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On the socle of a Jordan pair

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ABSTRACT

The socle of a nondegenerate Jordan pair, defined as the sum of all minimal inner ideals, is shown to have dcc on principal inner ideals. Furthermore, the socle is the direct sum of simple ideals, and can be represented as a direct limit of subpairs of finite capacity, imbedded as inner ideals.

Introduction

Let $V = (V^+, V^-)$ be a Jordan pair over an arbitrary commutative ring k of scalars. An inner ideal $M \subset V^\sigma (\sigma = \pm)$ is called *trivial* if $Q_M V^{-\sigma} = 0$, and *simple* if it is nontrivial and minimal among all nonzero inner ideals. The *socle* of V is $\operatorname{Soc} V = (\operatorname{Soc} V^+, \operatorname{Soc} V^-)$ where $\operatorname{Soc} V^\sigma$ is the sum of all simple and all trivial inner ideals of V^σ. This definition is an extension of the usual one in the nondegenerate case (where 0 is the only trivial inner ideal). In §1, we show that $\operatorname{Soc} V$ is an ideal of V, compatible with diagonal Peirce spaces. The proofs follow closely those of [4] for Jordan triple systems.

The main result of §2 (Th. 1) is the following characterization of the elements of the socle: If V is nondegenerate then $u \in \operatorname{Soc} V^+$ if and only if the inner ideal $(u) = ku + Q_u V^-$ generated by u has dcc on principal inner ideals. The same characterization then holds for Jordan algebras and triple systems. In these cases, the result is due to Fernández López and García Rus [2, 3], with some restrictions on the characteristic. While their proof of sufficiency carries over to Jordan pairs without change, the converse required Zel'manov's classification of simple Jordan algebras and triple systems. In contrast, the proof presented here is elementary and works for an arbitrary ring of scalars. It is based on a careful study of the Peirce decomposition of a simple inner ideal. Then we show inductively that every $u \in \operatorname{Soc} V^+$ is regular, and if $u = e_+$ is extended to a Jordan pair idempotent $e = (e_+, e_-)$ then $V_2(e)$ has finite capacity. By the results of [8], $(u) = V_2^+(e)$ has dcc on principal inner ideals.

In §3 we show, using results of McCrimmon [9] and following the proof for Jordan algebras by Osborn-Racine [11] that $\operatorname{Soc} V$ is a direct sum of simple ideals (Th. 2). A different proof was given in the Jordan triple case in [4]. Then we prove the analogue of Litoff's Theorem for Jordan pairs (Th. 3): A simple nondegenerate Jordan pair equal to its socle is a direct limit of finite capacity subpairs imbedded as inner ideals. The analogous result for linear Jordan algebras is due to Ánh [1] who used Zel'manov's classification. The proof given here is again elementary and also intrinsic in the sense that it avoids imbedding into associative algebras.

Terminology and notation follows [7]. When applying results on Jordan triple systems to Jordan pairs, we will sometimes identify V with the polarized Jordan triple system $V^+ \oplus V^-$. Expressions like $Q_V W$ are to be interpreted componentwise as $(Q(V^+)W^-, Q(V^-)W^+)$ or in the polarized triple systems.

§1. The socle of a Jordan pair

Let $V = (V^+, V^-)$ be a Jordan pair. Recall that an *inner ideal* of V^σ is a k-submodule M of V^σ such that $Q_M V^{-\sigma} \subset M$. Since we can always replace V by $V^{op} = (V^-, V^+)$ it will usually be sufficient to consider inner ideals of V^+. If $x \in V^+$ then $[x] = Q_x V^-$ is an inner ideal, the *principal inner ideal* determined by x. In general, $x \notin [x]$; in fact, this is the case if and only if x is (von Neumann) regular. The *inner ideal generated* by x, that is, the smallest inner ideal containing x, is easily seen to be $(x) = kx + [x]$. We remark that $u \in (x)$ implies $[u] \subset [x]$, a consequence of the identity $\{x, y, Q_x z\} = Q_x\{yxz\}$.

An inner ideal M is called *trivial* if $Q_M V^- = 0$; equivalently, every element of M is trivial (or an absolute zero divisor). An inner ideal M is *simple* if it is not trivial and minimal among non-zero inner ideals of V^+.

Lemma 1

The following conditions on an inner ideal M are equivalent.

(i) M is simple;

(ii) $M = V_2^+(d)$ where d is a division idempotent of V;

(iii) $M \neq 0$, and $M = [x]$ for all $0 \neq x \in M$.

Proof. (i) $\Longrightarrow$ (ii): By [7, 10.5], either $M = V_2^+(d)$, where d is a division idempotent, or M contains a non-zero trivial element z. The second case is impossible or else $(z) = M$ by minimality and M would be trivial.

