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A note on bornivorous barrels of C(X,E)

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Abstract

If X is a compact topological space and E is a locally convex space, we prove that the sets $\{\phi \in C(X,E) : \phi(X) \subset T\}$, T running through the set of bornivorous barrels in E, form a base of bornivorous barrels in $C(X,E)$ if and only if E'_β has prtoperty (P) of Pietsch.

In this note X is a completely regular and Hausdorff topological space and E is a Hausdorff locally convex space. We denote by $C(X,E)$ the space of all continuous functions on X with values in E, endowed with the compact-open topology.

If X is compact and $T \subset E$ is a bornivorous barrel, the set

$$C(X,T) := \{\phi \in C(X,E) : \phi(X) \subset T\}$$

is obviously a bornivorous barrel of $C(X,E)$. Schmets [4, Proposition I.6.5] proves that the sets $C(X,T)$, T running through the set of bornivorous barrels in E, form a base of bornivorous barrels in $C(X,E)$ if E has a fundamental sequence of bounded subsets. Here we prove that the sets $C(X,T)$ form a base of bornivorous barrels if and only if E'_β has property (B) of Pietsch.

For terminology and notations used in this note we refer to [2] and [3].

1. The compact case

Throughout this section X is a compact and Hausdorff topological space. We say that $C(X,E)$ has the bornivorous barrel property (b.b. property), if the sets $C(X,T)$ form a base of bornivorous barrels in $C(X,E)$.

Proposition 1

The following are equivalent:

i) There exists an infinite and compact space X_0 such that $C(X_0,E)$ has the b.b. property.

ii) $C(X,E)$ has the b.b. property for every compact space X.

Proof. i) $\Rightarrow$ ii) Let us suppose that there exists an infinite and compact space X_0 such that $C(X_0,E)$ has the b.b. property. Let X be a compact space and let $\mathcal{T}$ be a bornivorous barrel in $C(X,E)$. Let $\mathcal{H}$ be the family of the absolutely convex subsets D of E so that $C(X,D) \subset \mathcal{T}$. We define :

$$\mathcal{T}_0 := \bigcup_{D\in\mathcal{H}} C(X_0,D).$$

In [2,Proposition 2.5(i) $\Rightarrow$ (ii)] is proved that $\mathcal{T}_0$ is a bornivorous barrel in $C(X_0,E)$. Hence there exists a bornivorous barrel T in E satisfying $C(X_0,T) \subset \mathcal{T}_0$.

Now, following Mendoza's proof, it can be proved that $C(X,T) \subset \mathcal{T}$. □

The preceding result allow us to study the space $C_0(E)$ formed of all sequences (x_n) in E convergent to zero, endowed with the uniform convergence topology (it is well known that there exists a topological isomorphism from $C_0(E)$ onto $C(\mathbb{N}^*,E)$, where $\mathbb{N}^* = \mathbb{N}\cup\{\infty\}$ is the Alexandroff compactification of $\mathbb{N}$). If $C_0(E)'_\beta$ denotes the dual space of $C_0(E)$ endowed with the topology $\beta[C_0(E)',C_0(E)]$, following [1, Proposition 1.6] we shall identify $C_0(E)'_\beta$ with a suspace of $\ell^1_\pi\{E'_\beta\}$.

If $C_0(T)$ denotes the set $\{(x_n)\in C_0(E) : x_n \in T\ (\forall n)\}$, we have the following

Proposition 2

The following statements are equivalent :

i) For every bornivorous barrel $\mathcal{T}$ of $C_0(E)$, there exists a bornivorous barrel T of E such that $C_0(T) \subset \mathcal{T}$.

ii) E'_β has property (B).

Proof. ii) $\Rightarrow$ i) Let $\mathcal{T}$ be a bornivorous barrel of $C_0(E)$. If $*$ denotes the polar in $\langle C_0(E), C_0(E)'\rangle$, then $\mathcal{T}^*$ is a strongly bounded subset of $\ell^1_\pi\{E'_\beta\}$. By hypothesis, there is a strongly bounded set $B \subset E'$, which can be chosen absolutely convex and $\sigma(E', E)$-closed, such that

$$\sum_{n=1}^{\infty} p_B(x'_n) \leq 1 \qquad \text{for all } (x'_n) \in \mathcal{T}^* \tag{1}$$

Thus, for each $(x'_n) \in \mathcal{T}^*$ and $(x_n) \in C_0(B^\circ)$, from (1) we obtain

$$|\langle (x_n), (x'_n)\rangle| \leq \sum_{n=1}^{\infty} |\langle x_n, x'_n\rangle| \leq \sum_{n=1}^{\infty} p_B(x'_n) \leq 1$$

Hence, $C_0(B^\circ) \subset \mathcal{T}^{**} = \mathcal{T}$.

