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An analytic study on the self-similar fractals: Differentiation of integrals

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Abstract

Let $E \subset \mathbb{R}^n$ be a self-similar fractal of Hausdorff dimension s, such that its Hausdorff measure H^s is finite and positive. In this paper, we define two differentiation bases on E which are density bases for (E, H^s). For these differentiation bases, we study covering properties of Vitali type, and we prove that they differentiate $L^1(E, H^s)$.

1. Introduction

Let E be an s-set, that is, a subset of the Euclidean n-space $\mathbb{R}^n$ which is measurable with respect to the s-dimensional Hausdorff measure H^s and for which $0 < H^s(E) < \infty$. In the Mandelbrot's terminology [4], a fractal is an s-set for which s is fractional, or s is integer and its geometric properties are completely opposite to the properties of the nice s-dimensional surfaces. The aim of this paper is the definition and the study of two differentiation bases on the self-similar s-sets, that is s-sets which are a finite union of disjoint subsets, each of them similar to the whole set.

In the second section of this paper we present some definitions and results about Hausdorff measures and self-similar sets, and notations.

Given $E \subset \mathbb{R}^n$, a self-similar s-set, in section 3 we define two differentiation bases, $\mathcal{B}_1$ and $\mathcal{B}_2$, on E. In the theorems 3.1 and 3.2, we prove covering properties, of Vitali type, for these bases and, in the theorem 3.3, we prove that they are density

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bases for (E, H^s). In the theorem 3.4, we shall see that the maximal operator, associated to $\mathcal{B}$ ($\mathcal{B}_1$ or $\mathcal{B}_2$), satisfies an inequality of weak type $(1,1)$. Using the preceding results we prove, in the theorem 3.5, that basis $\mathcal{B}$ ($\mathcal{B}_1$ or $\mathcal{B}_2$) differentiates to $L^1(E, H^s)$.

2. Preliminaries

Given a subset $E \subset \mathbb{R}^n$, for s, $0 \le s \le n$, we define:

$$H^s(E) = \liminf_{\delta \to 0^+} \left\{ \sum_{i=1}^{\infty} d(A_i)^s : E \subset \bigcup_{i=1}^{\infty} A_i,\ 0 < d(A_i) \le \delta \right\}$$

where $d(A)$ denote the diameter of the set A. For each s, $0 \le s \le n$, the application H^s is an outer measure on $\mathbb{R}^n$ which we call *Hausdorff s-dimensional outer measure*. The restriction of H^s to the σ-field of H^s-measurable sets is called *Hausdorff s-dimensional measure*. This measure H^s is Borel regular. We shall say that E is an *s-set* if E is H^s-measurable and $0 < H^s(E) < \infty$.

Given $E \subset \mathbb{R}^n$ there is a unique number s, $0 \le s \le n$, such that:

$$H^t(E) = \begin{cases} \infty, & \text{if } t < s \\ 0, & \text{if } t > s \end{cases}$$

which is called the *Hausdorff dimension* of E, and we shall write $s = \dim E$.

More details about Hausdorff measures may be consulted in [1] or [5].

A wide and important class of s-sets is that formed by the self-similar sets, which are defined as follows:

2.1. DEFINITION. We shall say that a compact set $E \subset \mathbb{R}^n$ is *self-similar* if there is a finite family $\mathcal{S} = \{S_1, \ldots, S_l\}$ of similitudes ($S_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $|S_i(x) - S_i(y)| = r_i\,|x-y|$, for all $x, y \in \mathbb{R}^n$, $0 < r_i < 1$) and a number s, $0 \le s \le n$, satisfying:

a) $E = \bigcup_{i=1}^{l} S_i(E)$

b) $0 < H^s(E) < \infty$

c) $H^s(S_i(E) \cap S_j(E)) = 0$, for $1 \le i < j \le l$.

