Consider the following sets of identities

$$\left\{ \begin{array}{c} t^j \exp(\lambda_{k'}t) + t^j \exp(\lambda_{k''}t) = 2t^j \exp(\beta_{k'}t)t^j \cos(\gamma_{k'}t) \\ t^j \exp(\lambda_{k'}t) - t^j \exp(\lambda_{k''}t) = 2it^j \exp(\beta_{k'}t)t^j \sin(\gamma_{k'}t) \\ (j = 0, 1, \ldots, \rho_k - 1) \end{array} \right\} \tag{10}$$

$$\left\{ \begin{array}{c} \lambda_{k'}^r + \lambda_{k''}^r = 2\sum_{m=0}^{p} (-1)^m \binom{r}{2m} \beta_{k'}^{r-2m} \gamma_{k'}^{2m} \\ p = \begin{cases} r/2, & r \text{ even} \\ (r-1)/2, & \text{otherwise} \end{cases} \\ \lambda_{k'}^r - \lambda_{k''}^r = 2i\sum_{m=0}^{q} (-1)^m \binom{r}{2m+1} \beta_{k'}^{r-(2m+1)} \gamma_{k'}^{2m+1} \\ q = \begin{cases} r/2 - 1, & r \text{ even} \\ (r-1)/2, & \text{otherwise} \end{cases} \\ (r = 1, 2, \ldots, \rho - 1) \end{array} \right\} \tag{11}$$

Thus, from (2) and (8) through (11), it follows that $P_{k'} + P_{k''}$, $(P_{k'} - P_{k''})/i$ and $Q_{k'} + Q_{k''}$, $(Q_{k'} - Q_{k''})/i$ are, respectively, of orders $\rho_k \times \rho$ and $\rho_k \times 1$ and of real entries. The equivalence between (8) and (9) completes the proof. □

3. Expansions of exp(At) using Chebyshev systems

This section is structured in two susbsections. In subsection 3.1 some topics on Chebyshev systems of subsequent interest are presented, and the subsection 3.2 is devoted to the characterization of the set of functions $\{\alpha_u(t) : u = 0, 1, \ldots, p-1\}$ as a Chebyshev system on real intervals by using the results obtained in section 2.

3.1. Topics about Chebyshev systems

It is well known [3, 4] the fact that for whatever a linear normed space F may be, for every $x \in F$ and any set $X = \{x_k \in F : k = 0, 1, \ldots, n\}$ of linearly independent elements, there exists a generalized polynomial, non necessarily unique,

$$P^*(x, x_k) = \sum_{k=0}^{n} c_k^* x_k$$

with coefficients c_k^* in $\mathbf{K}$ (in particular, $\mathbb{R}$ or $\mathbb{C}$) such that

$$\left\| x - \sum_{k=0}^{n} c_k^* x_k \right\| \leq \left\| x - \sum_{k=0}^{n} c_k x_k \right\|,$$

for any generalized polynomial

$$P^*(x, x_k) = \sum_{k=0}^{n} c_k x_k$$

with coefficients in $\mathbf{K}$. P^* is the best approximation on x by generalized polynomials P with coefficients in $\mathbf{K}$. In [1, 2], the following basic definition and theorems about Chebyshev systems are useful.

DEFINITION 1. Let F be a linear normed space and consider the set $X = \{x_k \in F : k = 0, 1, \ldots, n\}$ of linearly independent elements. If the generalized polynomial of least deviation from any $x \in F$ is unique, then X is called a Chebyshev system.

From the Theory of Approximation the following result (condition of Haar) is useful for testing a Chebyshev system in the space C of continuous functions on $[a, b] \subset \mathbb{R}$ with

$$\|f\| = \max_{a \leq t \leq b} (|f(t)|).$$

Theorem 3

A set $\{x_k(t) \in C : k = 0, 1, \ldots, n\}$ of linearly independent continuous functions on $[a, b]$ is a Chebyshev system in the space C if and only if every nontrivial generalized polynomial

$$P_n(t) = \sum_{k=0}^{n} c_k x_k(t)$$

has not more than n distinct zeros on this interval.

A result equivalent to Theorem 3 which is useful from an algebraic point of view, in the context of this paper, is the following (Haar's condition). See [4] for details.