(ii) $\Longrightarrow$ (iii): Every non-zero x in $V_2^+(d)$ is invertible in $V_2(d)$ hence

$$V_2^+(d) = Q_x V_2^-(d) = [x].$$

(iii) $\Longrightarrow$ (i): Immediate from the definitions. □

We shall call an element x in V^+ (or V^-) *simple* if (x) is a simple inner ideal. By Lemma 1, x is simple if and only if x is a non-zero element of some simple inner ideal, and if and only if $x = d_+$ can be extended to a division idempotent $d = (d_+, d_-)$ of V. In particular, simple elements are regular. The *socle* of V is $\operatorname{Soc} V = (\operatorname{Soc} V^+, \operatorname{Soc} V^-)$ where $\operatorname{Soc} V^\sigma$ is the sum of all simple and all trivial inner ideals. Equivalently, the elements of the socle are those of the form

$$s_1 + \cdots + s_n + t_1 + \cdots + t_m$$

where the s_i are simple and the t_j are trivial.

It will be convenient to single out a class of subpairs of V that behaves well with respect to these concepts. A subpair $U = (U^+, U^-)$ of V is called *full* if $Q_x V^{-\sigma} = Q_x U^{-\sigma}$, for all $x \in U^\sigma$, $\sigma = \pm$. For example, diagonal Peirce spaces $V_2(c)$, $V_0(c)$ are full, as follows from the Peirce relations. The proof of the following properties of full subpairs is straightforward and left to the reader.

Proposition 1

Let U be a full subpair of V.

(a) The simple elements (trivial elements, division idempotents) of U are precisely the simple elements (trivial elements, division idempotents) of V contained in U.

(b) $\operatorname{Soc} U \subset U \cap \operatorname{Soc} V$.

(c) If V is nondegenerate so is U.

(d) If T is a full subpair of U then T is a full subpair of V.

(e) U^+ and U^- are inner ideals of V. Conversely if $W = (W^+, W^-)$ is a subpair consisting of inner ideals and W is regular then W is full.

In order to prove that the socle is an ideal it is useful to introduce structural transformations [4]. For Jordan pairs, this concept takes the following form. A *structural transformation* between Jordan pairs V and W is a pair of maps (f,g) where $f : V^+ \to W^+$ and $g : W^- \to V^-$ (in the opposite direction!) are k-linear maps satisfying $Q(f(x)) = fQ_x g$ and $Q(g(y)) = gQ_y f$, for all $x \in V^+$ and $y \in W^-$. We indicate this by $(f,g) : V \rightleftarrows W$. The main examples are the *inner structural transformations*: If $u \in V^+$ and $v \in V^-$ then

$$(Q_v, Q_u) : V^{op} \rightleftarrows V,$$

$$(Q_v, Q_v) : V \rightleftarrows V^{op},$$

and

$$(B(u,v), B(v,u)) : V \rightleftarrows V$$

are structural, and so is the composition of structural transformations. If $(\varphi_+, \varphi_-) : V \to W$ is an isomorphism then $(\varphi_+, \varphi_-^{-1}) : V \rightleftarrows W$ is structural.

Finally, if (f,g) is structural so is $(g,f) : W^{op} \rightleftarrows V^{op}$.

Lemma 2

Let $(f,g) : V \rightleftarrows W$ be structural.

(a) If $M \subset V^+$ is an inner ideal so is $f(M) \subset W^+$, and if M is trivial so is $f(M)$.

(b) If $M \subset V^+$ is a simple inner ideal then $f(M)$ is either simple or trivial. In the first case, the restriction of f to M is injective.

Proof. (a) is immediate from the definitions. For (b), suppose that $Q_x g(W^-) = 0$ for some $0 \neq x \in M$. Since $M = [x]$ we then have

$$Q_M g(W^-) = Q_x Q(V^-) Q_x g(W^-) = 0$$

hence

$$0 = f(Q_M g(W^-)) = Q(f(M))W^-$$

and $f(M)$ is trivial. Otherwise, $Q_x g(W^-) \neq 0$ for all $0 \neq x \in M$. By (a) (applied to (g,f)), $g(W^-)$ is an inner ideal of V^- hence $Q_x g(W^-)$ is an inner ideal contained in M which must equal M by simplicity. Thus

$$f(M) = f(Q_x g(W^-)) = Q(f(x))W^-$$

for all $0 \neq x \in M$. By Lemma 1, either $f(M) = 0$ or $f(M)$ is simple and f is injective on M. □

Proposition 2

(a) If $(f,g) : V \rightleftarrows W$ *is structural then* $f(\operatorname{Soc} V^+) \subset \operatorname{Soc} W^+$ *and* $g(\operatorname{Soc} W^-) \subset \operatorname{Soc} V^-$.

(b) $\operatorname{Soc} V$ *is an ideal of* V, *invariant under all structural transformations.*

(c) Let c *be an idempotent, and let* $p^\sigma : V^\sigma \to V_j^\sigma(c)$ *and* $i^\sigma : V_j^\sigma(c) \to V^\sigma$ *be the Peirce projections and inclusions, respectively. Then, for* $j = 0, 2$,

$$(p^+, i^-) : V \rightleftarrows V_j(c)$$

and

$$(i^+, p^-) : V_j(c) \rightleftarrows V$$

are structural, and $\operatorname{Soc} V_j(c) = V_j(c) \cap \operatorname{Soc} V$.