The number s which give us the Hausdorff dimension of the self-similar set E is the unique positive number which satisfies $\sum_{i=1}^{l} r_i^s = 1$. From the definition one deduce immediately that if $l = 1$ the set E contains a single point, and $\dim E = 0$ and $H^0(E) = 1$. We shall assume then, in this paper, that $l \ge 2$.

2.2. Notation. Let l be a fixed positive integer number. For every positive integer number k, we denote by $\mathcal{S}_k^l$ the set of all k-tuples formed by the first l positive integer numbers, that is:

$$\mathcal{S}_k^l = \{(i_1,\dots,i_k) : 1 \le i_j \le l,\ 1 \le j \le k\}$$

and analogously $\mathcal{S}_\infty^l$ will denote the set of all the infinite sequences formed with the ℓ first positive integer numbers:

$$\mathcal{S}_\infty^l = \{(i_1,\dots,i_k\dots) : 1 \le i_j \le l, j \ge 1\}.$$

If $\alpha = (i_1\dots,i_k) \in \mathcal{S}_k^l$ and $\beta = (j_1,\dots,j_q) \in \mathcal{S}_q^l$, we define the *concatenation* of α and β by:

$$\alpha\beta = (i_1,\dots,i_k,j_1,\dots j_q) \in \mathcal{S}_{k+q}^l,$$

and for every p, $1 \le p \le k$, we denote by $\alpha[p]$ the p-tuple formed by the p first coordinates of α, that is:

$$\alpha[p] = (i_1,\dots i_p) \in \mathcal{S}_p^l.$$

Moreover, if $k < q$, we shall say that $\alpha \subset \beta$ if $\alpha = \beta[k]$, that is, if the k-tuple α are the k first coordinates of β.

2.3. Basic results about self-similar sets

Let $E \subset \Re^n$ be the self-similar set associated to the family of similitudes $\mathcal{S} = \{S_1,\dots,S_l\}$, with ratios $\{r_1,\dots,r_l\}$, and with Hausdorff dimension s ($\sum_{i=1}^l r_i^s = 1$). For every $k \ge 1$ and $\alpha = (i_1,\dots,i_k) \in \mathcal{S}_k^l$ we denote:

$$E_\alpha = S_\alpha(E) = S_{i_1} \circ \dots \circ S_{i_k}(E),$$

$$r_\alpha = r_{i_1} \cdots r_{i_k},$$

where r_α will be the similitude ratio of S_α. It is easy to show that:

(a) $E = \bigcup_{\alpha\in\mathcal{S}_k^l} E_\alpha$, for all $k \ge 1$.

(b) $E_\alpha = \bigcup_{\beta\in\mathcal{S}_p^l} E_{\alpha\beta}$, for every $\alpha \in \mathcal{S}_k^l$, $k \ge 1$ and $p \ge 1$. In particular $E_\alpha = \bigcup_{i=1}^l E_{\alpha i}$.

(c) $H^s(E_\alpha \cap E_\beta) = 0$, for every $k \ge 1$, $\alpha,\beta \in \mathcal{S}_k^l$ and $\alpha \ne \beta$.

(d) If $\alpha \in \mathcal{S}_p^l$, $\beta \in \mathcal{S}_q^l$ and $p < q$, we have that:

(d1) $E_\beta \subset E_\alpha$, if $\alpha \subset \beta$.

(d2) $H^s(E_\alpha \cap E_\beta) = 0$, if $\alpha \not\subset \beta$.

For more details about self-similar sets one can see [1] or [3].

3. Differentiation Bases on the Self-similar Fractals

In this section $E \subset \mathbb{R}^n$ will be the self-similar set, of dimension s, associated to the family of similitudes $\mathcal{S} = \{S_1, \ldots, S_l\}$ with ratios $\{r_1, \ldots, r_l\}$ ($\sum_{i=1}^{l} r_i^s = 1$). We shall call *subsets of the generation* k, $k \geq 1$, the elements of the following family:

$$\mathcal{E}_k = \left\{ E_\alpha : \alpha \in \mathcal{S}_k^l \right\}$$

and we shall also consider the following families of sets:

$$\mathcal{A}_1 = \mathcal{E}_1 \cup \left\{ E^j = \bigcup_{i=1}^{j} E_i : 1 < j < l \right\}$$

$$\mathcal{A}_k = \mathcal{E}_k \cup \left\{ E_\alpha^j = \bigcup_{i=1}^{j} E_{\alpha i} : \alpha \in \mathcal{S}_{k-1}^l,\ 1 < j < l \right\}, \qquad \text{for all } k > 1.$$