Theorem 4

A set X defined as in Theorem 3 is a Chebyshev system if and only if for any distinct points $t_0, t_1, \ldots, t_n$ in $[a, b]$

$$\det \begin{bmatrix} x_0(t_0) & x_0(t_1) & \ldots & x_0(t_n) \\ x_1(t_0) & x_1(t_1) & \ldots & x_1(t_n) \\ . & . & . & . \\ x_n(t_0) & x_n(t_1) & \ldots & x_n(t_n) \end{bmatrix} \neq 0.$$

We prove the next preliminary technical results which are of applicability in subsection 3.2.

Lemma 1

Let $F = \{t^k \exp(\lambda t) : k = 0, 1, \ldots, \rho - 1\}$ be any finite set of functions on a real interval I and let H be any subset of F. F is a Chebyshev system on I so that H is also a Chebyshev system on I.

Proof (Outline). If follows by matrix factorization immediately from Theorem 4, since $\{t^k : k = 0, 1, \ldots, \rho - 1\}$ is a Chebyshev system and $\exp(\lambda t)$ is a continuous and differentiable function on any real interval I having no zeros within such an interval [4]. □

Lemma 2

Assume F and I being defined as in Lemma 1 and $G = (g_{ij}) \in C^{\rho\times\rho}$ any nonsingular matrix. If F is a Chebyshev system on I, then

$$\hat{F} = \left\{ \sum_{j=0}^{\rho-1} g_{ij}t^j : i = 0,1,\ldots,\rho-1 \right\}$$

is also a Chebyshev system on I.

Proof. Take any arbitrary finite set

$$\{t_j \in I : j = 0,1,\ldots,\rho-1;\ t_i \neq t_j\,(i \neq j)\}$$

and define for $f_j(t) = t^j$ the matrices

$$\Phi = \begin{bmatrix} f_0(t_0) & f_0(t_1) & \ldots & f_0(t_{\rho-1}) \\ f_1(t_0) & f_1(t_1) & \ldots & f_1(t_{\rho-1}) \\ \cdot & \cdot & \cdot & \cdot \\ f_{\rho-1}(t_0) & f_{\rho-1}(t_1) & \ldots & f_{\rho-1}(t_{\rho-1}) \end{bmatrix} \tag{12}$$

$$\Psi = \begin{bmatrix} \sum_{j=0}^{\rho-1} g_{0j}f_j(t_0) & \sum_{j=0}^{\rho-1} g_{0j}f_j(t_1) & \ldots & \sum_{j=0}^{\rho-1} g_{0j}f_j(t_{\rho-1}) \\ \sum_{j=0}^{\rho-1} g_{1j}f_j(t_0) & \sum_{j=0}^{\rho-1} g_{1j}f_j(t_1) & \ldots & \sum_{j=0}^{\rho-1} g_{1j}f_j(t_{\rho-1}) \\ \cdot & \cdot & \cdot & \cdot \\ \sum_{j=0}^{\rho-1} g_{\rho-1,j}f_j(t_0) & \sum_{j=0}^{\rho-1} g_{\rho-1,j}f_j(t_1) & \ldots & \sum_{j=0}^{\rho-1} g_{\rho-1,j}f_j(t_{\rho-1}) \end{bmatrix} \tag{13}$$

Since F is a Chebyshev system, $\det\Phi \neq 0$, from Theorem 4. Since G is nonsingular, it follows from (12)–(13) that $\Psi = G\Phi$ and $\det\Psi = \det G \cdot \det\Phi \neq 0$. Applying again Theorem 4 to $\hat{F}$, one concludes that $\hat{F}$ is also a Chebyshev system on I and the proof is complete. □

Lemma 3

Assume F and I as in Lemma 1 and $\tilde{G} = (\tilde{g}_{ij}) \in C^{\tilde{\rho}\times\rho}$, with $1 \leq \tilde{\rho} < \rho$, being full row rank. If F is a Chebyshev system on I, then

$$\hat{\hat{F}} = \left\{ \sum_{j=0}^{\rho-1} \tilde{g}_{ij}t^j : i = 0,1,\ldots,\rho-1 \right\}$$

is also a Chebyshev system on I.