Proof. (a) is immediate from Lemma 2. In particular, $\operatorname{Soc} V$ is invariant under all inner structural transformations, hence it is an outer ideal (observe $\{xyz\} = z + Q_xQ_yz - B(x,y)z$). Since the socle is a sum of inner ideals, the outer ideal property implies that it is in fact an ideal, proving (b). For part (c), (p^+, i^-) is structural by the Peirce relations. Hence $\operatorname{Soc} V_j^-(c) \subset \operatorname{Soc} V^-$ and $p^+(\operatorname{Soc} V^+) \subset \operatorname{Soc} V_j^+(c)$ which implies $V_j^+(c) \cap \operatorname{Soc} V^+ \subset \operatorname{Soc} V_j^+(c)$. The assertion follows by interchanging the roles of $+$ and $-$. □

§ 2. Characterization of the socle

In the following three Lemmas, V denotes a nondegenerate Jordan pair. We first analyze the Peirce decomposition of a simple inner ideal M. Let c be an idempotent with Peirce spaces $V_j = V_j(c)$, $j = 0, 1, 2$. For $j = 0, 2$, V_j^+ is an inner ideal hence so is $M \cap V_j^+$, and by simplicity of M, either $M \cap V_j^+ = 0$ or $M \subset V_j^+$. Let $p_i : V^+ \to V_i^+$ be the Peirce projection and $M_i = p_i(M)$. If $i = 0$ or $i = 2$ then by Lemma 2 and Prop. 2(c), either $M_i = 0$ or M_i is simple and $p_i : M \to M_i$ is injective. In particular, if

$$x = x_2 + x_1 + x_0$$

is the Peirce decomposition of a non-zero element $x \in M$ then $x_i = 0$ implies $M_i = 0$, and if $x_i \neq 0$ then x_i is simple. For $i = 1$, the situation is more complicated.

Lemma 3

(a) If $x_1 = 0$ *then either* $x_2 = 0$ *or* $x_0 = 0$ *as well.*

(b) If $x_1 \neq 0$ *and* $x_j = 0$ *for* $j = 0$ *or* $j = 2$ *then* x_1 *is simple,* M_1 *is a simple inner ideal, and* $M = Q_x V_1^-$.

Proof. (a) Suppose $x_j \neq 0$ ($j = 0$ or $j = 2$). Then x_j is simple in V_j hence $x_j = Q(x_j)y_j$ for some $y_j \in V_j^-$. By orthogonality of V_j and V_{2-j},

$$Q_x y_j = Q(x_j)y_j = x_j \in M \cap V_j^+$$

whence $M \subset V_j^+$ and $x_{2-j} = 0$.

(b) From $x_j = 0$ we have $M_j = 0$ which means that M is contained in $V_{2-j}^+ \oplus V_1^+$, the kernel of p_j. We first show

$$(1) \qquad Q_M V_j^- = Q(M_1)V_j^- = Q(M_1)V_{2-j}^- = 0.$$

Indeed, by the Peirce relations,

$$Q_M V_j^- = Q(M_1)V_j^- \subset M \cap Q(V_1^+)V_j^- \subset M \cap V_{2-j}^+ = 0$$

since $x_1 \neq 0$ implies $M \not\subset V_{2-j}^+$. Also, $Q(M_1)V_{2-j}^- \subset V_j^+$ hence

$$Q(Q(M_1)V_{2-j}^-)V^- \subset Q(Q(M_1)V_{2-j}^-)V_j^- \subset Q(M_1)Q(V_{2-j}^-)Q(M_1)V_j^- = 0$$

implies $Q(M_1)V_{2-j}^- = 0$ by nondegeneracy.

Next, we claim

$$(2) \qquad p_1(Q_u V_1^-) = Q(u_1)V^-,$$

for all $u = u_{2-j} + u_1 \in M$. Indeed,

$$Q(u_{2-j} + u_1)V_1^- = \{u_{2-j}, V_1^-, u_1\} + Q(u_1)V_1^- \subset V_{2-j}^+ \oplus V_1^+$$

by the Peirce relations, hence

$$p_1(Q_u V_1^-) = Q(u_1)V_1^- = Q(u_1)V^-$$

by (1).

Now $Q_u V_1^-$ is an inner ideal:

$$\begin{aligned} Q(Q_u V_1^-)V^- &= Q_u Q(V_1^-)Q_u V^- \\ &\subset Q_u Q(V_1^-)(V_{2-j}^+ \oplus V_1^+) \\ &\subset Q_u(V_j^- \oplus V_1^-) \\ &= Q_u V_1^-, \end{aligned}$$

by (1) and the Peirce relations. Also, $u_1 \neq 0$ implies $Q(u_1)V^- \neq 0$ by nondegeneracy, and thus $Q_u V_1^- \neq 0$ by (2). Since M is simple, $M = Q_u V_1^-$, and by (2),

$$p_1(M) = M_1 = Q(u_1)V^-$$

for all non-zero $u_1 \in M_1$, showing that M_1 is a simple inner ideal. In particular, x_1 is a simple element, and $M = Q_x V_1^-$. This completes the proof. □

Lemma 4

Let $x_0 \neq 0$, and extend $x_0 = d_+$ to a division idempotent $d \in V_0(c)$. Then $x \in V_2^+(e)$ where $e = c + d$, and $c_+ + x$ is conjugate to e_+ by a d-elementary automorphism.