It is easy to check that $\mathcal{B}_1 = \bigcup_{k=1}^{\infty} \mathcal{E}_k$ and $\mathcal{B}_2 = \bigcup_{k=1}^{\infty} \mathcal{A}_k$ are two *differentiation bases* for (E, H^s), being:

$$\mathcal{B}_1(x) = \left\{ V \in \mathcal{B}_1 : x \in V \right\}, \qquad \text{for all } x \in E$$

$$\mathcal{B}_2(x) = \left\{ V \in \mathcal{B}_2 : x \in V \right\}, \qquad \text{for all } x \in E.$$

We are going to prove now a covering theorem, of Vitali type, for each one of the preceding bases ($\mathcal{B}_1$ and $\mathcal{B}_2$).

3.1 Theorem

Let $F \subset E$, and for every $x \in F$ let $U(x) \in \mathcal{B}_1(x)$. Then, we can choose a sequence $\{V_k\}_{k\geq 1} \subset (U(x))_{x\in F}$ (possibly finite) satisfying:

(a) $H^s(V_p \cap V_q) = 0$, if $p \neq q$ (almost disjoint).

(b) $F \subset \bigcup_{k\geq 1} V_k$.

Proof. We construct the sequence $\{V_k\}_{k\geq 1}$ in the following way:

Let

$$i_1 = \min\left\{ i : \mathcal{E}_i \cap \{U(x) : x \in F\} \neq \emptyset \right\}.$$

We choose all the sets of $\mathcal{E}_{i_1} \cap \{U(x) : x \in F\}$, and we denote these sets by $V_1, \ldots, V_p$.

If $F \subset \bigcup_{i=1}^{p} V_k$, this process is finished. In other case, let

$$i_2 = \min\left\{ i : i > i_1, \left\{ U : U \in \mathcal{E}_i \cap \{U(x) : x \in F\},\ U \not\subset \bigcup_{j=1}^{p} V_j \right\} \neq \emptyset \right\},$$

and we choose all the sets of $\mathcal{E}_{i_2} \cap \{U(x) : x \in F\}$ such that they are not contained in $\bigcup_{j=1}^{p} V_j$. We denote this sets by $V_{p+1}, \dots, V_q$. And so on.

It is obvious that the elements of the sequence $\{V_k\}_{k\geq 1}$, obtained in this way and which may be finite or infinite, are almost disjoint. Moreover, for every $x \in F$ the set $U(x)$ either has been chosen with its corresponding generation or it is contained in some of the already chosen sets. Then, in any case:

$$x \in U(x) \subset \bigcup_{k\geq 1} V_k$$

and therefore $F \subset \bigcup_{k\geq 1} V_k$. □

3.2. Theorem

Let $F \subset E$, and for every $x \in F$ let $U(x) \in \mathcal{B}_2(x)$. Then, we can choose a sequence $\{V_k\}_{k\geq 1} \subset (U(x))_{x\in F}$ (possibly finite) satisfying:

(a) $H^s(V_p \cap V_q) = 0$, if $p \neq q$ (almost disjoint).

(b) $F \subset \bigcup_{k\geq 1} V_k$.