Proof. Construct a nonsingular matrix $G = \left[\tilde{G}^T L^T\right] = (g_{ij}) \in \mathbb{C}^{\rho\times\rho}$ by completing $\tilde{G}$ with a non unique appropiate matrix $L \in \mathbb{C}^{(\rho-\tilde{\rho})\times\rho}$. Then, from Lemma 2,

$$\hat{F} = \left\{ \sum_{j=0}^{\rho-1} \tilde{g}_{ij} f_j(t) : i = 0, 1, \ldots, \rho - 1 \right\}$$

is a Chebyshev system on I. From Lemma 1, $\hat{\hat{F}} \subset F$ is also a Chebyshev system on I. □

3.2. Characterization of the sets $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$ as Chebyshev systems

In the sequel, it is assumed that the spectrum Λ of A verifies that if $\lambda_k \in \Lambda$ then $\bar{\lambda}_k \in \Lambda$ ($\bar{\lambda}_k$ denotes the complex conjugate of λ_k) with the entries of A in $\mathbb{R}$ or $\mathbb{C}$. $c(\lambda_k)$ and μ_k denote, respectively, the multiplicities of the eigenvalue λ_k in the characteristic and the minimal polynomials $P_c(\lambda)$ and $P_m(\lambda)$ of the matrix A. $\partial P_c(A)$ and $\partial P_m(A)$ are the degrees of the polynomials $P_c(\lambda)$ and $P_m(\lambda)$ of A.

Remark 3.1. Note, from results in [4], that Lemmas 1 and 3 may be extended to systems F defined by linear combinations of functions $t^j \exp(\lambda_k t)$ with j, k being integers and λ_k a real number, by assuming intervals $I_1 \subset I$, where the new deleted (Lemma 1) or added (Lemma 3) functions have no zeros. Thus, it suffices to exclude such zeros to form the new intervals where new Chebyshev systems may be defined. Note, in this context, that the expansions of $\exp(At)$ may be computed according to eqn. (2) with non unique choices of ρ leading to non unique Chebyshev systems on different intervals depending on ρ, namely $I_\rho \subset I$.

Further generaliztions may be applied to other added or deleted continuous functions having no zeros on certain intervals of interest since it is obvious that Lemma 1 is not applicable to Chebyshev systems F by considering arbitrarily deleted functions to form the subsystem H.

The next result is given in [5].

Lemma 4

Assume $A \in \mathbb{R}^{n\times n}$ with spectrum Λ in $\mathbb{C}$ and the matrix function

$$\exp(At) = \sum_{u=0}^{\mu-1} \alpha_u(t) A^u,$$

with

$$\partial P_m(A) = \mu = \sum_{k=1}^{\sigma} \mu_k$$

and the set of unique real functions of real variable $\{\alpha_u(t) : u = 0, 1, \ldots, \mu - 1\}$ *being computed from (1). Then, this set is a Chebyshev system on any bounded real interval* $I = (\gamma, \gamma + \pi/\omega)$ *with*

$$\omega = \max_{1 \leq k \leq \sigma} \Im\lambda_k,$$

$\Im\lambda_k$ *being the imaginary part of each* λ_k *of* Λ *and any* $\gamma \in \mathbf{R}$.

We now extend and combine the above results with some of those given in section 2.

Lemma 5

Consider:

(a) Any integer number $\rho > 0$ *and a choice of positive integer numbers* $(\rho_1, \rho_2, \ldots, \rho_\sigma)$ *such that* $\rho = \sum_{k=1}^{\sigma} \rho_k$.

(b) The set of square matrices $\mathcal{A}(\Lambda) \subset \mathbf{R}^{\rho \times \rho}$ *with spectrum* Λ *and* $c(\lambda_k) = \rho_k$ $(k = 1, 2, \ldots, \sigma)$, *with* $\rho_k = \rho_{k'}$ *if* λ_k *and* $\lambda_{k'}$ *are conjugate complex eigenvalues of* $A \in \mathcal{A}(\Lambda)$.

(c) A real interval I as defined in Lemma 4.

Then, the following propositions hold.

(i) For any $A \in \mathcal{A}(\Lambda)$, *there exists a unique set of real functions of real variable* $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$, *calculated from (2), which only depends on* Λ, ρ *and* $\{\rho_1, \rho_2, \ldots, \rho_\sigma\}$ *(but not on the particular matrix A) and*

$$\exp(At) = \sum_{u=0}^{\rho-1} \alpha_u(t) A^u$$

on I.

(ii) This set is a Chebyshev system on I.