(See [8, §1], for the definition of elementary automorphisms).

Proof. Let V_{ij} be the Peirce spaces relative to the orthogonal system $e_1 = c$, $e_2 = d$, and decompose $x = \sum x_{ij}$ accordingly. Then by definition, $x_0 = x_{22} = e_2^+$, $x_2 = x_{11}$, and $x_1 = x_{12}+x_{10}$. The Peirce decomposition of x relative to d is now $x = u_2+u_1+u_0$ ($u_i \in V_i^+(d)$) where $u_2 = d_+$ is invertible in $V_2(d)$ with inverse $u_2^{-1} = d_-$, $u_1 = x_{12}$, and $u_0 = x_{11} + x_{10}$. By [8, Lemma 1], the d-elementary transform $x' = B(u_1, d_-)x$ is given by

$$x' = u_2 + (u_0 - Q(u_1)d_-) \in V_2^+(d) \oplus V_0^+(d).$$

Since x' is simple along with x and $u_2 \neq 0$, Lemma 3 (a) yields

$$0 = u_0 - Q(u_1)d_- = (x_{11} - Q(x_{12})d_-) + x_{10} \in V_{11}^+ \oplus V_{10}^+.$$

Thus $x_{10} = 0$ and

$$x = x_{11} + x_{12} + x_{22} \in V_2^+(e).$$

Furthermore, $B(u_1, d_-)x = d_+$ which implies

$$B(u_1, d_-)(c_+ + x) = c_+ + d_+ = e_+$$

by orthogonality of c and d. □

Lemma 5

If $x_1 \neq 0$ and $x_0 = 0$ then $x_1 = d_+$ can be extended to a division idempotent $d \in V_1(c)$ such that $V_2(d) \subset V_1(c)$. Furthermore, $c_+ + x$ is conjugate to c_+ by a d-elementary automorphism.

Proof. By Lemma 3 (b), x_1 is simple and $[x_1] = M_1 \subset V_1^+(c)$. Also, $x = Q_x y_1$ for some $y_1 \in V_1^-(c)$. Since $x = x_2 + x_1$, this means that $x_2 = \{x_1 y_1 x_2\}$ and $x_1 = Q(x_1)y_1$. Setting $y_1' = Q(y_1)x_1$ we have $y_1' = Q(y_1')x_1$ and $x_1 = Q(x_1)y_1'$ [7, 5.2], and

$$\{x_1, Q(y_1)x_1, x_2\} = \{Q(x_1)y_1, y_1, x_2\} = \{x_1 y_1 x_2\} = x_2.$$

Thus $d = (x_1, y_1') \in V_1(c)$ is an idempotent extending x_1, it is a division idempotent since x_1 is simple, and $\{d_+ d_- x_2\} = x_2$ whence $x_2 \in V_1^+(d)$. Also,

$$V_2^+(d) = M_1 \subset V_1^+(c)$$

and therefore

$$V_2^-(d) = Q(y_1')V_2^+(d) \subset V_1^-(c).$$

Furthermore, by the Peirce relations.

$$\begin{aligned} B(x_2, d_-)(c_+ + x) &= c_+ + x_2 + B(x_2, d_-)d_- \\ &= c_+ + x_2 + d_- - x_2 \\ &= c_+ + d_+, \end{aligned}$$

and

$$\begin{aligned} B(d_+, c_-)(c_+ + d_+) &= c_+ - \{d_+ c_- c_+\} + Q(d_+)Q(c_-)c_+ \\ &\quad + d_+ - 2Q(d_+)c_- + Q(d_+)Q(c_-)d_+ \\ &= c_+ - d_+ + d_+ \\ &= c_+, \end{aligned}$$

since $Q(d_+)c_- \in M_1 \cap V_0^+(c) = 0$. This shows that $\varphi_+(c_+ + x) = c_+$, where $\varphi = \beta(d_+, c_-)\beta(x_2, d_-)$. It remains to show that φ is d-elementary. This is clear for the second factor by $x_2 \in V_1^+(d)$. Also, $Q(d_+)c_- = 0$ shows that

$$c_- = v_1 + v_0 \in V_1^-(d) \oplus V_0^-(d).$$

Hence

$$\beta(d_+, c_-) = \beta(d_+, v_1 + v_0) = \beta(d_+, v_1)$$

is d-elementary as well. □

Remark. If we decompose $c_+ = u_1 + u_0 \in V_1^+(d) \oplus V_0^+(d)$, then it follows from [10, 1.9] that $c_1 = (u_1, v_1)$ and $c_0 = (u_0, v_0)$ are orthogonal idempotents and c_1 is collinear to d.

Before stating the main result of this section, we recall from [8] that the following finiteness conditions on a degenerate Jordan pair V are equivalent.

(i) finite capacity: there exists a strong frame, i.e., a finite set of orthogonal division idempotents whose common Peirce-0-space is zero;

(ii) finite length: the lengths of chains of principal inner ideals are bounded;

(iii) both chain conditions on principal inner ideals.