i) $\Rightarrow$ ii) Let $\mathcal{A}$ be a bounded subset of $\ell^1_\pi\{E'_\beta\}$. If we put

$$\mathcal{A}_0 = \{\hat{x}'(p) : p \in \mathbb{N}, \hat{x}' \in \mathcal{A}\},$$

then $\mathcal{A}_0 \subset C_0(E)'$ and it is also a bounded subset of $\ell^1_\pi\{E'_\beta\}$ (here, $\hat{x}'(p)$ denotes the sequence $(x'_1, \ldots, x'_p, 0, 0, \ldots)$). Therefore, it is enough to carry out the proof supposing that $\mathcal{A}$ is a bounded subset of $\ell^1_\pi\{E'_\beta\}$ contained in $C_0(E)'$. By (i), there exists a bornivorous barrel T of E such that $C_0(T) \subset \mathcal{A}^*$. Then $\mathcal{A} \subset C_0(T)^*$. Hence, given $(x_n) \in T^{\mathbb{N}}$, $(x'_n) \in C_0(T)^*$ and $p \in \mathbb{N}$, we have (for suitable complex numbers α_n, $|\alpha_n| = 1$)

$$\sum_{n=1}^{p} |\langle x_n, x'_n\rangle| = \left|\sum_{n=1}^{p} \langle \alpha_n x_n, x'_n\rangle\right| = |\langle(\alpha_1 x_1, \ldots, \alpha_p x_p, 0, 0, \ldots), (x'_n)\rangle| \leq 1$$

because $(\alpha_1 x_1, \ldots, \alpha_p x_p, 0, 0, \ldots) \in C_0(T)$. Thus, we have the inequality

$$\sum_{n=1}^{\infty} |\langle x_n, x'_n\rangle| \leq 1 \qquad \text{for all } (x_n) \in T^{\mathbb{N}} \text{ and } (x'_n) \in C_0(T)^*.$$

This proves that $\sum_{n=1}^{\infty} p_{T^\circ}(x'_n) \leq 1$ for all $(x'_n) \in \mathcal{A}$. $\square$

2. The completely regular case

Now we consider the completely regular case. If X is completely regular and Hausdorff, we denote by $C_0(X,E)$ the space of all continuous functions $\phi : X \to E$ such that $\phi(X)$ is relatively compact, endowed with the uniform convergence topology. Again, if $T \subset E$ is absolutely convex, we put $C_0(X,T) := \{\phi \in C_0(X,E) : \phi(X) \subset T\}$. We say that $C(X,E)$ has the b.b. property if, for every bornivorous barrel $\mathcal{T}$ of $C(X,E)$, there exists a bornivorous barrel $T \subset E$ such that $C_0(X,T) \subset \mathcal{T}$.

Proposition 3

If E'_β has property (B), then $C(X,E)$ has the b.b. property

Proof. As the canonical linear map $C_0(X,E) \to C(X,E)$ is continuous, $\mathcal{T} \cap C_0(X,E)$ is a bornivorous barrel in $C_0(X,E)$ for every bornivorous barrel $\mathcal{T}$ of $C(X,E)$. Since the spaces $C_0(X,E)$ and $C(\beta X,E)$ are isomorphic, the conclusion follows from section 1. □

Proposition 4

If X is a completely regular space such that $C(X,E)$ has the b.b. property, then $C(K,E)$ has the b.b. property for every compact subset K of X.

Proof. Let us denote by $R : C(X,E) \to C(K,E)$ the restriction map. If $\mathcal{T}$ is a bornivorous barrel of $C(K,E)$, then $R^{-1}(\mathcal{T})$ so is in $C(X,E)$. Hence, there exists a bornivorous barrel $T \subset E$ such that $C_0(X,T) \subset R^{-1}(\mathcal{T})$, i.e.,

$$R(C_0(X,T)) \subset \mathcal{T}. \tag{2}$$

Now we shall prove that $C(K,T) \subset \mathcal{T}$. For this, by [2, 2.3(i)], it is enough to prove that $\mathcal{P}(K,T) \subset \mathcal{T}$, where $\mathcal{P}(K,T)$ denotes the set of all functions $\phi = \sum_{i=1}^n \varphi_i x_i$, with $\varphi_i \in C(K)$, $x_i \in T$, and $\sum |\varphi_i(t)| \leq 1$ for all $t \in K$. Let $\phi = \sum_{i=1}^n \varphi_i x_i$ be such a function. For each $i = 1, \dots, n$, we choose a continuous extension $\bar{\varphi}_i : X \to \mathcal{K}$ of φ_i (here $\mathcal{K}$ denotes the scalar field). It is easy to check that each function $\psi_i : X \to \mathcal{K}$ defined by

$$\psi_i(t) = \begin{cases} \bar{\varphi}_i(t), & \text{if } \sum_{i=1}^n |\bar{\varphi}_i(t)| \leq 1; \\ \bar{\varphi}_i(t) \left(\sum_{i=1}^n |\bar{\varphi}_i(t)| \right)^{-1}, & \text{otherwise,} \end{cases}$$

is a continuous extension of φ_i and the inequality $\sum_{i=1}^n |\psi_i(t)| \leq 1$ holds for all $t \in X$. Then $\psi = \sum_{i=1}^n \psi_i x_i$ belongs to $C_0(X,T)$ and it follows from (2) that $R(\psi) = \phi$ is an element of $\mathcal{T}$. □

In view of the above Propositions, we may state the following

Theorem

If X is a completely regular and Hausdorff topological space containing a compact and infinite subset, then the following are equivalent:

i) $C(X,E)$ has the b.b. property.

ii) E'_β has property (B).

Remark. a) It is obvious that $C(X,E)$ has the b.b. property if it is quasibarrelled.

b) If every compact subset of X is finite, there are spaces $C(X,E)$ which have the b.b. property but E'_β has not property (B). Indeed, if we take $X = \mathbb{N}$ and $E = \Phi$, then $C(X,E)$ is quasibarrelled [2, Theorem 2.10] and $E'_\beta = \Omega$ has not property (B) [3, p. 31].

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