Proof. Take eqn. (2). For the set $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$: Unicity follows from Theorem 1. Independence of the particular $A \in \mathcal{A}(\Lambda)$, but not of Λ, ρ and $\{\rho_1, \rho_2, \ldots, \rho_\sigma\}$, follows from Corollary 1.1. Realness follows from Theorem 2 since $c(\lambda_k) = c(\lambda_{k'})$ if λ_k and $\lambda_{k'}$ are conjugate complex eigenvalues. This proves (i).

To prove that the set $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$ is a Chebyshev system, take any $A^* \in \mathcal{A}(\Lambda)$ with $c(\lambda_k) = \mu_k = \rho_k$ $(k = 1, 2, \ldots, \sigma)$. The sets $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$ in the expansions of $\exp(At)$ and $\exp(A^*t)$ are identical since the spectrum and $\rho_k = \mu_k = c(\lambda_k)$ $(k = 1, 2, \ldots, \sigma)$ are identical for both matrices. Now, by computing the set $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$ for the matrix A^* and using Lemma 4, it follows that $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$ is a Chebyshev system on I and the proof is complete by using Theorem 2 for the expansion of $\exp(At)$. This completes the proof of (ii). □

The above result can be extended for a class of matrices having complex entries as follows.

Lemma 6

Consider:

(a) Any positive integer numbers μ and ρ.

(b) Two choices of positive integer numbers $(\mu_1, \mu_2, \ldots, \mu_\sigma)$ and $(\rho_1, \rho_2, \ldots, \rho_\sigma)$ with

$$\mu = \sum_{k=1}^{\sigma} \mu_k \qquad \text{and} \qquad \rho = \sum_{k=1}^{\sigma} \rho_k.$$

(c) The set of square matrices of any order $\mathcal{A}(\Lambda)$ with complex entries and spectrum Λ, such that for any $A \in \mathcal{A}(\Lambda)$, $\partial P_m(A) = \mu$ and the multiplicity of each $\lambda_k \in \Lambda$ in $P_m(\lambda)$ is μ_k $(k = 1, 2, \ldots, \sigma)$.

(d) The set of square matrices $\hat{\mathcal{A}}(\Lambda) \subset \mathbf{C}^{\rho \times \rho}$ with spectrum Λ and $c(\lambda_k) = \rho_k$ $(k = 1, 2, \ldots, \sigma)$.

(e) For any pair of conjugate complex eigenvalues λ_k and $\lambda_{k'}$ of any $A \in \mathcal{A}(\Lambda)$ (or $A \in \hat{\mathcal{A}}(\Lambda)$) their multiplicities in $P_m(\lambda)$ and $P_c(\lambda)$ fulfill $\mu_k = \mu_{k'}$ and $\rho_k = \rho_{k'}$.

Then, results in Lemma 5 (i) hold for the sets of functions $\{\alpha_u(t) : u = 0, 1, \ldots, \mu - 1\}$ and $\{\hat{\alpha}_u(t) : u = 0, 1, \ldots, \rho - 1\}$ being assocaited, respectively, with $\mathcal{A}(\Lambda)$ and $\hat{\mathcal{A}}(\Lambda)$.

Proof. The proof of Lemma 5, under the current hypothesis, is given using Theorems 1 and 2. This proof folows since Theorem 1 is applicable to complex matrices A with spectrum Λ. □

Remark 3.2. We have shown for the moment that unique sets of Chebyshev systems of cardinals $\partial P_m(A)$ and $\partial P_c(A)$ can be used to compute $\exp(At)$ if A has real entries (Lemmas 4 and 5) or if it has complex entries and spectrum Λ when the multiplicities of the eigenvalues which are complex conjugate either in $P_m(\lambda)$ or $P_c(\lambda)$ are identical (Lemma 6). In all these cases the Chebyshev systems are formed by real functions of real variable.

A problem which remains unsolved is that which occurs when the sets $\{\alpha_u(t) : u = 0, 1, \dots, \rho - 1\}$ contain complex functions (see the example of Section 2). Our next main result in this section extends previous results to this context for the case when A has real entries but

$$\rho = \sum_{k=1}^{\sigma} \rho_k, \qquad \rho_k \geq \mu_k$$

with $\rho_k \neq \rho_{k'}$ if $\lambda_k = \bar{\lambda}_{k'}$ $(k, k' = 1, 2, \dots, \sigma)$.