If these conditions hold, then the capacity (that is, the cardinality of a strong frame) equals the maximum length of a chain of principal inner ideals, and V is regular.

Theorem 1

Let V be a nondegenerate Jordan pair and $u \in V^+$. Then the following conditions are equivalent.

(i) $u \in \mathrm{Soc}\, V^+$;

(ii) u is regular, and if $u = e_+$ for some idempotent e then $V_2(e)$ has finite capacity;

(iii) (u) has dcc on principal inner ideals.

If these conditions hold, then the maximum length of a chain of principal inner ideals in (u) is finite and equals the capacity of $V_2(e)$.

Remark. Any two extensions of u to idempotents are conjugate by an automorphism [8, Cor. of Prop. 2], hence (ii) depends only on u.

Proof. (i) $\Longrightarrow$ (ii): Let $u = s_1 + \cdots + s_n$ where the s_i are simple elements. We prove (ii) by induction on n. For $n = 0$ there is nothing to prove. Now suppose that $c_+ = s_1 + \cdots + s_n$ is regular, and $V_2(c)$ has finite capacity, where $c = (c_+, c_-)$ is an idempotent. Let $x = s_{n+1}$ be a simple element, and decompose $x = x_2 + x_1 + x_0$ with respect to c. We distinguish the following cases.

Case 1. $x_0 \neq 0$. With the notation of Lemma 4, $c_+ + x$ is conjugate to $c_+ + d_+ = e_+$. Hence it suffices to show that $W = V_2(e)$ has finite capacity. Now W inherits nondegeneracy from V. If $\{e_1, \ldots, e_r\}$ is a strong frame of $V_2(c)$ then [8, Prop. 3]

$$c \approx e_1 + \cdots + e_r,$$

where $\approx$ means that the idempotents are associated in the sense that they have the same Peirce spaces. It follows that

$$e \approx e_1 + \cdots + e_r + d$$

whence $\{e_1, \ldots, e_r, d\}$ is a strong frame of W.

Case 2. $x_1 \neq 0 = x_0$. Then $c_+ + x$ is conjugate to c_+ by Lemma 5, and we are done by induction hypothesis.

Case 3. $x_0 = x_1 = 0$. Then $c_+ + x = c_+ + x_2 \in V_2^+(c)$. By regularity of $V_2(c)$ we have $c_+ + x = e_+$ for an idempotent $e \in V_2(c)$, and by [8, Prop. 3(b)] $V_2(e)$, which is the same as the Peirce space of e in $V_2(c)$, has again finite capacity.

(ii) $\Longrightarrow$ (iii): By regularity, $(u) = [u] = V_2^+(e)$. Now (iii) and the last assertion follow from [8, Th. 3].

(iii) $\Longrightarrow$ (i): This can be proved as in [2]. For completeness, we present the argument, adapted to the Jordan pair case. Let $\mathcal{M}$ be the set of all $[x]$ where $x \in (u)$

and $u - x \in \operatorname{Soc} V^+$. Then $[u] \in \mathcal{M}$ so $\mathcal{M}$ is not empty. Choose a minimal $M = [x]$ in $\mathcal{M}$. If $x = 0$ then $u \in \operatorname{Soc} V^+$ and we are done. Now suppose $x \neq 0$. By the dcc for (u) and [7, 10.5], there exists a division idempotent d such that $d_+ \in M$. Decompose $x = x_2 + x_1 + x_0$ relative to d. The socle is an ideal containing d hence

$$x_2 = Q(d_+)Q(d_-)x$$

and

$$x_1 = \{d_+ d_- x\} - 2x_2$$

are in the socle, and so is

$$u - x_0 = (u - x) + (x_2 + x_1).$$

Also, $x_2, x_1 \in (x)$ since $d_+ \in M \subset (x)$ and

$$\{d_+ d_- x\} = Q(d_+ + x)d_- - Q(d_+)d_- - Q_x d_- \in (x).$$

It follows that $x_0 \in (x) \subset (u)$ as well, showing $[x_0] \in \mathcal{M}$ and $[x_0] \subset [x]$ (cf. the remark made at the beginning of §1). By minimality of M we have

$$M = [x_0] = Q(x_0)V^- \subset V_0^+(d),$$

contradicting $0 \neq d_+ \in M \cap V_2^+(d)$. This completes the proof. □

Corollary 1

The following conditions on a degenerate Jordan pair V are equivalent.
(i) $V = \operatorname{Soc} V$;
(ii) V has dcc on principal inner ideals;
(iii) V is the union of full subpairs of finite capacity.

This is immediate from Th. 1. Note that a subpair of finite capacity will be full as soon as it consists of inner ideals, by regularity and Prop. 1.

Let J be a Jordan algebra or triple system. Then $V = (J, J)$ is a Jordan pair, and (principal) inner ideals and the socle are the same for J and V^+. This shows

Corollary 2

Let J be a nondegenerate Jordan algebra or triple system. Then $u \in \operatorname{Soc} J$ if and only if (u) has dcc on principal inner ideals. In this case, u is regular.