Proof. We construct the sequence $\{V_k\}_{k\geq 1}$ in the following way:

Let

$$i_1 = \min\{i : \mathcal{A}_i \cap \{U(x) : x \in F\} \neq \emptyset\}.$$

We choose, for every $\alpha \in \mathcal{S}^l_{i_1-1}$ (if $i_1 = 1$, we consider $\mathcal{S}^l_0 = \{\emptyset\}$), the set $E^{j_0}_\alpha$ with:

$$j_0 = \max\{j : E^j_\alpha \in (U(x))_{x\in F}\}$$

and the remainder of the sets of $\mathcal{A}_{i_1} \cap \{U(x) : x \in F\}$ which are almost disjoint with $E^{j_0}_\alpha$ (if $\{E^j_\alpha : 1 < j < l,\ E^j_\alpha \in (U(x))_{x\in F}\} = \emptyset$, then we choose all the different sets of $\mathcal{A}_{i_1} \cap \{U(x) : x \in F\}$). We denote the selected sets by $V_1, \dots, V_p$.

If $F \subset \bigcup_{k=1}^{p} V_k$, this process is finished. In other case, let

$$i_2 = \min\left\{ i : i > i_1, \left\{ U : U \in \mathcal{A}_i \cap \{U(x) : x \in F\},\ U \not\subset \bigcup_{j=1}^{p} V_j \right\} \neq \emptyset \right\}.$$

Now we perform an analogous operation replacing the set $\mathcal{A}_{i_1} \cap \{U(x) : x \in F\}$ by the set $\{U : U \in \mathcal{A}_{i_2} \cap \{U(x) : x \in F\},\ U \not\subset \bigcup_{j=1}^{p} V_j\}$ and denote the selected sets by $V_{p+1}, \dots, V_q$. And so on.

The sequence $\{V_k\}_{k\geq 1}$, obtained in this way, satisfies (a) and (b) by the same arguments of the proof of the preceding theorem. □

In the next theorems, $\mathcal{B}$ denote any of the considered bases ($\mathcal{B}_1$ or $\mathcal{B}_2$).

Given a differentiation basis $\mathcal{B}$ on E, a set $F \subset E$ and $x \in E$, we define the *density of F at x* by the limit:

$$D(F,x) = \lim_{\substack{d(U)\to 0 \\ U\in\mathcal{B}(x)}} \frac{H^s(F\cap U)}{H^s(U)}$$

when this limit exists. Otherwise, we shall say that the density does not exist.

3.3. Theorem

The basis $\mathcal{B}$ is a density basis for (E, H^s).

Proof. We have to show that for every s-set $F \subset E$ one has:

$$D(F,x) = \chi_F(x) \qquad \text{for} \qquad H^s - \text{a.e. } x \in E.$$

(a) $D(F,x) = 0$ at H^s-a.e. $x \in E \setminus F$.

To prove this, it is enough to show that for every $\alpha > 0$ the set:

$$G_\alpha = \left\{ x \in E\setminus F : \exists\{U_k(x)\}_{k\geq 1} \subset \mathcal{B}(x),\ d(U_k(x)) \longrightarrow 0, \right.$$
$$\left. \frac{H^s(F\cap U_k(x))}{H^s(U_k(x))} > \alpha,\ \forall k \geq 1 \right\}$$

has H^s-measure zero. Let $\varepsilon > 0$. Using the Besicovitch and Moran Theorem [1], we can choose a closed and bounded set $F_1 \subset F$ such that $H^s(F\setminus F_1) < \varepsilon$. For every $x \in G_\alpha$, there is $U(x) \in \mathcal{B}(x)$ such that:

$$\frac{H^s(F\cap U(x))}{H^s(U(x))} \qquad \text{and} \qquad U(x)\cap F_1 = \emptyset \tag{1}$$

and so, applying Theorem 3.1 or 3.2 (if $\mathcal{B} = \mathcal{B}_1$ or $\mathcal{B} = \mathcal{B}_2$, respectively) to $(U(x))_{x\in G_\alpha}$, we can choose an almost disjoint sequence $\{V_k\}_{k\geq 1} \subset (U(x))_{x\in G_\alpha}$ such that $G_\alpha \subset \bigcup_{k\geq 1} V_k$ and so, by (1), we have that:

$$\frac{H^s(F\cap V_k)}{H^s(V_k)} > \alpha \qquad \text{and} \qquad V_k \cap F_1 = \emptyset \qquad \text{for all } k \geq 1. \tag{2}$$