Theorem 5

Consider:

(a) Definitions (a) to (c) of Lemma 6 with $\mu = \rho$, except for the fact that matrices A have order $n \times n$ and have real entries. The set of matrices is denoted by $\mathcal{A}_n(\Lambda)$.

(b) A real interval I defined as in Lemma 4.

Then, there exists a unique set of (in general complex) functions of real variable $\{\alpha_u(t) : u = 0, 1, \dots, \rho - 1\}$, which are uniquely determined from (2), such that

$$\exp(At) = \sum_{u=1}^{\rho-1} \alpha_u(t) A^u,$$

which are a Chebyshev system on I, which are dependent on ρ, and identical for each particular $A \in \mathcal{A}_n(\Lambda)$, provided that the same integer is used for the expansion.

Proof. For any integer number $\tilde{\rho} \geq \rho$ and any set of integer numbers $(\tilde{\rho}_1, \tilde{\rho}_2, \dots, \tilde{\rho}_\sigma)$ such that

$$\tilde{\rho} = \sum_{k=1}^{\sigma} \tilde{\rho}_k \qquad \text{and} \qquad \tilde{\rho}_k \geq \mu_k \quad (k = 1, 2, \dots, \sigma)$$

define the set of square matrices $\tilde{\mathcal{A}}_{\tilde{\rho}}(\Lambda)$ such that for any $\tilde{A} \in \tilde{\mathcal{A}}_{\tilde{\rho}}(\Lambda)$:

$$\left\{ \begin{array}{cc} \Lambda = \text{spectrum } \tilde{A}; & \partial\big(P_m(\tilde{A})\big) = \rho \geq \mu \\ c(\lambda_k) = \tilde{\rho}_k \geq \mu_k \quad (k = 1, 2, \dots, \sigma); & \partial P_c(\tilde{A}) = \tilde{\rho} = \sum_{k=1}^{\sigma} \tilde{\rho}_k \end{array} \right\} \qquad (14)$$

and assume the additional constraint $\tilde{\rho}_k = \tilde{\rho}_{k'}$ if $\lambda_k, \lambda_{k'} \in \Lambda$ with $\lambda_{k'} = \bar{\lambda}_k$ any integer $1 \leq k \leq \sigma$.

It is obvious from Theorem 1 that there exist unique sets

$$\{\alpha_u^*(t) : u = 0, 1, \dots, \mu - 1\},$$

$$\{\alpha_u(t) : u = 0, 1, \dots, \rho - 1\},$$

$$\{\tilde{\alpha}_u(t) : u = 0, 1, \dots, \tilde{\rho} - 1\}$$

of functions of real variable which satisfy

$$\exp(At) = \sum_{u=0}^{\mu-1} \alpha_u^*(t) A^u = \sum_{u=0}^{\rho-1} \alpha_u(t) A^u = \sum_{u=0}^{\tilde{\rho}-1} \tilde{\alpha}_u(t) A^u \tag{15}$$

$$\exp(\tilde{A}t) = \sum_{u=0}^{\rho-1} \alpha_u(t) \tilde{A}^u = \sum_{u=0}^{\tilde{\rho}-1} \tilde{\alpha}_u(t) \tilde{A}^u \tag{16}$$

for any matrices $A \in \mathcal{A}_n(\Lambda)$, $\tilde{A} \in \tilde{\mathcal{A}}_{\tilde{\rho}}(\Lambda)$. Now, consider the two polynomials of coefficients in $\mathbf{C}$

$$q(\lambda) = \sum_{u=0}^{\rho-1} \alpha_u(t) \lambda^u; \qquad \tilde{q}(\lambda) = \sum_{u=0}^{\rho-1} \tilde{\alpha}_u(t) \lambda^u \tag{17}$$

for any $t \in \mathbf{R}$. Since $\partial P_m(\tilde{A}) = \rho$, it follows [5] from (16) and (17) for any $\lambda_k \in \Lambda$ $(k = 1, 2, \dots, \sigma)$ that

$$q(\lambda_k) = \tilde{q}(\lambda_k); \qquad q^{(j)}(\lambda_k) = \tilde{q}^{(j)}(\lambda_k) \quad (j = 1, 2, \dots, \rho_k - 1;\ k = 1, 2, \dots, \sigma). \tag{18}$$