Note, however, that condition (ii) of Th. 1 has no natural counterpart for algebras or triple systems: Even if $V = (J, J)$, $V_2(e)$ will in general not be the Jordan pair associated with a subalgebra or subtriple of J.

We show next that the socle behaves well relative to ideals and full subpairs. For a different proof in the Jordan triple case, see [4].

Proposition 3

Let V be nondegenerate and $I = (I^+, I^-)$ an ideal or a full subpair of V. Then the simple elements (division idempotents) of I are just the simple elements (division idempotents) of V contained in I, and $\operatorname{Soc} I = I \cap \operatorname{Soc} V$.

Proof. Let first I be an ideal. Then for any regular element $x = Q_x y \in I^+$ we have

$$Q_x V^- = Q_x Q_y Q_x V^- \subset Q_x Q_y I^+ \subset Q_x I^- \subset Q_x V^-,$$

hence the principal inner ideal $[x]$ is the same computed in V or in I. Simple elements are regular and I is nondegenerate [7, 4.13]. Hence the first assertion and the inclusion $\operatorname{Soc} I \subset I \cap \operatorname{Soc} V$ follow easily. The corresponding statement for a full subpair holds by Prop. 1. Conversely, let $x \in I^+ \cap \operatorname{Soc} V^+$. Then x is regular by Th. 1, say, $x = Q_x y$, and we may even assume $y \in I^-$ and $y = Q_y x$ (replace y by $Q_y x$, having chosen $y \in I^-$ in case of a full subpair). Now $x = \sum s_i$ where the s_i are simple. Hence $y = \sum Q_y s_i$ and the summands are zero or simple by Lemma 2 and belong to I^- as I^- is an inner ideal. Thus $y \in \operatorname{Soc} I^-$, showing $x = Q_x y \in \operatorname{Soc} I^+$ since $\operatorname{Soc} I$ is an ideal in I. □

Corollary

If $V = \operatorname{Soc} V$ is nondegenerate, then a full nonzero subpair of V contains a division idempotent.

§3. Structure of the socle

Recall that two orthogonal idempotents e and d of a Jordan pair are *connected* if $V_1(e) \cap V_1(d)$ contains invertible elements of $V_2(e+d)$, and V is called connected [12] if any two orthogonal division idempotents of $V/\operatorname{Rad} V$ are connected. We denote the ideal of V generated by a subset X by $\langle X \rangle$.

Lemma 8

Let c and d be orthogonal division idempotents of a nondegenerate Jordan pair V. Then the following conditions are equivalent.

(i) c and d are connected;

(ii) $V_1(c) \cap V_1(d) \neq 0$;

(iii) $d \in \langle c \rangle$;

(iv) $\langle d \rangle = \langle c \rangle$.

Proof. Let $e_1 = c$, $e_2 = d$, $e = e_1 + e_2$, and denote the Peirce spaces of V relative to e_1, e_2 by V_{ij}. Then the equivalence of (i) and (ii) follows from [6, Th. 6.3.1], applied to the unital Jordan algebra $J = V_2^+(e)$ (with unit $1 = e_+$ and U-operators $U_x = Q(x)Q(e_-)$) which is nondegerate and has e_1^+, e_2^+ as supplementary set of orthogonal division idempotents. If c and d are connected then

$$d \in V_{22} = Q(V_{12})V_{11} \subset \langle c \rangle.$$

If $V_{12} = 0$, we have $V_2(c) = V_{11}$, $V_1(c) = V_{10}$, hence

$$\begin{aligned} \langle c \rangle &= V_2(c) \oplus V_1(c) \oplus Q(V_1(c))V_2(c) \\ [9,\ 2.13] \qquad &= V_{11} \oplus V_{10} \oplus Q(V_{10})V_{11} \\ &\subset V_{11} \oplus V_{10} \oplus V_{00}, \end{aligned}$$

and $d \in V_{22}$, showing $d \notin \langle c \rangle$. Thus we have (i) $\Longleftrightarrow$ (iii). The equivalence with (iv) follows by symmetry. □

Theorem 2

Let V be nondegenerate.

(a) A simple element generates a simple ideal of V.

(b) Soc V is the direct sum of simple ideals of V.

(c) A nonzero ideal of Soc V is simple if and only if it is connected.

Proof. (a) Let I be the ideal of V generated by a simple element $x \in V^+$. Extend $x = c_+$ to a division idempotent c and let $V = V_2 \oplus V_1 \oplus V_0$ relative to c. Then $c_- = Q(c_-)c_+ \in I^-$ hence