Then, we have:

$$\begin{aligned} H^s(G_\alpha) &\le H^s\left(\bigcup_{k\ge1} V_k\right) \\ &= \sum_{k\ge1} H^s(V_k) \\ &< \frac{1}{\alpha}\sum_{k\ge1} H^s(F\cap V_k) \\ &= \frac{1}{\alpha} H^s\left(F\cap \bigcup_{k\ge1} V_k\right) \\ &= \frac{1}{\alpha} H^s\left((F\setminus F_1)\cap \bigcup_{k\ge1} V_k\right) \\ &\le \frac{1}{\alpha} H^s(F\setminus F_1) \\ &< \frac{\varepsilon}{\alpha}. \end{aligned}$$

But this is true for all $\varepsilon > 0$, and so $H^s(G_\alpha) = 0$.

(b) $D(F,x) = 1$ at H^s-a.e. $x \in F$.

It is an easy consequence of (a) and:

$$\frac{H^s(F\cap U)}{H^s(U)} = 1 - \frac{H^s((E\setminus F)\cap U)}{H^s(U)}. \square$$

We define the *maximal operator* associated to the basis $\mathcal{B}$, in the usual way, by:

$$\mathcal{M}f(x) = \sup_{U\in\mathcal{B}(x)} \frac{1}{H^s(U)}\int_U |f(y)|\, dH^s(y) \qquad \text{for all } x\in E$$

for every function $f \in L^1(E, H^s)$. We shall see now that this maximal operator satisfies an inequality of weak type (1,1).

3.4. Theorem

For every function $f \in L^1(E, H^s)$ and every number $\lambda > 0$, we have that:

$$H^s(\{x\in E : \mathcal{M}f(x) > \lambda\}) \le C\,\frac{\|f\|_1}{\lambda} \tag{3}$$

where $C > 0$ is a constant independent of λ and f.

Proof. Let $A_\lambda = \{x \in E : Mf(x) > \lambda\}$. Using the definition of the maximal operator, we have that, for every $x \in A_\lambda$, there is $U(x) \in \mathcal{B}(x)$ such that:

$$\frac{1}{H^s(U(x))} \int_{U(x)} |f(y)| \, dH^s(y) > \lambda \tag{4}$$

and, applying Theorem 3.1 or 3.2 (if $\mathcal{B} = \mathcal{B}_1$ or $\mathcal{B} = \mathcal{B}_2$, respectively) to $(U(x))_{x \in A_\lambda}$, we can find an almost disjoint sequence $\{V_k\}_{k \geq 1} \subset (U(x))_{x \in A_\lambda}$ such that $A_\lambda \subset \bigcup_{k \geq 1} V_k$, and their elements satisfy the inequality (4). Hence:

$$\begin{aligned} H^s(A_\lambda) &\leq H^s\left(\bigcup_{k \geq 1} V_k\right) \\ &= \sum_{k \geq 1} H^s(V_k) \\ &< \frac{1}{\lambda} \sum_{k \geq 1} \int_{V_k} |f(y)| \, dH^s(y) \\ &= \frac{1}{\lambda} \int_{\bigcup_{k \geq 1} V_k} |f(y)| \, dH^s(y) \\ &\leq \frac{1}{\lambda} \int_E |f(y)| \, dH^s(y) \\ &= \frac{\|f\|_1}{\lambda} \end{aligned}$$

and so we obtain (3) ($C = 1$, and it is independent of λ and f). □

Following the usual techniques of differentiation of integrals [2], and using the theorems 3.2 and 3.3, one can show easily the following differentiation theorem:

3.5. Theorem

For every function $f \in L^1(E, H^s)$ one has:

$$\lim_{\substack{d(U) \to 0 \\ U \in \mathcal{B}(x)}} \frac{1}{H^s(U)} \int_U f(y) \, dH^s(y) = f(x)$$

for H^s-a.e. $x \in E$.

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