From (17) and (18), one obtains the following identities in $\alpha_u(t)$, $\tilde{\alpha}_{u'}(t)$ $(u = 0, 1, \dots, \rho - 1;\ u' = 0, 1, \dots, \tilde{\rho} - 1)$

$$\left\{ \begin{array}{c} \displaystyle\sum_{u=0}^{\rho-1} \alpha_u(t) \lambda_k^u = \sum_{u=0}^{\tilde{\rho}-1} \tilde{\alpha}_u(t) \lambda_k^u \\ \displaystyle\sum_{u=j}^{\rho-1} \prod_{e=0}^{j-1} (u - e) \alpha_u(t) \lambda_k^{u-j} = \sum_{u=j}^{\tilde{\rho}-1} \prod_{e=0}^{j-1} (u - e) \tilde{\alpha}_u(t) \lambda_k^{u-j} \\ (k = 1, 2, \dots, \sigma;\ j = 1, 2, \dots, \rho_k - 1). \end{array} \right\} \tag{19}$$

Equations (19) can be rewritten for each $t \in \mathbf{R}$ as

$$P\alpha(t) = \tilde{P}\tilde{\alpha}(t) \tag{20}$$

where $P \in \mathbb{C}^{\rho\times\rho}$ and $\alpha(t) \in \mathbb{C}^{\rho\times 1}$, both being defined in (4) (Theorem 1), $\tilde{P} \in \mathbb{C}^{\rho\times\tilde{\rho}}$ is defined in the same way by making the change $\rho \mapsto \tilde{\rho}$ in (4), and $\alpha(t) \mapsto \tilde{\alpha}(t) \in \mathbb{C}^{\rho\times 1}$ with

$$\tilde{\alpha}(t) = \begin{bmatrix}\tilde{\alpha}_0(t) & \tilde{\alpha}_1(t) & \dots & \tilde{\alpha}_{\tilde{\rho}-1}(t)\end{bmatrix}^T.$$

Because of its structure, P is nonsingular. Therefore $M = P^{-1}\tilde{P}$ such that $\alpha(t) = M\tilde{\alpha}(t)$ exists and is full row rank. Since, for all $\tilde{A} \in \tilde{\mathcal{A}}_\rho(\Lambda)$, $\tilde{\rho}_k = \tilde{\rho}_{k'}$ holds if $\lambda_{k'} = \bar{\lambda}_k$ $(k, k' = 1, 2, \dots, \sigma;\ k \neq k')$, the functions $\{\tilde{\alpha}_u(t) : u = 0, 1, \dots, \tilde{\rho} - 1\}$ are real and a Chebyshev system on $I = I_{\tilde{\rho}}$ from Lemma 6. Now, from Lemma 3, since M is full row rank, the set of complex functions of real variable $\{\alpha_u(t) : u = 0, 1, \dots, \rho - 1\}$ is also a Chebyshev system on $I_\rho \subset I$ and the proof is complete. □

From Lemma and Theorem 5, the two following results follow trivially.

Corollary 5.1

If, in Theorem 5, $\rho_k = \rho_{k'}$ for each $\lambda_k, \lambda_{k'} \in \Lambda$ such that $\lambda_{k'} = \bar{\lambda}_k$, then the set $\{\alpha_u(t) : u = 0, 1, \dots, \rho - 1\}$ is a Chebyshev system on I and has real elements.

Proof. It is obvious since $M = P^{-1}\tilde{P}$ and the set $\{\tilde{\alpha}_u(t) : u = 0, 1, \dots, \tilde{\rho} - 1\}$ have, respectively, real entries and elements. □

Corollary 5.2

Define $\mathcal{A}_n(\Lambda)$ as in Theorem 5 with the modification that $a_{ij} \in \mathbb{C}$ $(i, j = 1, 2, \dots, n)$. Then, Theorem 5 applies mutatis-mutandis.

Proof. Proceeding as in the proof of Theorem 5, a full row rank matrix M can be found which maps the Chebyshev system $\{\tilde{\alpha}_u(t) : u = 0, 1, \dots, \tilde{\rho} - 1\}$ into $\{\alpha_u(t) : u = 0, 1, \dots, \rho - 1\}$. □

In this proof it is crucial from 'definition of Λ' to note that $A \in \mathcal{A}_n(\Lambda)$ has always pairs of complex conjugate eigenvalues (non necessarily having the same multiplicities).