$$V_2 = \big(Q(c_+)V^-, Q(c_-)V^+\big) \subset I,$$

and $I = \langle c \rangle = \langle V_2 \rangle$. By [9,2.13], $I = V_2 \oplus V_1 \oplus Q(V_1)V_2$. This shows that the ideal of I generated by c (or x) is I itself. Also, I is nondegenerate [7, 4.13]. Thus we may replace V by I and then have to show: If V is generated (as an ideal) by a division idempotent c then V is simple. An ideal K of V decomposes $K = K_2 \oplus K_1 \oplus K_0$, where $K_i = K \cap V_i$. If $K_2 \neq 0$ then $K_2 = V_2$ since V_2 is a division pair. Thus $V = \langle c \rangle = K$ in this case. Now suppose $K_0 \neq 0$. Clearly $V = \operatorname{Soc} V$. Since the socle is regular (Th. 1) so is K. By Proposition 1, K is full in V, and K_0 is full in K as a Peirce-0-space. By the Corollary of Proposition 3, K_0 contains a division idempotent d. Then $d \in \langle c \rangle = V$ hence $V = \langle d \rangle = K$ by Lemma 6. Finally, suppose $K_2 = K_0 = 0$. Then

$$Q(K_1)V_2 \subset K_0 = 0,$$

$$Q(K_1)V_0 \subset K_2 = 0,$$

and by [9, formula (P1)],

$$Q(K_1)V_1 \subset \{\{K_1V_1V_2\}V_2K_1\} + \{Q(K_1)V_2, V_1, V_2\} = 0$$

since $\{K_1V_1V_2\} \subset K_2 = 0$. Hence $K_1 = 0$ by nondegeneracy of V.

(b) Since $\operatorname{Soc} V$ is spanned by simple elements this is immediate from (a) (directness of the sum follows from a standard argument).

(c) If I is a simple ideal of $\operatorname{Soc} V$ then $I = \langle c \rangle$ for any division idempotent $c \in I$. By Lemma 6, any division idempotent d of I, orthogonal to c, is connected to c. Now

$$\operatorname{Rad} I = I \cap \operatorname{Rad}(\operatorname{Soc} V) = 0$$

by regularity of the socle. Thus I is connected. If I is a nonsimple ideal of $\operatorname{Soc} V$ then (by (a) and (b)) it contains two different simple ideals $\langle c \rangle \neq \langle d \rangle$ generated by (necessarily orthogonal) division idempotents c and d which are not connected by Lemma 6. This completes the proof. □

By Theorem 2, the study of the socle reduces to the case where it is simple. Thus assume that V is simple, nondegenerate, and equal to its socle; equivalently, V has dcc on principal inner ideals. If V has acc on principal inner ideals as well, then it has finite capacity and the structure of V is well known [7, 12.12]. To handle the case of infinite capacity, we first prove

Lemma 7

Let V be a simple Jordan pair with idempotents c and d such that $0 \neq V_2(d) \subset V_1(c)$. Then $V_0(c) \cap V_1(d) = 0$ implies $V_0(c) = 0$.

Proof. Since $d \in V_1(c)$ there is a simultaneous Peirce decomposition $V = \sum V(ij)$ where $V(ij) = V_i(c) \cap V_j(d)$ [10, 1.8]. Now $V_2(d) \subset V_1(c)$ implies $V(22) = V(02) = 0$, and $V(01) = 0$ by hypothesis. Hence

$$V_2(d) = V(12),$$
$$V_1(d) = V(21) \oplus V(11),$$
$$V_0(d) = V(20) \oplus V(10) \oplus V(00),$$

and

$$V_0(c) = V(00).$$

Simplicity of V implies

$$V = \langle d \rangle = V_2(d) \oplus V_1(d) \oplus Q\big(V_1(d)\big)V_2(d)$$

hence

$$\begin{aligned} V_0(d) &= Q\big(V(21) + V(11)\big)V(12) \\ &= Q\big(V(21)\big)V(12) + \big\{V(21), V(12), V(11)\big\} + Q\big(V(11)\big)V(12) \\ &\subset 0 + V(20) + V(10) \end{aligned}$$

from the Peirce relations. It follows that $V(00) = V_0(c) = 0$. □

We now prove the analogue of Litoff's theorem [5, p. 90, Th. 3] for Jordan pairs. An associative coordinate algebra means a triple $C = (R, j, R_0)$ where R is an associative unital k-algebra with involution j and ample subspace R_0 [6, 5.4]. We denote by $H_n(C)$ the Jordan algebra of $n \times n$ hermitian matrices over R with diagonal elements in R_0, and by $(H_n(C), H_n(C))$ the associated Jordan pair.

Theorem 3

Let $V = \mathrm{Soc}\, V$ be a simple nondegenerate Jordan pair of infinite capacity.

(a) If e is any nonzero idempotent then $V_2(e)$ is simple of finite capacity, and $V_0(e) \neq 0$. The $V_2(e)$ form a directed set with respect to inclusion, and $V = \varinjlim V_2(e)$.

(b) There exists an associative coordinate algebra C such that every $V_2(e)$ of capacity n is isomorphic to $(H_n(C), H_n(C))$, where $C = (R, j, R_0)$ is one of the following:

- *(I) $R = D \oplus D^{op}$ where D is a division ring, j is the exchange involution, and R_0 is the diagonal;*
- *(II) R is a split quaternion algebra over a field K, with standard involution and $R_0 = K \cdot 1$;*
- *(III) R is a division ring.*

Remark. $(H_n(C), H_n(C))$ is isomorphic to the Jordan pair of all $n \times n$ matrices over D in case (I), and to the Jordan pair of $2n \times 2n$ alternating matrices over K in case (III) [7, 12.8].