4. Applications in some Control Theory problems

The preceding results of sections 2 and 3 have applicability fields in discretization of the equations in Linear Control Theory. For instance, consider a square matrix $A \in \mathbf{R}^{n\times n}$ and a free linear and time-invariant differential system:

$$\dot{x}(t) = Ax(t), \quad \forall t \in [\hat{t}_0, \hat{t}], \qquad x(\hat{t}_0) = x_0.$$

From Theorem 1, for any integer $\rho \geq \partial P_m(A)$, the solution is

$$x(t) = \exp\big(A(t-\hat{t}_0)\big)x_0 = \sum_{u=0}^{\rho-1} \alpha_u(t-\hat{t}_0)A^u x_0, \qquad \forall t \in [\hat{t}_0, \hat{t}].$$

From Lemma 4 and Theorem 5, the set of functions $\{\alpha_u(t) : u = 0, 1, \ldots, \rho-1\}$ is a Chebyshev system on $I = [\hat{t}_0, \hat{t}] \cap (\delta, \delta + \pi/\omega)$, where

$$\omega = \max_{1\leq k\leq \sigma} \Im\lambda_k,$$

and the λ_k's belong to the spectrum of A in $\mathbf{C}$, for any real number δ such that I is nonempty.

This result is useful to improve the errors associated with the numerical results in the observability problem (namely, computation of $x(\hat{t}_0)$ from $x(t)$) when approximating the problem via discretization by using a set of samples

$$\big\{x(t_0), x(t_1), \ldots, x(t_{\rho-1}) : t_i \neq t_j;\ i,j = 0,1,\ldots,\rho-1;\ i \neq j\big\}$$

of $x(t)$ on $(\hat{t}_0, \hat{t}]$ which belongs to I even if $\rho \geq \partial P_m(A)$. The Chebyshev system structure of $\{\alpha_u(t) : u = 0, 1, \ldots, \rho-1\}$ ensures (Theorem 4) that

$$\det\big(\alpha_i(t_j)\big)_{0\leq i,j\leq \rho-1} \neq 0$$

so that the discretized system is observable when the continuous one is observable (results for $\rho = \partial P_m(A)$ were proved in [5]).

The parallel control problems of controllability for a non-free differential system and identifiability were studied in [6] and [7], respectively, although the structure of Chebyshev system was not proved.

5. Conclusions

Using standard algebraic tools, several new results concerning the expansion of the matrix function $\exp(At)$ and Chebyshev systems have been proved. In particular, we have proved that

(1) For any matrix A having σ different eigenvalues,

$$\exp(At) = \sum_{u=0}^{\rho-1} \alpha_u(t) A^u,$$

with unique functions of real variable which can be complex even if A is real and which are only dependent on $\rho \geq \mu$, and on a certain weak condition for the choices of sets of integers $(\rho_1, \ldots, \rho_\sigma)$ with

$$\sum_{k=1}^{\sigma} \rho_k = \rho.$$

(2) The set $\{\alpha_u(t) : u = 0, 1, \ldots, \rho - 1\}$ is a Chebyshev system on certain real intervals even if ρ is arbitrarily large and the $\alpha_u(t)$ are complex, provided that A has real entries. These properties also stand when A has complex entries and the spectrum of A contains all the possible pairs of complex conjugate eigenvalues.

Up till now, the only known results on the subject were concerned with A having real entries and ρ = degree of the minimal or characteristic polynomial of A. Therefore, the results presented can be very useful in extensions of certain Control Theory problems.

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References

1. L. A. Zadeh and C. A. Desoer, *Linear System Theory. The State Space Approach*, McGraw-Hill, New York, 1963.
2. F. R. Gantmacher, *Théorie des Matrices*, Dunod, Paris, 1966.
3. E. W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966.
4. A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, New York, 1963.

5. I. Troch, Sampling with arbitrary choice of the sampling points, *Automatica* **9** (1973), 117–124.
6. M. de la Sen, S. Dormido and M. Mellado, Controlabilidad del estado y de la salida con elección aperiódica de los instantes de muestreo, *Revista de Informática y Automática* **43** (1980), 12–17.
7. M. de la Sen, Application of the non-periodic sampling to the identifiability and model matching problems in dynamic systems, *Int. J. Systems Sci.* **14** (1983), 367–383.