Proof. (a) By Theorem 1 and Theorem 2, $V_2(e)$ has finite capacity and is connected since connectedness is inherited by diagonal Pierce spaces. Thus $V_2(e)$ is simple. If $V_0(e) = 0$ then V would have finite capacity by [8, Theorem 3], contrary to the hypothesis. Furthermore, V is the union of the $V_2(e)$ by Theorem 1 (ii) . Thus it remains to show: For any two idempotents c, d, there exists an idempotent e such that $V_2(c) \cup V_2(d) \subset V_2(e)$. Since $V_2(e)$ is full, it suffices to have c and d in $V_2(e)$, and since c_+, c_-, d_+, d_- are finite sums of simple elements it is enough to prove:

(*) Every finite subset X of $V^+ \cup V^-$ consisting of simple elements is contained in $V_2^+(e) \cup V_2^-(e)$, for some idempotent e.

We show this by induction on the cardinality of X. If X is empty let $e = 0$. Now suppose $X \subset V_2^+(c) \cup V_2^-(c)$ and let x be a simple element (which we may take in V^+, passing to V^{op} if necessary). Also, we may assume $x \notin V_2^+(c)$ or else $X \cup \{x\} \subset V_2^+(c) \cup V_2^-(c)$. Decompose $x = x_2 + x_1 + x_0$ with respect to c. Then there are the following cases.

Case 1. $x_0 \neq 0$. With the notation of Lemma 4, let $e = c + d$. Then $x \in V_2^+(e)$ and $V_2(c) \subset V_2(e)$, proving $X \cup \{x\} \subset V_2^+(e) \cup V_2^-(e)$.

Case 2. $x_0 = 0$ and $x_1 \neq 0$. Let $d \in V_1(c)$ be as in Lemma 5, and set $W = V_0(c)$. Then the Peirce decompositions relative to c and d are compatible [10, 1.8], hence $W = W_2 \oplus W_1 \oplus W_0$ where $W_i = W \cap V_i(d)$, and $W_2 = 0$ by $V_2(d) \subset V_1(c)$. Also, $W \neq 0$ implies $W_1 \neq 0$ by Lemma 7, and $Q(W_1)W_0 \subset W_2 = 0$. It follows that W_1 is a full subpair of W and hence of V. By the Corollary of Proposition 3, W_1 contains a division idempotent d' and $Q(d')d = 0$ implies that d and d' are collinear [10, 3.7]. Now $e = c + d'$ is an idempotent since d' is orthogonal to c, and $d \in V_1(c) \cap V_1(d') \subset V_2(e)$ (by collinearity of d and d'). In particular, $d_+ = x_1 \in V_2^+(e)$. Also, $V_2(c) \subset V_2(e)$ which implies $x = x_2 + x_1 \in V_2^+(e)$. Thus $X \cup \{x\} \subset V_2^+(e) \cup V_2^-(e)$. This completes the proofs of (*) and (a).

(b) Let $V_2(c)$ and $V_2(d)$ be Peirce-2-spaces of the same capacity n, both contained in some $V_2(e)$. By [8, Prop. 3, Cor. 3 of Th. 2], there exists an elementary automorphism of $V_2(e)$ mapping $V_2(c)$ isomorphically onto $V_2(d)$. In particular, any two Peirce-2-spaces of the same capacity are isomorphic. On the other hand, V contains Peirce-2-spaces of arbitrary capacity: If $V_2(e)$ has capacity n then $V_0(e)$ contains a division idempotent f and $V_2(e + f)$ has capacity $n + 1$. Now (b) follows easily from [7, 12.12]. □

Corollary 1

The socle of a nondegenerate Jordan pair is a direct limit of full subpairs of finite capacity.

Proof. Since $S = \operatorname{Soc} V$ is a regular ideal (Theorem 1), it is full in V (Proposition 1). Decompose S into a direct sum of simple ideals (Theorem 2). Let $\mathcal{U}$ be the collection of all subpairs of V which are finite sums of ideals of finite capacity of S, plus finite sums of Peirce-2-spaces in the simple ideals of infinite capacity of S. Then $\mathcal{U}$ is directed by inclusion with union S. □

Corollary 2

Let $V = \operatorname{Soc} V$ be nondegenerate and simple. Then any two finite ordered sets of orthogonal division idempotents of the same cardinality are conjugate up to association by an elementary automorphism, and the group $E(V)$ of elementary automorphisms equals $E(d, V)$ where d is any division idempotent.

Proof. In case of finite capacity, this is [8, Cor. 3 of Th. 2]. In the infinite case, it follows from Theorem 3 and [8, Prop. 1 (c)]. □

Remarks.

(1) I don't know if the set of all full subpairs of finite capacity is directed.

(2) Suppose that $V = (J, J)$ is the Jordan pair associated with a simple nondegenerate Jordan algebra or triple system equal to its socle. Although V is then a direct limit of subpairs $V_2(e)$ which all contain invertible elements and are therefore associated with unital Jordan algebras, I don't know if these can be chosen compatible with the algebra or triple structure except for linear Jordan algebras, see [1].

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